Wuhan University
Journal of Natural Sciences

2024, Vol.29 No.6, 547-557

Article ID 1007-1202(2024)06-0547-11 DOI https://doi.org/10.1051/wujns/2024296547

Cite this article: WANG Cai, SONG Chuanjing. Symmetries of Fractional Constrained Hamiltonian System Described by the Singular Lagrangian[J]. *Wuhan Univ J of Nat Sci*, 2024, 29(6): 547-557.

Symmetries of Fractional Constrained Hamiltonian System Described by the Singular Lagrangian

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Abstract: Singular systems within combined fractional derivatives are established. Firstly, the fractional Lagrange equation is analyzed. Secondly, the fractional primary constraint is given. Thirdly, the Noether and Lie symmetry methods of fractional constrained Hamiltonian system are studied. Finally, the obtained results are illustrated with an example.

Key words: combined fractional derivative; constrained Hamilton equation; conserved quantity; Noether symmetry; Lie symmetry

CLC number: O316

0 Introduction

With the continuous development of mathematics and physics, fractional calculus has been widely discussed in recent years. Fractional-order model enables us to describe complex dynamics and physical behaviors better. Fractional calculus has important applications in fluid mechanics, nuclear magnetic resonance imaging, mechanics of complex viscoelastic materials, and so on[1-5]. Riewe[6,7] introduced fractional derivatives to deal with dissipative forces. After that, research on fractional derivatives has mainly focused on the left and right Riemann-Liouville, the left and right Caputo, the Riesz-Riemann-Liouville and the Riesz-Caputo fractional derivatives. Recently, a more general definition of fractional derivative, which is called combined fractional derivative, was introduced. For example, Malinowska and Torres[8] studied fractional calculus of variations based on a combined Caputo fractional derivative. Zhang[9] established the fractional differential equations of motion using a combined Riemann-Liouville fractional derivative. In 2015, Luo *et al*[10] presented the dynamics of a Birkhoffian system with both combined Caputo fractional derivatives and combined Riemann-Liouville fractional derivatives. Specifically, the four derivatives mentioned above are all special cases of the combined fractional derivative. Therefore, the combined fractional derivative is general.

The singular system refers to the procedure described by the singular Lagrangian. The singular system plays a vital role in modern quantum field theory, such as gravity theory, string (membrane) field theory, Yang-Mills theory, supersymmetry, and supergravity[11]. When a singular system is described by the canonical variables, there are several in-

Received date: 2023-09-28

Foundation item: Supported by the National Natural Science Foundation of China (12172241, 12272248) and the Qing Lan Project of Colleges and Universities in Jiangsu Province

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herent constraints. In this case, the system is called the constrained Hamiltonian system, which is essential in condensed matter theory, gauge field theory and many other aspects[12-14].

Noether symmetry and Lie symmetry are two commonly used methods to study constrained mechanics systems. Noether symmetry was put forward by Noether[15]. Lie symmetry was introduced by Lutzky[16]. Then Noether and Lie symmetry were studied further[17,18]. Particularly, for singular systems, Lie symmetry and conserved quantity based on Riemann-Liouville fractional derivative[19], Noether symmetry and conserved quantity based on mixed order derivative and Caputo fractional derivative[20] are studied. Besides, based on the combined fractional derivative, Noether symmetry and conserved quantity for the Birkhoffian system are studied[21]. In this paper, we intend to study Noether symmetry and Lie symmetry for the singular system based on the combined fractional derivative.

Section 1 gives the preliminaries. In Section 2, the fractional Lagrange equation is analyzed using the combined fractional derivative. In Section 3 and Section 4, the fractional primary constraint is established and the Hamilton equation of the fractional constraint is given. In Section 5 and Section 6, the fractional Noether symmetry, Lie symmetry, and conserved quantities are studied, respectively. An example is given in Section 7 and the conclusion is given in Section 8.

1 Preliminaries

The following preliminaries are about the combined fractional derivatives, which can be seen in Ref. [8], Ref. [22] and Ref. [23]. The combined fractional derivative includes the combined Riemann-Liouville fractional derivative and the combined Caputo fractional derivative.

The combined Riemann-Liouville fractional derivative and combined Caputo fractional derivative are

$$ {}^{\mathrm{RL}}D_{\gamma}^{\alpha,\beta} f(t)=\gamma {}_{t_1}^{\mathrm{RL}}D_t^{\alpha} f(t)+(-1)^n(1-\gamma){}_t^{\mathrm{RL}}D_{t_2}^{\beta} f(t), \tag{1} $$

$$ {}^{C}D_{\gamma}^{\alpha,\beta} f(t)=\gamma {}_{t_1}^{C}D_t^{\alpha} f(t)+(-1)^n(1-\gamma){}_t^{C}D_{t_2}^{\beta} f(t), \tag{2} $$

where ${}_{t_1}^{\mathrm{RL}}D_t^{\alpha} f(t)$, ${}_t^{\mathrm{RL}}D_{t_2}^{\beta} f(t)$, ${}_{t_1}^{C}D_t^{\alpha} f(t)$ and ${}_t^{C}D_{t_2}^{\beta} f(t)$ are the left and right Riemann-Liouville and Caputo fractional derivatives, $t\in[t_1,t_2]$, $0\leqslant\gamma\leqslant 1$, $n-1\leqslant\alpha,\beta<n$.

