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Common Fixed Points for Multiplicative Contractions in Multiplicative Metric Spaces

□ SONG Jiping¹, LUO Tianqi¹, LEI Lei²

1. College of Mathematics and Physics, Leshan Normal University, Leshan 614000, Sichuan, China;
2. College of Mathematics, Chengdu Normal University, Chengdu 610000, Sichuan, China

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Abstract: The study delves into multiplicative contractions, exploring the existence and uniqueness of common fixed points for a weakly compatible pair of mappings. Those mappings adhere to specific multiplicative contraction conditions characterized by exponents expressed as fraction multiplicative metric spaces. It is noted that a metric can induce a multiplicative metric, and conversely, a multiplicative metric can give a rise to a metric on a nonempty set. As an application, another proof of the existence and uniqueness of the solution of a multiplicative initial problem is given.

Key words: weakly compatible pair of mappings; common fixed point; multiplicative metric space

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0 Introduction

Fixed point theory provides essential tools for solving the existence of unique solutions to many problems in mathematics and applications. The Banach contraction mapping principle, one of the fundamental and most widely applied fixed point theorems, has been generalized. These expansions generally proceed along two lines: one is extending the domain of mappings, and another is considering a more general contractive condition on mappings; the fixed points or common fixed points of mappings satisfying certain contraction conditions on a specific space have received much research^[1-6]. Bashirov *et al* introduced the notion of multiplicative metric spaces and studied some fundamental theorems of multiplicative calculus^[7]. Özavsar and Cevikel studied some topological properties of multiplicative metric spaces and proved an analogous result to the Banach contraction principle in multiplicative metric spaces^[8]. Since then, some fixed-point and common fixed-point results have been obtained in multiplicative metric spaces^[9-13].

This paper shows some common fixed-point results for two mappings satisfying specific multiplicative contraction conditions with exponents of fraction expression in multiplicative metric spaces.

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Biography: SONG Jiping, male, Professor, research direction: functional analysis and its application. E-mail: songjp@lsnu.edu.cn

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1 Preliminaries

Definition 1^[7] Let X be a nonempty set. If mapping $d: X \times X \rightarrow \mathbb{R}^+$ satisfies the following conditions:

- (1) $d(x, y) \geq 1$ for all $x, y \in X$, and $d(x, y) = 1$ if and only if $x = y$;
- (2) $d(x, y) = d(y, x)$ for all $x, y \in X$;
- (3) $d(x, z) \leq d(x, y)d(y, z)$ for all $x, y, z \in X$ (multiplicative triangle inequality).

We say d is a multiplicative metric on X , and (X, d) a multiplicative metric space.

Definition 2^[8] Let (X, d) be a multiplicative metric space, $\{x_n\} \subset X, x \in X, \{x_n\}$ is said to be multiplicative convergent to x , if for arbitrary $\varepsilon > 1$, there exists a natural number N such that $d(x_n, x) < \varepsilon$ for all $n > N$, denoted by $x_n \rightarrow x (n \rightarrow \infty)$.

Definition 3^[8] Let (X, d) be a multiplicative metric space, $\{x_n\} \subset X, \{x_n\}$ is called a multiplicative Cauchy sequence, if for arbitrary $\varepsilon > 1$, there exists a natural number N such that $d(x_n, x_m) < \varepsilon$ for all $n, m > N$.

We say that (X, d) is complete if every multiplicative Cauchy sequence in (X, d) is multiplicative convergent to $x \in X$.

Proposition 1^[8] The uniqueness of the limit holds for a convergent sequence in a multiplicative metric space.

Definition 4^[7] Multiplicative absolute value function $|\cdot|^*: \mathbb{R}^+ \rightarrow \mathbb{R}^+$ is defined as:

$$|x|^* = \begin{cases} x, & x \geq 1 \\ \frac{1}{x}, & x < 1 \end{cases}$$

Remark 1 Multiplicative absolute value function $|\cdot|^*: \mathbb{R}^+ \rightarrow \mathbb{R}^+$ satisfies :

- 1) $|x|^* \geq 1$; 2) $|\frac{x}{x_0}|^* < a \Leftrightarrow \frac{x_0}{a} < x < ax, x_0, x \in \mathbb{R}^+$.

Proposition 2^[8] $(\mathbb{R}^+, |\cdot|^*)$ is a complete multiplicative metric space.

Proposition 3^[8] Let (X, d) be a multiplicative metric space, $\{x_n\} \subset X, x \in X, \{x_n\}$ multiplicative converges to x if and only if $d(x_n, x) \rightarrow 1 (n \rightarrow \infty)$.

$\{x_n\}$ is a multiplicative Cauchy sequence if and only if $d(x_n, x_m) \rightarrow 1 (n, m \rightarrow \infty)$.

Definition 5^[7] The multiplicative derivative of a function $f: \mathbb{R} \rightarrow \mathbb{R}$ is defined by $\lim_{h \rightarrow 0} \left(\frac{f(x+h)}{f(x)} \right)^{\frac{1}{h}}$. Denoted it by $f^*(x)$.