Under the condition $0<\alpha,\beta<1$, we have

$$ {}_{t_1}^{\mathrm{RL}}D_t^{\alpha} f(t)={}_{t_1}^{C}D_t^{\alpha} f(t)-\frac{1}{\Gamma(1-\alpha)}\frac{f(t_1)}{(t-t_1)^{\alpha}}, \tag{3} $$

$$ {}_t^{\mathrm{RL}}D_{t_2}^{\beta} f(t)={}_t^{C}D_{t_2}^{\beta} f(t)+\frac{1}{\Gamma(1-\beta)}\frac{f(t_2)}{(t_2-t)^{\beta}}. \tag{4} $$

Additionally, the fractional partial integral formulae are

$$ \int_{t_1}^{t_2}[*]{}_{t_1}^{\mathrm{RL}}D_t^{\alpha}\eta\mathrm{d}t=\int_{t_1}^{t_2}\eta\,{}_t^{C}D_{t_2}^{\alpha}[*]\mathrm{d}t-\sum_{j=0}^{n-1}(-1)^{n+j}\,{}_{t_1}^{\mathrm{RL}}D_t^{\alpha+j-n}\eta(t)D^{n-1-j}[*]\bigg|_{t_1}^{t_2}, \tag{5} $$

$$ \int_{t_1}^{t_2}[*]{}_t^{\mathrm{RL}}D_{t_2}^{\beta}\eta\mathrm{d}t=\int_{t_1}^{t_2}\eta\,{}_{t_1}^{C}D_t^{\beta}[*]\mathrm{d}t-\sum_{j=0}^{n-1}{}_t^{\mathrm{RL}}D_{t_2}^{\beta+j-n}\eta(t)D^{n-1-j}[*]\bigg|_{t_1}^{t_2}, \tag{6} $$

$$ \int_{t_1}^{t_2}[*]{}_{t_1}^{C}D_t^{\alpha}\eta\mathrm{d}t=\int_{t_1}^{t_2}\eta\,{}_t^{\mathrm{RL}}D_{t_2}^{\alpha}[*]\mathrm{d}t+\sum_{j=0}^{n-1}{}_t^{\mathrm{RL}}D_{t_2}^{\alpha+j-n}[*]D^{n-1-j}\eta(t)\bigg|_{t_1}^{t_2}, \tag{7} $$

$$ \int_{t_1}^{t_2}[*]{}_{t_1}^{C}D_t^{\beta}\eta\mathrm{d}t=\int_{t_1}^{t_2}\eta\,{}_{t_1}^{\mathrm{RL}}D_t^{\beta}[*]\mathrm{d}t+\sum_{j=0}^{n-1}(-1)^{n+j}\,{}_{t_1}^{\mathrm{RL}}D_t^{\beta+j-n}[*]D^{n-1-j}\eta(t)\bigg|_{t_1}^{t_2}, \tag{8} $$

$$ \int_{t_1}^{t_2}[*]{}_{t_1}^{R}D_{t_2}^{\alpha}\eta\mathrm{d}t=(-1)^n\int_{t_1}^{t_2}\eta\,{}_{t_1}^{RC}D_{t_2}^{\alpha}[*]\mathrm{d}t-\sum_{j=0}^{n-1}(-1)^{n+j}\,{}_{t_1}^{R}D_{t_2}^{\alpha+j-n}\eta(t)D^{n-1-j}[*]\bigg|_{t_1}^{t_2}, \tag{9} $$

$$ \int_{t_1}^{t_2}[*]{}_{t_1}^{RC}D_{t_2}^{\alpha}\eta\mathrm{d}t=(-1)^n\int_{t_1}^{t_2}\eta\,{}_{t_1}^{R}D_{t_2}^{\alpha}[*]\mathrm{d}t+\sum_{j=0}^{n-1}(-1)^{j}\,{}_{t_1}^{R}D_{t_2}^{\alpha+j-n}[*]D^{n-1-j}\eta(t)\bigg|_{t_1}^{t_2}, \tag{10} $$

where $D = \mathrm{d}/\mathrm{d}t$ represents the integer order derivative.

It should be noted that in this paper, we set $0<\alpha,\beta<1$.

2 Fractional Lagrange Equation

In this section, the fractional variational problems under combined fractional derivatives are studied.

We give a function

$$I_{\mathrm{RL}}\left[\boldsymbol{q}_{\mathrm{RL}}(\cdot)\right]=\int_{t_1}^{t_2} L_{\mathrm{RL}}\left(t, \boldsymbol{q}_{\mathrm{RL}}, {}^{\mathrm{RL}}\boldsymbol{D}_{\gamma}^{\alpha,\beta}\boldsymbol{q}_{\mathrm{RL}}\right)\mathrm{d}t,\ \boldsymbol{q}_{\mathrm{RL}}(t_1)=\boldsymbol{q}_{\mathrm{RL}1},\ \boldsymbol{q}_{\mathrm{RL}}(t_2)=\boldsymbol{q}_{\mathrm{RL}2}, \tag{11}$$

where $\boldsymbol{q}_{\mathrm{RL}}=\left(q_{\mathrm{RL}1}, q_{\mathrm{RL}2}, \cdots, q_{\mathrm{RL}n}\right)$, ${}^{\mathbf{RL}}\boldsymbol{D}_{\gamma}^{\alpha,\beta}\boldsymbol{q}_{\mathrm{RL}}=\left({}^{\mathrm{RL}}D_{\gamma}^{\alpha,\beta}q_{\mathrm{RL}1}, {}^{\mathrm{RL}}D_{\gamma}^{\alpha,\beta}q_{\mathrm{RL}2}, \cdots, {}^{\mathrm{RL}}D_{\gamma}^{\alpha,\beta}q_{\mathrm{RL}n}\right)$, $\boldsymbol{q}_{\mathrm{RL}1}=\left(q_{\mathrm{RL}11}, q_{\mathrm{RL}12}, \cdots, q_{\mathrm{RL}1n}\right)$, $\boldsymbol{q}_{\mathrm{RL}2}=\left(q_{\mathrm{RL}21}, q_{\mathrm{RL}22}, \cdots, q_{\mathrm{RL}2n}\right)$, and the Lagrangian L_{RL} is assumed to be a C^2 function. If $\boldsymbol{q}_{\mathrm{RL}}(\cdot)$ is an extremal of Eq. (11), we obtain

$$\left.\frac{\mathrm{d}}{\mathrm{d}\varepsilon_{\mathrm{RL}}} I_{\mathrm{RL}}\left[\boldsymbol{q}_{\mathrm{RL}}+\varepsilon_{\mathrm{RL}}\boldsymbol{h}_{\mathrm{RL}}\right]\right|_{\varepsilon_{\mathrm{RL}}=0}=0, \tag{12}$$

where $\varepsilon_{\mathrm{RL}}$ is a small parameter, $\boldsymbol{h}_{\mathrm{RL}}=\left(h_{\mathrm{RL}1}, h_{\mathrm{RL}2}, \cdots, h_{\mathrm{RL}n}\right)$, $\boldsymbol{h}_{\mathrm{RL}}(t_1)=\boldsymbol{h}_{\mathrm{RL}}(t_2)=0$. Further, we get

$$\int_{t_1}^{t_2}\left(\frac{\partial L_{\mathrm{RL}}}{\partial q_{\mathrm{RL}k}}\cdot h_{\mathrm{RL}k}+\frac{\partial L_{\mathrm{RL}}}{\partial\, {}^{\mathrm{RL}}D_{\gamma}^{\alpha,\beta}q_{\mathrm{RL}k}}\cdot {}^{\mathrm{RL}}D_{\gamma}^{\alpha,\beta}h_{\mathrm{RL}k}\right)\mathrm{d}t=0,\ k=1,2,\cdots,n. \tag{13}$$

Using fractional partial integral formulae (Eqs. (5) and (6)) in the second terms of Eq. (13), we obtain

$$\begin{aligned}\int_{t_1}^{t_2}\frac{\partial L_{\mathrm{RL}}}{\partial\, {}^{\mathrm{RL}}D_{\gamma}^{\alpha,\beta}q_{\mathrm{RL}k}}\cdot {}^{\mathrm{RL}}D_{\gamma}^{\alpha,\beta}h_{\mathrm{RL}k}\,\mathrm{d}t=&-\int_{t_1}^{t_2}\left(h_{\mathrm{RL}k}\cdot {}^{C}D_{1-\gamma}^{\beta,\alpha}\frac{\partial L_{\mathrm{RL}}}{\partial^{\mathrm{RL}}D_{\gamma}^{\alpha,\beta}q_{\mathrm{RL}k}}\right)\mathrm{d}t+\frac{\gamma}{\Gamma(1-\alpha)}\int_{t_1}^{t_2}\left(t_2-t\right)^{-\alpha}h_{\mathrm{RL}k}\mathrm{d}t\cdot\frac{\partial L_{\mathrm{RL}}(t_2)}{\partial^{\mathrm{RL}}D_{\gamma}^{\alpha,\beta}q_{\mathrm{RL}k}}\\&-\frac{1-\gamma}{\Gamma(1-\beta)}\int_{t_1}^{t_2}\left(t-t_1\right)^{-\beta}h_{\mathrm{RL}k}\,\mathrm{d}t\cdot\frac{\partial L_{\mathrm{RL}}(t_1)}{\partial^{\mathrm{RL}}D_{\gamma}^{\alpha,\beta}q_{\mathrm{RL}k}},\end{aligned} \tag{14}$$

where $\dfrac{\partial L_{\mathrm{RL}}(t_1)}{\partial^{\mathrm{RL}}D_{\gamma}^{\alpha,\beta}q_{\mathrm{RL}k}}=\dfrac{\partial L_{\mathrm{RL}}\left(t_1, \boldsymbol{q}_{\mathrm{RL}}(t_1), {}^{\mathrm{RL}}\boldsymbol{D}_{\gamma}^{\alpha,\beta}\boldsymbol{q}_{\mathrm{RL}}(t_1)\right)}{\partial^{\mathrm{RL}}D_{\gamma}^{\alpha,\beta}q_{\mathrm{RL}k}}$, $\dfrac{\partial L_{\mathrm{RL}}(t_2)}{\partial^{\mathrm{RL}}D_{\gamma}^{\alpha,\beta}q_{\mathrm{RL}k}}=\dfrac{\partial L_{\mathrm{RL}}}{\partial^{\mathrm{RL}}D_{\gamma}^{\alpha,\beta}q_{\mathrm{RL}k}}\left(t_2, \boldsymbol{q}_{\mathrm{RL}}(t_2), {}^{\mathrm{RL}}\boldsymbol{D}_{\gamma}^{\alpha,\beta}\boldsymbol{q}_{\mathrm{RL}}(t_2)\right)$. Substituting Eq. (14) into Eq. (13), we obtain

$$\int_{t_1}^{t_2}\left[\frac{\partial L_{\mathrm{RL}}}{\partial q_{\mathrm{RL}k}}-{}^{C}D_{1-\gamma}^{\beta,\alpha}\frac{\partial L_{\mathrm{RL}}}{\partial^{\mathrm{RL}}D_{\gamma}^{\alpha,\beta}q_{\mathrm{RL}k}}+\frac{\gamma\left(t_2-t\right)^{-\alpha}}{\Gamma(1-\alpha)}\frac{\partial L_{\mathrm{RL}}(t_2)}{\partial^{\mathrm{RL}}D_{\gamma}^{\alpha,\beta}q_{\mathrm{RL}k}}-\frac{(1-\gamma)\left(t-t_1\right)^{-\beta}}{\Gamma(1-\beta)}\frac{\partial L_{\mathrm{RL}}(t_1)}{\partial^{\mathrm{RL}}D_{\gamma}^{\alpha,\beta}q_{\mathrm{RL}k}}\right]\cdot h_{\mathrm{RL}k}\mathrm{d}t=0. \tag{15}$$

According to the fundamental lemma of variational calculation[24], we can deduce that

$$\frac{\partial L_{\mathrm{RL}}}{\partial q_{\mathrm{RL}k}}-{}^{C}D_{1-\gamma}^{\beta,\alpha}\frac{\partial L_{\mathrm{RL}}}{\partial^{\mathrm{RL}}D_{\gamma}^{\alpha,\beta}q_{\mathrm{RL}k}}+\frac{\gamma\left(t_2-t\right)^{-\alpha}}{\Gamma(1-\alpha)}\frac{\partial L_{\mathrm{RL}}(t_2)}{\partial^{\mathrm{RL}}D_{\gamma}^{\alpha,\beta}q_{\mathrm{RL}k}}-\frac{(1-\gamma)\left(t-t_1\right)^{-\beta}}{\Gamma(1-\beta)}\frac{\partial L_{\mathrm{RL}}(t_1)}{\partial^{\mathrm{RL}}D_{\gamma}^{\alpha,\beta}q_{\mathrm{RL}k}}=0,\quad k=1,2,\cdots,n. \tag{16}$$

Eq. (16) is called the fractional Lagrange equation within the combined Riemann-Liouville fractional derivative.