If $f(x)$ is a positive function and its derivative at x exists, then $f^*(x) = e^{(\ln f(x))'}$.

Definition 6^[7] Let $f: [a, b] \rightarrow \mathbb{R}$ be a positive bounded function, $P = \{x_0, x_1, \dots, x_n\}$ be a partition of $[a, b]$, and $\xi_i \in [x_{i-1}, x_i]$. The function f is said to be integral in the multiplicative sense if there exists a number P having the properties: for every $\varepsilon > 0$, there exists a partition P_ε of $[a, b]$ such that $|\prod_{i=1}^n f(\xi_i)^{(x_i - x_{i-1})} - P| < \varepsilon$ for every refinement P of P_ε independently on selection of the numbers $\xi_i \in [x_{i-1}, x_i] (i = 1, 2, \dots, n)$. P is called the multiplicative integral of f on $[a, b]$, we denote it with $\int_a^b f(x)^{dx}$.

It is easily seen that if f is positive and Riemann integral on $[a, b]$, then $\int_a^b f(x)^{dx} = e^{\int_a^b \ln f(x) dx}$.

Let X be a nonempty set, recall that mappings $f, g: X \rightarrow X$ are weakly compatible if, for every $x \in X, fgx = ggx$ holds whenever $fx = gx$. If f and g are weakly compatible and have a unique point of coincidence $w = fx = gx$, then w is the unique common fixed point of f and g .

2 Common Fixed Points

Theorem 1 Let (X, d) be a multiplicative metric space, mappings $f, g: X \rightarrow X$ satisfy: for all $x, y \in X$,

$$d(fx, fy) \leq d(gx, gy)^{\lambda_1} d(fx, gx)^{\lambda_2} d(fy, gy)^{\lambda_3} d(fx, gy)^{\lambda_4} d(fy, gx)^{\lambda_5} \quad (1)$$

holds, where $\lambda_i \in (0, 1) (i = 1, 2, 3, 4, 5)$, $\lambda_1 + \lambda_4 + \lambda_5 < 1$, $\lambda_3 + \lambda_5 < 1$, $\lambda_1 + \lambda_2 + \lambda_3 + 2\lambda_4 < 1$.

If $g(X)$ is a complete subspace of X , $f(X) \subset g(X)$ and f and g are weakly compatible, then f and g have a unique common fixed point.

Proof Let x_0 is an arbitrary point in X , since $f(X) \subset g(X)$, there exists $x_1, x_2 \in X$ such that $fx_0 = gx_1, fx_1 = gx_2$. Continuing this process, we can obtain a sequence $\{x_n\} \subset X$ such that $fx_n = gx_{n+1} (n = 0, 1, 2, \dots)$. From (1), we have

$$\begin{aligned} d(gx_{n+1}, gx_n) &= d(fx_n, fx_{n-1}) \leq d(gx_n, gx_{n-1})^{\lambda_1} d(fx_n, gx_n)^{\lambda_2} d(fx_{n-1}, gx_{n-1})^{\lambda_3} d(fx_n, gx_{n-1})^{\lambda_4} d(fx_{n-1}, gx_n)^{\lambda_5} \\ &= d(gx_n, gx_{n-1})^{\lambda_1} d(gx_{n+1}, gx_n)^{\lambda_2} d(gx_n, gx_{n-1})^{\lambda_3} d(gx_{n+1}, gx_{n-1})^{\lambda_4} d(gx_n, gx_n)^{\lambda_5} \\ &\leq d(gx_n, gx_{n-1})^{\lambda_1 + \lambda_3} d(gx_{n+1}, gx_n)^{\lambda_2} d(gx_{n+1}, gx_n)^{\lambda_4} d(gx_n, gx_{n-1})^{\lambda_5} \\ &= d(gx_{n+1}, gx_n)^{\lambda_2 + \lambda_4} d(gx_n, gx_{n-1})^{\lambda_1 + \lambda_3 + \lambda_5} \end{aligned}$$

which implies that $d(gx_{n+1}, gx_n)^{1 - \lambda_2 - \lambda_4} \leq d(gx_n, gx_{n-1})^{\lambda_1 + \lambda_3 + \lambda_5}$, and so $d(gx_{n+1}, gx_n) \leq d(gx_n, gx_{n-1})^{\frac{\lambda_1 + \lambda_3 + \lambda_5}{1 - \lambda_2 - \lambda_4}}$.

Let $q = \frac{\lambda_1 + \lambda_3 + \lambda_5}{1 - \lambda_2 - \lambda_4}$, then

$$d(gx_{n+1}, gx_n) \leq d(gx_n, gx_{n-1})^q \leq d(gx_{n-1}, gx_{n-2})^{q^2} \leq \dots \leq d(gx_1, gx_0)^{q^n} \quad (2)$$

for all $n = 1, 2, \dots$. Since $\lambda_1 + \lambda_2 + \lambda_3 + 2\lambda_4 < 1$, we get $q < 1$.