The fractional Lagrange equation has another form, which can be obtained from Eqs. (3), (4), (7), (8) and (13) as

$$\frac{\partial L_{\mathrm{RL}}}{\partial q_{\mathrm{RL}k}}-{}^{\mathrm{RL}}D_{1-\gamma}^{\beta,\alpha}\frac{\partial L_{\mathrm{RL}}}{\partial\, {}^{\mathrm{RL}}D_{\gamma}^{\alpha,\beta}q_{\mathrm{RL}k}}=0,\quad k=1,2,\cdots\cdot n. \tag{17}$$

Similarly, the fractional Lagrange equation within the combined Caputo fractional derivative can also be obtained as

$$\frac{\partial L_{C}}{\partial q_{Ck}}-{}^{\mathrm{RL}}D_{1-\gamma}^{\beta,\alpha}\frac{\partial L_{C}}{\partial^{C}D_{\gamma}^{\alpha,\beta}q_{Ck}}=0,\ k=1,2,\cdots,n, \tag{18}$$

where the Lagrangian $L_C=L_C\left(t, \boldsymbol{q}_C, {}^{C}\boldsymbol{D}_{\gamma}^{\alpha,\beta}\boldsymbol{q}_C\right)$, L_C is assumed to be a C^2 function, $\boldsymbol{q}_C=\left(q_{C1}, q_{C2}, \cdots, q_{Cn}\right)$, ${}^{C}\boldsymbol{D}_{\gamma}^{\alpha,\beta}\boldsymbol{q}_C=\left({}^{C}D_{\gamma}^{\alpha,\beta}q_{C1}, {}^{C}D_{\gamma}^{\alpha,\beta}q_{C2}, \cdots, {}^{C}D_{\gamma}^{\alpha,\beta}q_{Cn}\right)$.

Remark 1 Eqs. (16) and (18) are general and universal, so we can select different values of γ to get different results.

3 Fractional Primary Constraint

This section presents fractional primary constraints within the combined Riemann-Liouville fractional derivative and combined Caputo fractional derivative.

Define fractional generalized momentum and Hamiltonian as

$$p_{\mathrm{RL}k}=\frac{\partial L_{\mathrm{RL}}\left(t,\boldsymbol{q}_{\mathrm{RL}},{}^{\mathrm{RL}}\boldsymbol{D}_{\gamma}^{\alpha,\beta}\boldsymbol{q}_{\mathrm{RL}}\right)}{\partial\,{}^{\mathrm{RL}}D_{\gamma}^{\alpha,\beta}q_{\mathrm{RL}k}}, \tag{19}$$

$$H_{\mathrm{RL}}=p_{\mathrm{RL}k}\cdot{}^{\mathrm{RL}}D_{\gamma}^{\alpha,\beta}q_{\mathrm{RL}k}-L_{\mathrm{RL}}\left(t,\boldsymbol{q}_{\mathrm{RL}},{}^{\mathrm{RL}}\boldsymbol{D}_{\gamma}^{\alpha,\beta}\boldsymbol{q}_{\mathrm{RL}}\right),\ k=1,2,\cdots,n. \tag{20}$$

Here, we consider that the $L_{\mathrm{RL}}\left(t,\boldsymbol{q}_{\mathrm{RL}},{}^{\mathrm{RL}}\boldsymbol{D}_{\gamma}^{\alpha,\beta}\boldsymbol{q}_{\mathrm{RL}}\right)$ is singular, meaning that only a portion of the terms ${}^{\mathrm{RL}}D_{\gamma}^{\alpha,\beta}q_{\mathrm{RL}k}$, $k=1,2,\cdots,n$ can be solved. In this case, we assume the number is R, $0\leqslant R<n$.

Firstly, when $1\leqslant R<n$, we have

$${}^{\mathrm{RL}}D_{\gamma}^{\alpha,\beta}q_{\mathrm{RL}\sigma}=f_{\mathrm{RL}}^{\sigma}\left(t,\boldsymbol{q}_{\mathrm{RL}},\boldsymbol{p}_{\mathrm{RL}E},{}^{\mathrm{RL}}\boldsymbol{D}_{\gamma}^{\alpha,\beta}\boldsymbol{q}_{\mathrm{RL}\boldsymbol{\rho}}\right),\ \sigma,E=1,2,\cdots,R,\ \rho=R+1,\cdots,n, \tag{21}$$

where $\boldsymbol{p}_{\mathrm{RL}\boldsymbol{E}}=\left(p_{\mathrm{RL}1},p_{\mathrm{RL}2},\cdots,p_{\mathrm{RL}R}\right)$, ${}^{\mathrm{RL}}\boldsymbol{D}_{\gamma}^{\alpha,\beta}\boldsymbol{q}_{\mathrm{RL}\rho}=\left({}^{\mathrm{RL}}D_{\gamma}^{\alpha,\beta}q_{\mathrm{RL}(R+1)},{}^{\mathrm{RL}}D_{\gamma}^{\alpha,\beta}q_{\mathrm{RL}(R+2)},\cdots,{}^{\mathrm{RL}}D_{\gamma}^{\alpha,\beta}q_{\mathrm{RL}n}\right)$, $1\leqslant R<n$. Substituting Eq. (21) into Eq. (19), we have

$$p_{\mathrm{RL}k}=g_{\mathrm{RL}k}\left(t,\boldsymbol{q}_{\mathrm{RL}},f_{\mathrm{RL}}^{\sigma}\left(t,\boldsymbol{q}_{\mathrm{RL}},\boldsymbol{p}_{\mathrm{RLE}},{}^{\mathrm{RL}}\boldsymbol{D}_{\gamma}^{\alpha,\beta}\boldsymbol{q}_{\mathrm{RL}\rho}\right),{}^{\mathrm{RL}}\boldsymbol{D}_{\gamma}^{\alpha,\beta}\boldsymbol{q}_{\mathrm{RL}\rho}\right)=g_{\mathrm{RL}k}\left(t,\boldsymbol{q}_{\mathrm{RL}},\boldsymbol{p}_{\mathrm{RL}E},{}^{\mathrm{RL}}\boldsymbol{D}_{\gamma}^{\alpha,\beta}\boldsymbol{q}_{\mathrm{RL}\rho}\right),k=1,2,\cdots,n. \tag{22}$$