For arbitrary natural numbers $n, m, n > m$, from the multiplicative inequality and (2), we have

$$\begin{aligned} d(gx_n, gx_m) &\leq d(gx_n, gx_{n-1}) d(gx_{n-1}, gx_{n-2}) \dots d(gx_{m+1}, gx_m) \leq d(gx_1, gx_0)^{q^{n-1}} d(gx_1, gx_0)^{q^{n-2}} \dots d(gx_1, gx_0)^{q^m} \\ &= d(gx_1, gx_0)^{q^{n-1} + q^{n-2} + \dots + q^m} = d(gx_1, gx_0)^{\frac{q^m(1 - q^{n-m-1})}{1 - q}} < d(gx_1, gx_0)^{\frac{q^m}{1 - q}} \end{aligned}$$

For every $\varepsilon > 1$, by taking $N = \left\lceil \frac{\ln \frac{(1-q)\ln \varepsilon}{\ln(d(gx_1, gx_0))}}{\ln q} \right\rceil$, we have $\left| \frac{d(gx_n, gx_m)}{1} \right|^* < \varepsilon$ for $n > N$, that is, $\{gx_n\}$ is a multi-

plicative Cauchy sequence in $g(X)$.

Since $g(X)$ is a complete, there exists $p \in X$ such that $gx_n \rightarrow gp$. From (1) and the multiplicative inequality,

$$\begin{aligned} d(gx_n, fp) &= d(fx_{n-1}, fp) \leq d(gx_{n-1}, gp)^{\lambda_1} d(fx_{n-1}, gx_{n-1})^{\lambda_2} d(fp, gp)^{\lambda_3} d(fx_{n-1}, gp)^{\lambda_4} d(fp, gx_{n-1})^{\lambda_5} \\ &\leq d(gx_{n-1}, gp)^{\lambda_1} d(gx_n, gx_{n-1})^{\lambda_2} d(fp, gx_n)^{\lambda_3} d(gx_n, gp)^{\lambda_4} d(gx_n, gp)^{\lambda_5} d(fp, gx_n)^{\lambda_5} d(gx_n, gx_{n-1})^{\lambda_5} \end{aligned}$$

We get that $d(gx_n, fp)^{1 - \lambda_3 - \lambda_5} \leq d(gx_{n-1}, gp)^{\lambda_1} d(gx_n, gp)^{\lambda_3 + \lambda_4} d(gx_n, gx_{n-1})^{\lambda_2 + \lambda_5}$, that is

$$d(gx_n, fp) \leq d(gx_{n-1}, gp)^{\frac{\lambda_1}{1 - \lambda_3 - \lambda_5}} d(gx_n, gp)^{\frac{\lambda_3 + \lambda_4}{1 - \lambda_3 - \lambda_5}} d(gx_n, gx_{n-1})^{\frac{\lambda_2 + \lambda_5}{1 - \lambda_3 - \lambda_5}}.$$

Since $gx_n \rightarrow gp$, for every $\varepsilon > 1$, there exists a natural number N such that

$$\left| \frac{d(gx_{n-1}, gp)^{\frac{\lambda_1}{1 - \lambda_3 - \lambda_5}}}{1} \right|^* < \sqrt[3]{\varepsilon}, \quad \left| \frac{d(gx_n, gp)^{\frac{\lambda_3 + \lambda_4}{1 - \lambda_3 - \lambda_5}}}{1} \right|^* < \sqrt[3]{\varepsilon}, \quad \left| \frac{d(gx_n, gx_{n-1})^{\frac{\lambda_2 + \lambda_5}{1 - \lambda_3 - \lambda_5}}}{1} \right|^* < \sqrt[3]{\varepsilon}$$

for $n > N$. Therefore, we obtain that

$$\begin{aligned} \left| \frac{d(gx_n, fp)}{1} \right|^* &= d(gx_n, fp) \leq d(gx_{n-1}, gp)^{\frac{\lambda_1}{1 - \lambda_3 - \lambda_5}} d(gx_n, gp)^{\frac{\lambda_3 + \lambda_4}{1 - \lambda_3 - \lambda_5}} d(gx_n, gx_{n-1})^{\frac{\lambda_2 + \lambda_5}{1 - \lambda_3 - \lambda_5}} \\ &= \left| \frac{d(gx_{n-1}, gp)^{\frac{\lambda_1}{1 - \lambda_3 - \lambda_5}}}{1} \right|^* \left| \frac{d(gx_n, gp)^{\frac{\lambda_3 + \lambda_4}{1 - \lambda_3 - \lambda_5}}}{1} \right|^* \left| \frac{d(gx_n, gx_{n-1})^{\frac{\lambda_2 + \lambda_5}{1 - \lambda_3 - \lambda_5}}}{1} \right|^* < \sqrt[3]{\varepsilon} \cdot \sqrt[3]{\varepsilon} \cdot \sqrt[3]{\varepsilon} = \varepsilon \end{aligned}$$

for $n > N$, which implies that $gx_n \rightarrow fp$. From the uniqueness of the limit, $fp = gp$.