If $k=1,2,\cdots,R$, Eq. (22) obviously holds. If $k=R+1,\cdots,n$, we have

$$p_{\mathrm{RL}\rho}=g_{\mathrm{RL}\rho}\left(t,\boldsymbol{q}_{\mathrm{RL}},\boldsymbol{p}_{\mathrm{RL}E}\right)\text{ or }\boldsymbol{p}_{\mathrm{RL}F}=\boldsymbol{g}_{\mathrm{RL}F}\left(t,\boldsymbol{q}_{\mathrm{RL}},\boldsymbol{p}_{\mathrm{RL}E}\right),\ 1\leqslant R<n, \tag{23}$$

where $\boldsymbol{p}_{\mathrm{RL}F}=\left(p_{\mathrm{RL}(R+1)},\cdots,p_{\mathrm{RL}n}\right)$, $\boldsymbol{g}_{\mathrm{RL}F}=\left(g_{\mathrm{RL}(R+1)},\cdots,g_{\mathrm{RL}n}\right)$, $\rho=R+1,\cdots,n$. Another form of Eq. (23) is

$$\phi_{\mathrm{RL}}\left(t,\boldsymbol{q}_{\mathrm{RL}},\boldsymbol{p}_{\mathrm{RL}}\right)=\boldsymbol{p}_{\mathrm{RL}F}-\boldsymbol{g}_{\mathrm{RL}F}\left(t,\boldsymbol{q}_{\mathrm{RL}},\boldsymbol{p}_{\mathrm{RL}E}\right)=\boldsymbol{0}, \tag{24}$$

where $\phi_{\mathrm{RL}}=\left(\phi_{\mathrm{RL}1},\phi_{\mathrm{RL}2},\cdots,\phi_{\mathrm{RL}(n-R)}\right)$, $\boldsymbol{p}_{\mathrm{RL}}=\left(p_{\mathrm{RL}1},p_{\mathrm{RL}2},\cdots,p_{\mathrm{RL}n}\right)$, $1\leqslant R<n$.

Secondly, when $R=0$, ${}^{\mathrm{RL}}D_{\gamma}^{\alpha,\beta}q_{\mathrm{RL}k}$ cannot be solved. Therefore, from Eq. (19), we have

$$p_{\mathrm{RL}k}=g_{\mathrm{RL}k}\left(t,\boldsymbol{q}_{\mathrm{RL}}\right),\ k=1,2,\cdots,n. \tag{25}$$

$$\phi_{\mathrm{RL}a}\left(t,\boldsymbol{q}_{\mathrm{RL}},\boldsymbol{p}_{\mathrm{RL}}\right)=p_{\mathrm{RL}a}-g_{\mathrm{RL}a}\left(t,\boldsymbol{q}_{\mathrm{RL}}\right)=0,\ a=1,2,\cdots,n. \tag{26}$$

From Eq. (24) and Eq. (26), we can obtain

$$\phi_{\mathrm{RL}a}\left(t,\boldsymbol{q}_{\mathrm{RL}},\boldsymbol{p}_{\mathrm{RL}}\right)=0,\ 0\leqslant R<n,\ a=1,2,\cdots,n-R. \tag{27}$$

Eq. (27) is called fractional primary constraint within a combined Riemann-Liouville fractional derivative.

Similarly, we can define

$$p_{Ck}=\frac{\partial L_{C}\left(t,\boldsymbol{q}_{C},{}^{C}\boldsymbol{D}_{\gamma}^{\alpha,\beta}\boldsymbol{q}_{C}\right)}{\partial\,{}^{C}D_{\gamma}^{\alpha,\beta}q_{Ck}},\ k=1,2,\cdots,n, \tag{28}$$

$$H_{C}=p_{Ck}\,{}^{C}D_{\gamma}^{\alpha,\beta}q_{Ck}-L_{C}\left(t,\boldsymbol{q}_{C},{}^{C}\boldsymbol{D}_{\gamma}^{\alpha,\beta}\boldsymbol{q}_{C}\right),\ k=1,2,\cdots,n. \tag{29}$$

The fractional primary constraint within the combined Caputo fractional derivative can be obtained as

$$\phi_{Ca}\left(t,\boldsymbol{q}_{C},\boldsymbol{p}_{C}\right)=0,\ 0\leqslant R<n,\ a=1,2,\cdots,n-R,\ \boldsymbol{p}_{C}=\left(p_{C1},p_{C2},\cdots,p_{Cn}\right). \tag{30}$$

Remark 2 According to the definition of the fractional generalized momenta (Eqs. (19) and (28)) rather than the fractional Euler-Lagrange equations (Eqs. (16) and (18)), the fractional primary constraints (Eqs. (27) and (30)) are obtained.

4 Fractional Constrained Hamilton Equation

Firstly, the fractional constrained Hamilton equation within the combined Riemann-Liouville fractional derivative is studied.

From Eq. (20), we obtain

$$\delta H_{\mathrm{RL}}=\delta p_{\mathrm{RL}k}\cdot{}^{\mathrm{RL}}D_{\gamma}^{\alpha,\beta}q_{\mathrm{RL}k}-\frac{\partial L_{\mathrm{RL}}}{\partial q_{\mathrm{RL}k}}\delta q_{\mathrm{RL}k},\ k=1,2,\cdots,n. \tag{31}$$

Besides, because of the Hamiltonian $H_{\mathrm{RL}}=H_{\mathrm{RL}}(t,\boldsymbol{q}_{\mathrm{RL}},\boldsymbol{p}_{\mathrm{RL}})$, we have

$$\delta H_{\mathrm{RL}}=\frac{\partial H_{\mathrm{RL}}}{\partial q_{\mathrm{RL}k}}\cdot\delta q_{\mathrm{RL}k}+\frac{\partial H_{\mathrm{RL}}}{\partial p_{\mathrm{RL}k}}\cdot\delta p_{\mathrm{RL}k},\ k=1,2,\cdots,n. \tag{32}$$

It follows from Eqs. (31) and (32) that

$$\left({}^{\mathrm{RL}}D_{\gamma}^{\alpha,\beta}q_{\mathrm{RL}k}-\frac{\partial H_{\mathrm{RL}}}{\partial p_{\mathrm{RL}k}}\right)\delta p_{\mathrm{RL}k}-\left(\frac{\partial L_{\mathrm{RL}}}{\partial q_{\mathrm{RL}k}}+\frac{\partial H_{\mathrm{RL}}}{\partial q_{\mathrm{RL}k}}\right)\delta q_{\mathrm{RL}k}=0. \tag{33}$$

Making use of Eqs. (16) and (19), the term $\partial L_{\mathrm{RL}}/\partial q_{\mathrm{RL}k}$ in Eq. (33) can be replaced by ${}^{C}D_{1-\gamma}^{\beta,\alpha}p_{\mathrm{RL}k}-\frac{\gamma(t_2-t)^{-\alpha}}{\Gamma(1-\alpha)}p_{\mathrm{RL}k}(t_2)+\frac{(1-\gamma)(t-t_1)^{-\beta}}{\Gamma(1-\beta)}p_{\mathrm{RL}k}(t_1)$, thus we have

$$\left({}^{\mathrm{RL}}D_{\gamma}^{\alpha,\beta}q_{\mathrm{RL}k}-\frac{\partial H_{\mathrm{RL}}}{\partial p_{\mathrm{RL}k}}\right)\cdot\delta p_{\mathrm{RL}k}-\left[{}^{C}D_{1-\gamma}^{\beta,\alpha}p_{\mathrm{RL}k}-\frac{\gamma(t_2-t)^{-\alpha}}{\Gamma(1-\alpha)}p_{\mathrm{RL}k}(t_2)+\frac{(1-\gamma)(t-t_1)^{-\beta}}{\Gamma(1-\beta)}p_{\mathrm{RL}k}(t_1)+\frac{\partial H_{\mathrm{RL}}}{\partial q_{\mathrm{RL}k}}\right]\delta q_{\mathrm{RL}k}=0,\ k=1,2,\cdots,n, \tag{34}$$

where $p_{\mathrm{RL}k}(t_1)=\frac{\partial L_{\mathrm{RL}}(t_1)}{\partial\,{}^{\mathrm{RL}}D_{\gamma}^{\alpha,\beta}q_{\mathrm{RL}k}}$, $p_{\mathrm{RL}k}(t_2)=\frac{\partial L_{\mathrm{RL}}(t_2)}{\partial\,{}^{\mathrm{RL}}D_{\gamma}^{\alpha,\beta}q_{\mathrm{RL}k}}$. When the system (Eq. (16)) is singular, we have

$$\lambda_{\mathrm{RL}a}\frac{\partial\phi_{\mathrm{RL}a}}{\partial q_{\mathrm{RL}k}}\cdot\delta q_{\mathrm{RL}k}+\lambda_{\mathrm{RL}a}\frac{\partial\phi_{\mathrm{RL}a}}{\partial p_{\mathrm{RL}k}}\cdot\delta p_{\mathrm{RL}k}=0. \tag{35}$$

It follows from Eqs. (34) and (35) that

$$\begin{aligned}{}^{C}D_{1-\gamma}^{\beta,\alpha}p_{\mathrm{RL}k}&=-\frac{\partial H_{\mathrm{RL}}}{\partial q_{\mathrm{RL}k}}+\frac{\gamma(t_2-t)^{-\alpha}}{\Gamma(1-\alpha)}p_{\mathrm{RL}k}(t_2)-\frac{(1-\gamma)(t-t_1)^{-\beta}}{\Gamma(1-\beta)}p_{\mathrm{RL}k}(t_1)-\lambda_{\mathrm{RL}a}\frac{\partial\phi_{\mathrm{RL}a}}{\partial q_{\mathrm{RL}k}},\\ {}^{\mathrm{RL}}D_{\gamma}^{\alpha,\beta}q_{\mathrm{RL}k}&=\frac{\partial H_{\mathrm{RL}}}{\partial p_{\mathrm{RL}k}}+\lambda_{\mathrm{RL}a}\frac{\partial\phi_{\mathrm{RL}a}}{\partial p_{\mathrm{RL}k}},\ a=1,2,\cdots,n-R,\ 0\leqslant R<n,\ k=1,2,\cdots,n.\end{aligned} \tag{36}$$

Eq. (36) is called fractional constrained Hamilton equation within a combined Riemann-Liouville fractional derivative.

Similarly, we can also get the Hamilton equation with fractional constraints within the combined Caputo fractional derivative

$${}^{\mathrm{RL}}D_{1-\gamma}^{\alpha,\beta}p_{Ck}=-\frac{\partial H_C}{\partial q_{Ck}}-\lambda_{Ca}\frac{\partial\phi_{Ca}}{\partial q_{Ck}},\ {}^{C}D_{\gamma}^{\alpha,\beta}q_{Ck}=\frac{\partial H_C}{\partial p_{Ck}}+\lambda_{Ca}\frac{\partial\phi_{Ca}}{\partial p_{Ck}},\ k=1,2,\cdots,n,\ a=1,2,\cdots,n-R,\ 0\leqslant R<n. \tag{37}$$

Remark 3 Because of the various values of γ, different results can be obtained from Eqs. (36) and (37).

According to the above method of establishing the Hamilton equation with fractional constraints, it is very important to solve the Lagrange multiplier. Next, we will calculate the Lagrange multiplier by fractional Poisson bracket.

Let $F=F(t,\boldsymbol{q},\boldsymbol{p})$, $G=G(t,\boldsymbol{q},\boldsymbol{p})$, $\boldsymbol{q}=(q_1,q_2,\cdots,q_n)$, $\boldsymbol{p}=(p_1,p_2,\cdots,p_n)$, then we define

$$\{F,G\}=\frac{\partial F}{\partial q_k}\frac{\partial G}{\partial p_k}-\frac{\partial F}{\partial p_k}\frac{\partial G}{\partial q_k},\ k=1,2,\cdots,n. \tag{38}$$