If there exists another point $p^* \in X$ such that $fp^* = gp^*$. From (1), we get

$$\begin{aligned} d(gp, gp^*) &= d(fp, fp^*) \leq d(gp, gp^*)^{\lambda_1} d(fp, gp^*)^{\lambda_2} d(fp^*, gp^*)^{\lambda_3} d(fp, gp^*)^{\lambda_4} d(fp^*, gp^*)^{\lambda_5} \\ &= d(gp, gp^*)^{\lambda_1} d(gp, gp^*)^{\lambda_4} d(gp^*, gp^*)^{\lambda_5} = d(gp, gp^*)^{\lambda_1 + \lambda_4 + \lambda_5}. \end{aligned}$$

Since $\lambda_1 + \lambda_4 + \lambda_5 < 1$, the above inequality implies that $d(gp, gp^*) = 1$, so $gp = gp^*$. Since f and g are weakly compatible, $fp (= gp)$ is a unique common fixed point of f and g .

Corollary 1 Let (X, d) be a multiplicative metric space, mappings $f, g: X \rightarrow X$ satisfy the followings: for arbitrary $x, y \in X$, inequality (1) holds, where $\lambda_i (i = 1, 2, 3, 4, 5)$ are non-negative real numbers and $\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 + \lambda_5 \leq 1, \lambda_3 < \lambda_2, \lambda_4 < \lambda_5$. If $g(X)$ is a complete subspace of $X, f(X) \subset g(X)$ and f and g are weakly compatible. Then f and g have a unique common fixed point.

Proof Since $\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 + \lambda_5 \leq 1, \lambda_3 < \lambda_2, \lambda_4 < \lambda_5$, we get

$$\lambda_1 + \lambda_4 + \lambda_5 < \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 + \lambda_5 \leq 1, \lambda_3 + \lambda_5 < \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 + \lambda_5 \leq 1, \lambda_1 + \lambda_2 + \lambda_3 + 2\lambda_4 < \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 + \lambda_5 \leq 1.$$

The conditions in Theorem 1 are satisfied. The conclusion is true from Theorem 1.

Corollary 2 Let (X, d) be a multiplicative metric space, mappings $f, g: X \rightarrow X$ satisfy the following: for arbitrary $x, y \in X$,

$$d(fx, fy) \leq d(gx, gy)^{\alpha} (d(fx, gx)d(fy, gy))^{\beta} (d(fx, gy)d(fy, gx))^{\gamma}$$

holds, where α, β, γ are non-negative real numbers and $\alpha + 2\beta + 2\gamma < 1$. If $g(X)$ is a complete subspace of $X, f(X) \subset g(X)$ and f and g are weakly compatible. Then f and g have a unique common fixed point.

Proof Let $\lambda_1 = \alpha, \lambda_2 = \lambda_3 = \beta, \lambda_4 = \lambda_5 = \gamma$, it is easy to see that the conclusion is true from Theorem 1.

Corollary 3^[6] Let (X, d) be a multiplicative metric space, mappings $f: X \rightarrow X$ satisfy: for arbitrary $x, y \in X$,

$$d(fx, fy) \leq d(x, y)^{\lambda}$$

holds, where $0 < \lambda < 1$. Then f has a unique fixed point.

Proof By taking $g = I_X$ and $\lambda_1 = \lambda, \lambda_2 = \lambda_3 = \lambda_4 = \lambda_5 = 0$ in Theorem 1, we see that f has a unique fixed point.

Theorem 2 Let (X, d) be a multiplicative metric space, mappings $f, g: X \rightarrow X$ satisfying followings: for arbitrary $x, y \in X$,

$$d(fx, fy) \leq \max \left\{ d(gx, gy)^{\alpha}, \left(d(fx, gx)d(fy, gy) \right)^{\frac{\beta}{d(gx, gy)}}, \left(d(fy, gx)d(fx, gy) \right)^{\frac{\gamma d(fx, gy)}{d(fx, fy)d(gx, gy)d(fy, gy)}} \right\} \quad (3)$$

holds, where α, β, γ are non-negative real numbers and $2\max\{\alpha, \beta, \gamma\} < 1$. If $g(X)$ is a complete subspace of $X, f(X) \subset g(X)$, f and g are weakly compatible, then f and g have a unique common fixed point.

Proof As we do in the proof of Theorem 1, we can obtain a sequence $\{x_n\} \subset X$ such that $fx_n = gx_{n+1} (n = 0, 1, 2, \dots)$. From (3), we have

$$\begin{aligned} d(gx_{n+1}, gx_n) &= d(fx_n, fx_{n-1}) \\ &\leq \max \left\{ d(gx_n, gx_{n-1})^{\alpha}, \left(d(fx_n, gx_n)d(fx_{n-1}, gx_{n-1}) \right)^{\frac{\beta}{d(gx_n, gx_{n-1})}}, \left(d(fx_{n-1}, gx_n)d(fx_n, gx_{n-1}) \right)^{\frac{\gamma d(fx_n, gx_{n-1})}{d(fx_n, fx_{n-1})d(gx_n, gx_{n-1})d(fx_{n-1}, gx_{n-1})}} \right\}. \end{aligned}$$