Then we have

$$\begin{gathered}\left(\{\phi_{\mathrm{RL}a},H_{\mathrm{RL}}\}+\lambda_{\mathrm{RL}b}\{\phi_{\mathrm{RL}a},\phi_{\mathrm{RL}b}\}\right)\cdot\frac{\partial\phi_{\mathrm{RL}a}}{\partial q_{\mathrm{RL}j}}\dot{q}_{\mathrm{RL}j}+\left(\frac{\partial\phi_{\mathrm{RL}a}}{\partial t}+\frac{\partial\phi_{\mathrm{RL}a}}{\partial p_{\mathrm{RL}j}}\dot{p}_{\mathrm{RL}j}\right)\times\frac{\partial\phi_{\mathrm{RL}a}}{\partial q_{\mathrm{RL}k}}\,{}^{\mathrm{RL}}D_{\gamma}^{\alpha,\beta}q_{\mathrm{RL}k}-\left(\frac{\partial\phi_{\mathrm{RL}a}}{\partial p_{\mathrm{RL}k}}\cdot\frac{\partial\phi_{\mathrm{RL}a}}{\partial q_{\mathrm{RL}j}}\dot{q}_{\mathrm{RL}j}\right)\\ \left[{}^{C}D_{1-\gamma}^{\beta,\alpha}p_{\mathrm{RL}k}-\frac{\gamma(t_2-t)^{-\alpha}}{\Gamma(1-\alpha)}p_{\mathrm{RL}k}(t_2)+\frac{(1-\gamma)(t-t_1)^{-\beta}}{\Gamma(1-\beta)}p_{\mathrm{RL}k}(t_1)\right]=0,\ a,b=1,2,\cdots,n-R,\ k,j=\\ 1,2,\cdots,n,\ 0\leqslant R<n,\end{gathered} \tag{39}$$

and

$$\begin{gathered}\left(\{\phi_{Ca},H_C\}+\lambda_{Cb}\{\phi_{Ca},\phi_{Cb}\}\right)\frac{\partial\phi_{Ca}}{\partial q_{Cj}}\dot{q}_{Cj}+\left(\frac{\partial\phi_{Ca}}{\partial t}+\frac{\partial\phi_{Ca}}{\partial p_{Cj}}\cdot\dot{p}_{Cj}\right)\times\frac{\partial\phi_{Ca}}{\partial q_{Ck}}\,{}^{C}D_{\gamma}^{\alpha,\beta}q_{Ck}-\frac{\partial\phi_{Ca}}{\partial p_{Ck}}\cdot\frac{\partial\phi_{Ca}}{\partial q_{Cj}}\dot{q}_{Cj}\cdot{}^{\mathrm{RL}}D_{1-\gamma}^{\beta,\alpha}p_{Ck}=0,\\ a,b=1,2,\cdots,n-R,\ k,j=1,2,\cdots,n,\ 0\leqslant R<n.\end{gathered} \tag{40}$$

From Eqs. (39) and (40), the Lagrange multipliers can be calculated.

5 Noether Theorem

Definition 1 A quantity C is called a conserved quantity if and only if $\mathrm{d}C/\mathrm{d}t=0$.

5.1 Noether Theorem within the Combined Riemann-Liouville Fractional Derivative

The Hamilton action within the combined Riemann-Liouville fractional derivative is

$$I_{\mathrm{RL}}=\int_{t_1}^{t_2}\left[p_{\mathrm{RL}k}\,{}^{\mathrm{RL}}D_{\gamma}^{\alpha,\beta}q_{\mathrm{RL}k}-H_{\mathrm{RL}}\left(t,\boldsymbol{q}_{\mathrm{RL}},\boldsymbol{p}_{\mathrm{RL}}\right)\right]\mathrm{d}t. \tag{41}$$

The infinitesimal transformations are

$$\bar{t}=t+\Delta t,\ \bar{q}_{\mathrm{RL}k}(\bar{t})=q_{\mathrm{RL}k}(t)+\Delta q_{\mathrm{RL}k},\ \bar{p}_{\mathrm{RL}k}(\bar{t})=p_{\mathrm{RL}k}(t)+\Delta p_{\mathrm{RL}k},\ k=1,2,\cdots,n. \tag{42}$$

Namely,

$$\bar{t}=t+\theta_{\mathrm{RL}}\xi_{\mathrm{RL}0}\left(t,\boldsymbol{q}_{\mathrm{RL}},\boldsymbol{p}_{\mathrm{RL}}\right)+o\left(\theta_{\mathrm{RL}}\right),\ \bar{q}_{\mathrm{RL}k}(\bar{t})=q_{\mathrm{RL}k}(t)+\theta_{\mathrm{RL}}\xi_{\mathrm{RL}k}\left(t,\boldsymbol{q}_{\mathrm{RL}},\boldsymbol{p}_{\mathrm{RL}}\right)+o\left(\theta_{\mathrm{RL}}\right),$$
$$\bar{p}_{\mathrm{RL}k}(\bar{t})=p_{\mathrm{RL}k}(t)+\theta_{\mathrm{RL}}\eta_{\mathrm{RL}k}\left(t,\boldsymbol{q}_{\mathrm{RL}},\boldsymbol{p}_{\mathrm{RL}}\right)+o\left(\theta_{\mathrm{RL}}\right), \tag{43}$$

where θ_{RL} is a small parameter, $\xi_{\mathrm{RL}0}$, $\xi_{\mathrm{RL}k}$, and $\eta_{\mathrm{RL}k}$ are infinitesimal generators, and $o\left(\theta_{\mathrm{RL}}\right)$ means the higher order of θ_{RL}.

Letting ΔI_{RL} be the linear part of $\bar{I}_{\mathrm{RL}}-I_{\mathrm{RL}}$, and without considering the higher order of θ_{RL}, we obtain