Since

$$\begin{aligned} \left(d(fx_n, gx_n)d(fx_{n-1}, gx_{n-1}) \right)^{\frac{\beta}{d(gx_n, gx_{n-1})}} &= \left(d(gx_{n+1}, gx_n)d(gx_n, gx_{n-1}) \right)^{\frac{\beta}{d(gx_n, gx_{n-1})}} \\ &\leq \left(d(gx_{n+1}, gx_n)d(gx_n, gx_{n-1}) \right)^{\beta} \leq \left(d(gx_{n+1}, gx_n)d(gx_n, gx_{n-1}) \right)^{\max\{\alpha, \beta, \gamma\}}, \\ \left(d(fx_{n-1}, gx_n)d(fx_n, gx_{n-1}) \right)^{\frac{\gamma d(fx_n, gx_{n-1})}{d(fx_n, fx_{n-1})d(gx_n, gx_{n-1})d(fx_{n-1}, gx_{n-1})}} &\leq \left(d(gx_n, gx_n)d(gx_{n+1}, gx_{n-1}) \right)^{\frac{\gamma d(gx_{n+1}, gx_n)d(gx_n, gx_{n-1})}{d(gx_{n+1}, gx_n)d(gx_n, gx_{n-1})d(gx_n, gx_{n-1})}} \\ &= \left(d(gx_n, gx_n)d(gx_{n+1}, gx_{n-1}) \right)^{\frac{\gamma}{d(gx_n, gx_{n-1})}} \leq \left(d(gx_{n+1}, gx_n)d(gx_n, gx_{n-1}) \right)^{\max\{\alpha, \beta, \gamma\}}, \end{aligned}$$

and

$$d(gx_n, gx_{n-1})^a \leq (d(gx_{n+1}, gx_n)d(gx_n, gx_{n-1}))^a \leq (d(gx_{n+1}, gx_n)d(gx_n, gx_{n-1}))^{\max\{a, \beta, \gamma\}}.$$

We get

$$d(gx_{n+1}, gx_n) \leq (d(gx_{n+1}, gx_n)d(gx_n, gx_{n-1}))^{\max\{a, \beta, \gamma\}}$$

and so

$$d(gx_{n+1}, gx_n)^{1-\max\{a, \beta, \gamma\}} \leq d(gx_n, gx_{n-1})^{\max\{a, \beta, \gamma\}}.$$

Let $\lambda = \frac{\max\{a, \beta, \gamma\}}{1-\max\{a, \beta, \gamma\}}$, then $0 < \lambda < 1$ and

$$d(gx_{n+1}, gx_n) \leq d(gx_n, gx_{n-1})^\lambda \quad (4)$$

From (4), we get

$$d(gx_{n+1}, gx_n) \leq d(gx_1, gx_0)^{\lambda^n} \quad (5)$$

For natural numbers $n, m, n > m$, from the multiplicative triangle inequality and (5), we get

$$\begin{aligned} d(gx_n, gx_m) &\leq d(gx_n, gx_{n-1})d(gx_{n-1}, gx_{n-2}) \cdots d(gx_{m+1}, gx_m) \\ &\leq d(gx_1, gx_0)^{\lambda^{n-1} + \lambda^{n-2} + \cdots + \lambda^m} \leq d(gx_1, gx_0)^{\frac{\lambda^n(1-\lambda^{n-m-1})}{1-\lambda}} \leq d(gx_1, gx_0)^{\frac{\lambda^n}{1-\lambda}}. \end{aligned}$$

This implies that $d(gx_n, gx_m) \rightarrow 1$ ($m \rightarrow \infty$). Hence $\{gx_n\}$ is a multiplicative Cauchy sequence in $g(X)$. By the completeness of $g(X)$, there exists $p \in X$ such that $gx_n \rightarrow gp$ ($n \rightarrow \infty$). From (3), we have

$$\begin{aligned} d(gx_n, fp) &= d(fx_{n-1}, fp) \\ &\leq \max \left\{ d(gx_{n-1}, gp)^a, (d(fx_{n-1}, gx_{n-1})d(fp, gp))^{\frac{\beta}{d(gx_{n-1}, gp)}}, (d(fp, gx_{n-1})d(fx_{n-1}, gp))^{\frac{\gamma d(fx_{n-1}, gp)}{d(fx_{n-1}, fp)d(gx_{n-1}, gp)d(fp, gp)}} \right\} \\ &\leq \max \left\{ d(gx_{n-1}, gp)^a, (d(gx_n, gx_{n-1})d(fp, gp))^{\frac{\beta}{d(gx_{n-1}, gp)}}, (d(fp, gx_{n-1})d(gx_n, gp))^{\frac{\gamma d(gx_n, gp)}{d(gx_n, fp)d(gx_{n-1}, gp)d(fp, gp)}} \right\}. \end{aligned}$$

Since

$$\begin{aligned} d(gx_{n-1}, gp)^a &\leq d(gx_{n-1}, gp)^{\max\{a, \beta, \gamma\}} \leq (d(gx_n, gp)d(gx_{n-1}, gp)d(fp, gp))^{\max\{a, \beta, \gamma\}} \\ &\leq (d(gx_n, gp))^{2\max\{a, \beta, \gamma\}} (d(gx_{n-1}, gp))^{\max\{a, \beta, \gamma\}} (d(gx_n, fp))^{\max\{a, \beta, \gamma\}}, \\ (d(gx_n, gx_{n-1})d(fp, gp))^{\frac{\beta}{d(gx_{n-1}, gp)}} &\leq (d(gx_n, gp)d(gx_{n-1}, gp)d(fp, gp))^{\frac{\max\{a, \beta, \gamma\}}{d(gx_{n-1}, gp)}} \\ &\leq (d(gx_n, gp)d(gx_{n-1}, gp)d(fp, gp))^{\max\{a, \beta, \gamma\}} \leq (d(gx_n, gp))^{2\max\{a, \beta, \gamma\}} (d(gx_{n-1}, gp))^{\max\{a, \beta, \gamma\}} (d(gx_n, fp))^{\max\{a, \beta, \gamma\}}, \end{aligned}$$

and

$$\begin{aligned} (d(fp, gx_{n-1})d(gx_n, gp))^{\frac{\gamma d(gx_n, gp)}{d(gx_n, fp)d(gx_{n-1}, gp)d(fp, gp)}} &\leq (d(fp, gp)d(gx_{n-1}, gp)d(gx_n, gp))^{\frac{\max\{a, \beta, \gamma\}d(gx_n, gp)}{d(gx_n, fp)d(gx_{n-1}, gp)d(fp, gp)}} \\ &\leq (d(fp, gp)d(gx_{n-1}, gp)d(gx_n, gp))^{\frac{\max\{a, \beta, \gamma\}d(gx_n, fp)d(fp, gp)}{d(gx_n, fp)d(gx_{n-1}, gp)d(fp, gp)}} \\ &\leq (d(fp, gp)d(gx_{n-1}, gp)d(gx_n, gp))^{\max\{a, \beta, \gamma\}} \\ &\leq (d(gx_n, gp))^{2\max\{a, \beta, \gamma\}} (d(gx_{n-1}, gp))^{\max\{a, \beta, \gamma\}} (d(gx_n, fp))^{\max\{a, \beta, \gamma\}}, \end{aligned}$$

we get

$$d(gx_n, fp) \leq (d(gx_n, gp))^{2\max\{a, \beta, \gamma\}} (d(gx_{n-1}, gp))^{\max\{a, \beta, \gamma\}} (d(gx_n, fp))^{\max\{a, \beta, \gamma\}}.$$

That is,

$$d(gx_n, fp) \leq (d(gx_n, gp))^{\frac{2\max\{a, \beta, \gamma\}}{1-\max\{a, \beta, \gamma\}}} (d(gx_{n-1}, gp))^{\frac{\max\{a, \beta, \gamma\}}{1-\max\{a, \beta, \gamma\}}}.$$

Since $gx_n \rightarrow gp$ ($n \rightarrow \infty$), for every $\varepsilon > 1$, there exists a natural number N , such that $d(gx_n, fp) < \varepsilon$ for $n > N$. Hence, $gx_n \rightarrow fp$ ($n \rightarrow \infty$). From the uniqueness of the limit, $fp = gp$.

If there exists another point $q \in X$ such that $fq = gq$, from (3), we get

$$\begin{aligned} d(fp, fq) &\leq \max \left\{ d(gp, gq)^\alpha, \left(d(fp, gp) d(fq, gq) \right)^{\frac{\beta}{d(gp, gq)}}, \left(d(fq, gp) d(fp, gq) \right)^{\frac{\gamma d(fp, gq)}{d(fp, fq) d(gp, gq) d(fq, gq)}} \right\} \\ &= \max \left\{ d(fp, fq)^\alpha, \left(d(fp, fp) d(fq, fq) \right)^{\frac{\beta}{d(fp, fq)}}, \left(d(fp, fq) \right)^{\frac{2\gamma}{d(fp, fq) d(fq, fq)}} \right\} \\ &= \max \left\{ d(fp, fq)^\alpha, \left(d(fp, fq) \right)^{\frac{2\gamma}{d(fp, fq) d(fq, fq)}} \right\} \\ &\leq d(fp, fq)^{2\max\{\alpha, \beta, \gamma\}}. \end{aligned}$$

Since $2\max\{\alpha, \beta, \gamma\} < 1$, it is a contradiction. Because f and g are weakly compatible, f and g have a unique common fixed point.

3 Applications

Let (X, ρ) be a usual metric space, we define $d: X \times X \rightarrow [1, +\infty)$ as follows: for any $x, y \in X$,

$$d(x, y) = e^{\rho(x, y)} \quad (6)$$

It is easy to see that (X, d) is a multiplicative metric space. Conversely, if (X, d) is a multiplicative metric space, by defining $\rho: X \times X \rightarrow \mathbb{R}^+$: for any $x, y \in X$,

$$\rho(x, y) = \ln d(x, y) \quad (7)$$

We have a metric space (X, ρ) .

Theorem 3 Let X be a nonempty set, ρ and d as (6) or (7), and $A \subset X$.

- 1) (X, ρ) is a complete metric space if and only if (X, d) is a complete multiplicative metric space;
- 2) A is an open set in (X, ρ) if and only if A is a multiplicative open set in (X, d) ;
- 3) A is a closed set in (X, ρ) if and only if A is a multiplicative closed set in (X, d) .

Proof 1) Let (X, ρ) be a complete metric space and $\{x_n\}$ a multiplicative Cauchy sequence (X, d) . For every $\varepsilon > 0$, there exists a natural number N such that $d(x_n, x_m) < e^\varepsilon$ for $n, m > N$. So we have $\rho(x_n, x_m) < \varepsilon$ for $n, m > N$. This implies that $\{x_n\}$ is a Cauchy sequence in (X, ρ) . Since (X, ρ) is complete, there exists $x_0 \in X$. For any $\varepsilon > 1$, there exists a natural number N such that $\rho(x_n, x_0) < \ln \varepsilon$ for $n > N$. Hence, $d(x_n, x_0) < \varepsilon$ for $n > N$, and so $\{x_n\}$ multiplicative converges to x_0 in (X, d) , (X, d) is a complete multiplicative metric space.

Similarly, the inverse state is true.

2) Let A is an open set in (X, ρ) . For every $x \in A$, there exists $\varepsilon > 0$ such that $B_\rho(x, \varepsilon) \subset A$ ($B_\rho(x, \varepsilon)$ is an open ball of radius $\varepsilon > 0$ with center x in (X, ρ)). Let $y \in B_d(x, e^\varepsilon)$ ($B_d(x, e^\varepsilon)$ is an open ball of radius $e^\varepsilon > 1$ with center x in (X, d)), then $d(x, y) < e^\varepsilon$. From (7), $\rho(x, y) = \ln d(x, y) < \varepsilon$, that is $y \in B_\rho(x, \varepsilon)$. This implies $B_d(x, e^\varepsilon) \subseteq B_\rho(x, \varepsilon) \subset A$. Hence, A is a multiplicative open set in (X, d) .

Similarly, the inverse state is true.

3) From 2), we see that the conclusion in 3) is true.

Remark 2 Theorem 3 shows that metric space (X, ρ) has the same topological properties as multiplicative metric space (X, d) if ρ and d satisfies equality (6) or (7).

Consider multiplicative initial value problem:

$$\begin{cases} y^* = f(x, y) \\ y(x_0) = y_0 \end{cases} \quad (8)$$

where y^* is the multiplicative derivative of y , $f(x, y)$ is defined on some subset $G \subset \mathbb{R} \times \mathbb{R}^+$, and $f(x, y) > 0$, $(x_0, y_0) \in G$.

We say that $f(x, y)$ satisfy multiplicative Lipschitz type condition concerning second coordinate on G , if for any $(x, y), (x, z) \in G$, $\left| \frac{f(x, y)}{f(x, z)} \right|^* \leq L^{|y-z|}$ holds, where $L > 1$ is a constant^[7].

Theorem 4 Let $f(x, y)$ be a positive continuous function on a rectangular region $D = \left\{ (x, y) \mid |x - x_0| \leq a, \left| \frac{y}{y_0} \right|^* \leq b \right\}$ ($a > 0, b > 1$). If $f(x, y)$ satisfies the multiplicative Lipschitz type condition concerning the second coordinate on D , then multiplicative initial value problem (6) has a unique solution on $[x_0 - \delta, x_0 + \delta]$, where $\delta < \min \left\{ a, \frac{\ln b}{\ln M}, \frac{1}{y_0 b \ln L} \right\}$, $M = \max_{(x, y) \in D} f(x, y)$.

Proof Let $X = C[x_0 - \delta, x_0 + \delta]$. $\rho: X \times X \rightarrow \mathbb{R}$ is defined as $\rho(y_1(x), y_2(x)) = \max_{x \in [x_0 - \delta, x_0 + \delta]} |y_1(x) - y_2(x)|$ for $y_1(x), y_2(x) \in X$, since (X, ρ) is a complete metric space. (X, d) is a complete multiplicative metric space, where $d(y_1(x), y_2(x)) = e^{\rho(y_1(x), y_2(x))}$. We define $T: X \rightarrow X$ as $Ty(x) = y_0 \int_{x_0}^x f(x, y(x)) dx$.

Let $\tilde{C} = \left\{ y(x) \mid y(x) \in X, \left| \frac{Ty(x)}{y_0} \right|^* \leq M^\delta \right\}$. It is easy to see that $\tilde{C}$ is a closed set in (X, ρ) , $\tilde{C}$ is complete in (X, ρ) ,

and so it is complete in (X, d) . Since $f(x, y) \leq M$, we have $-\delta \ln M \leq \int_{x_0}^x \ln f(x, y(x)) dx \leq \delta \ln M$.

That is, $\frac{1}{M^\delta} = e^{-\delta \ln M} \leq e^{\int_{x_0}^x \ln f(x, y(x)) dx} \leq e^{\delta \ln M} = M^\delta$. Hence $\frac{1}{M^\delta} \leq \frac{Ty(x)}{y_0} \leq M^\delta$, which implies that $\left| \frac{Ty(x)}{y_0} \right|^* \leq M^\delta$. Then,

$Ty(x) \in \tilde{C}$.

For $y_1(x), y_2(x) \in \tilde{C}$, since $\left| \frac{f(x, y_1(x))}{f(x, y_2(x))} \right|^* \leq L^{|y_1(x) - y_2(x)|}$, we have :

$$-|y_1(x) - y_2(x)| \ln L \leq \ln \frac{f(x, y_1(x))}{f(x, y_2(x))} \leq |y_1(x) - y_2(x)| \ln L.$$

We get:

$$-\ln L \left| \int_{x_0}^x |y_1(x) - y_2(x)| dx \right| \leq \int_{x_0}^x \ln \frac{f(x, y_1(x))}{f(x, y_2(x))} dx \leq \ln L \left| \int_{x_0}^x |y_1(x) - y_2(x)| dx \right|.$$

That is

$$\left| \int_{x_0}^x \ln \frac{f(x, y_1(x))}{f(x, y_2(x))} dx \right| \leq \ln L \left| \int_{x_0}^x |y_1(x) - y_2(x)| dx \right| \leq \delta \rho(y_1(x), y_2(x)) \ln L.$$

Thus, we obtain that

$$\begin{aligned} & \left| e^{\int_{x_0}^x \ln f(x, y_1(x)) dx} - e^{\int_{x_0}^x \ln f(x, y_2(x)) dx} \right| = \left| e^{\int_{x_0}^x \ln f(x, y_2(x)) dx} \cdot \left(e^{\int_{x_0}^x \ln \frac{f(x, y_1(x))}{f(x, y_2(x))} dx} - 1 \right) \right| \\ & \leq e^{\left| \int_{x_0}^x \ln f(x, y_2(x)) dx \right|} \cdot \left| e^{\int_{x_0}^x \ln \frac{f(x, y_1(x))}{f(x, y_2(x))} dx} - 1 \right| \leq M^\delta \cdot \left| \int_{x_0}^x \ln \frac{f(x, y_1(x))}{f(x, y_2(x))} dx \right| < \delta M^\delta \rho(y_1(x), y_2(x)) \ln L. \end{aligned}$$

We conclude that

$$d(Ty_1(x), Ty_2(x)) = e^{\max_{x \in [x_0 - \delta, x_0 + \delta]} \{ |Ty_1(x) - Ty_2(x)| \}} = e^{\max_{x \in [x_0 - \delta, x_0 + \delta]} \left| y_0 \int_{x_0}^x f(x, y_1(x)) dx - y_0 \int_{x_0}^x f(x, y_2(x)) dx \right|}$$

$$= e^{\max_{s \in [x_0 - \delta, x_0 + \delta]} \left| \int_{x_0}^s \log(z, y_1(x)) dx - \int_{x_0}^s \log(z, y_2(x)) dx \right|} < e^{y_0 \delta M^{\delta} \ln L \rho(y_1(x), y_2(x))} = \left(e^{\rho(y_1(x), y_2(x))} \right)^{y_0 \delta M^{\delta} \ln L} = d(y_1(x), y_2(x))^{y_0 \delta M^{\delta} \ln L} = d(y_1(x), y_2(x))^{\lambda},$$

where $\lambda = y_0 \delta M^{\delta} \ln L < 1$. By Corollary 3, T has a unique fixed point in $\tilde{C}$, denoted it by $\tilde{y}(x)$, that is $\tilde{y}(x) = y_0 \int_{x_0}^x f(x, \tilde{y}(x))^{dx}$. Thus, $\tilde{y}^*(x) = f(x, \tilde{y}(x))$, and $\tilde{y}(x_0) = y_0$. Hence, $\tilde{y}(x)$ is a solution of (6).

If $\tilde{y}_1(x)$ is another solution of (6), since

$$\left(\frac{\tilde{y}_1(x)}{\int_{x_0}^x f(x, \tilde{y}_1(x))^{dx}} \right)^* = \frac{\tilde{y}_1^*(x)}{\left(\int_{x_0}^x f(x, \tilde{y}_1(x))^{dx} \right)^*} = 1,$$

we have $\frac{\tilde{y}_1(x)}{\int_{x_0}^x f(x, \tilde{y}_1(x))^{dx}} = C$ (a constant), then $\tilde{y}_1(x) = C \int_{x_0}^x f(x, \tilde{y}_1(x))^{dx}$. From $\tilde{y}_1(x_0) = y_0$, we get $C = y_0$, which im-

plies that $\tilde{y}_1(x)$ is also a fixed point of T . Therefore, $\tilde{y}(x) = \tilde{y}_1(x)$, we conclude that the solution of (6) is unique.

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