$$\begin{aligned}\Delta I_{\mathrm{RL}}=\theta_{\mathrm{RL}}\int_{t_1}^{t_2}\Bigg[&p_{\mathrm{RL}k}\,{}^{\mathrm{RL}}D_{\gamma}^{\alpha,\beta}\left(\xi_{\mathrm{RL}k}-\dot{q}_{\mathrm{RL}k}\xi_{\mathrm{RL}0}\right)+\left(p_{\mathrm{RL}k}\cdot{}^{\mathrm{RL}}D_{\gamma}^{\alpha,\beta}q_{\mathrm{RL}k}-H_{\mathrm{RL}}\right)\dot{\xi}_{\mathrm{RL}0}+\left(p_{\mathrm{RL}k}\frac{\mathrm{d}}{\mathrm{d}t}{}^{\mathrm{RL}}D_{\gamma}^{\alpha,\beta}q_{\mathrm{RL}k}-\frac{\partial H_{\mathrm{RL}}}{\partial t}\right)\xi_{\mathrm{RL}0}\\&-\frac{\partial H_{\mathrm{RL}}}{\partial q_{\mathrm{RL}k}}\xi_{\mathrm{RL}k}+\lambda_{\mathrm{RL}a}\frac{\partial\phi_{\mathrm{RL}a}}{\partial p_{\mathrm{RL}k}}\cdot\eta_{\mathrm{RL}k}+q_{\mathrm{RL}k}\left(t_2\right)\xi_{\mathrm{RL}0}\left(t_2\right)\times\frac{\left(1-\gamma\right)p_{\mathrm{RL}k}}{\Gamma\left(1-\beta\right)}\frac{\mathrm{d}}{\mathrm{d}t}\left(t_2-t\right)^{-\beta}-q_{\mathrm{RL}k}\left(t_1\right)\xi_{\mathrm{RL}0}\left(t_1\right)\frac{\gamma p_{\mathrm{RL}k}}{\Gamma\left(1-\alpha\right)}\frac{\mathrm{d}}{\mathrm{d}t}\left(t-t_1\right)^{-\alpha}\Bigg]\mathrm{d}t,\end{aligned} \tag{44}$$

where $\xi_{\mathrm{RL}0}\left(t_1\right)=\xi_{\mathrm{RL}0}\left(t_1,\boldsymbol{q}_{\mathrm{RL}}\left(t_1\right),\boldsymbol{p}_{\mathrm{RL}}\left(t_1\right)\right)$ and $\xi_{\mathrm{RL}0}\left(t_2\right)=\xi_{\mathrm{RL}0}\left(t_2,\boldsymbol{q}_{\mathrm{RL}}\left(t_2\right),\boldsymbol{p}_{\mathrm{RL}}\left(t_2\right)\right)$. Let $\Delta I_{\mathrm{RL}}=0$, then Eq. (44) gives

$$\begin{aligned}&p_{\mathrm{RL}k}\,{}^{\mathrm{RL}}D_{\gamma}^{\alpha,\beta}\left(\xi_{\mathrm{RL}k}-\dot{q}_{\mathrm{RL}k}\xi_{\mathrm{RL}0}\right)+\left(p_{\mathrm{RL}k}\,{}^{\mathrm{RL}}D_{\gamma}^{\alpha,\beta}q_{\mathrm{RL}k}-H_{\mathrm{RL}}\right)\dot{\xi}_{\mathrm{RL}0}+\lambda_{\mathrm{RL}a}\frac{\partial\phi_{\mathrm{RL}a}}{\partial p_{\mathrm{RL}k}}\cdot\eta_{\mathrm{RL}k}+\left(p_{\mathrm{RL}k}\frac{\mathrm{d}}{\mathrm{d}t}{}^{\mathrm{RL}}D_{\gamma}^{\alpha,\beta}q_{\mathrm{RL}k}-\frac{\partial H_{\mathrm{RL}}}{\partial t}\right)\xi_{\mathrm{RL}0}\\&-\frac{\partial H_{\mathrm{RL}}}{\partial q_{\mathrm{RL}k}}\xi_{\mathrm{RL}k}+q_{\mathrm{RL}k}\left(t_2\right)\xi_{\mathrm{RL}0}\left(t_2\right)\times\frac{\left(1-\gamma\right)p_{\mathrm{RL}k}}{\Gamma\left(1-\beta\right)}\frac{\mathrm{d}}{\mathrm{d}t}\left(t_2-t\right)^{-\beta}-q_{\mathrm{RL}k}\left(t_1\right)\xi_{\mathrm{RL}0}\left(t_1\right)\frac{\gamma p_{\mathrm{RL}k}}{\Gamma\left(1-\alpha\right)}\frac{\mathrm{d}}{\mathrm{d}t}\left(t-t_1\right)^{-\alpha}=0.\end{aligned} \tag{45}$$

Equation (45) is called the fractional Noether identity within the combined Riemann-Liouville fractional derivative.

Theorem 1 If $\xi_{\mathrm{RL}0}$, $\xi_{\mathrm{RL}k}$, and $\eta_{\mathrm{RL}k}$ satisfy Eq. (45), then there exists a conserved quantity

$$\begin{aligned}C_{\mathrm{RL}}=&\left(p_{\mathrm{RL}k}\,{}^{\mathrm{RL}}D_{\gamma}^{\alpha,\beta}q_{\mathrm{RL}k}-H_{\mathrm{RL}}\right)\xi_{\mathrm{RL}0}+\int_{t_1}^{t}\Bigg\{p_{\mathrm{RL}k}\,{}^{\mathrm{RL}}D_{\gamma}^{\alpha,\beta}\left(\xi_{\mathrm{RL}k}-\dot{q}_{\mathrm{RL}k}\xi_{\mathrm{RL}0}\right)\\&+\left(\xi_{\mathrm{RL}k}-\dot{q}_{\mathrm{RL}k}\xi_{\mathrm{RL}0}\right)\left[{}^{C}D_{1-\gamma}^{\beta,\alpha}p_{\mathrm{RL}k}-\frac{\gamma\left(t_2-\tau\right)^{-\alpha}}{\Gamma\left(1-\alpha\right)}p_{\mathrm{RL}k}\left(t_2\right)+\frac{\left(1-\gamma\right)\left(\tau-t_1\right)^{-\beta}}{\Gamma\left(1-\beta\right)}p_{\mathrm{RL}k}\left(t_1\right)\right]\Bigg\}\mathrm{d}\tau\\&+q_{\mathrm{RL}k}\left(t_2\right)\xi_{\mathrm{RL}0}\left(t_2\right)\frac{1-\gamma}{\Gamma\left(1-\beta\right)}\int_{t_1}^{t}p_{\mathrm{RL}k}\frac{\mathrm{d}}{\mathrm{d}\tau}\left(t_2-\tau\right)^{-\beta}\mathrm{d}\tau-q_{\mathrm{RL}k}\left(t_1\right)\xi_{\mathrm{RL}0}\left(t_1\right)\frac{\gamma}{\Gamma\left(1-\alpha\right)}\int_{t_1}^{t}p_{\mathrm{RL}k}\frac{\mathrm{d}}{\mathrm{d}\tau}\left(\tau-t_1\right)^{-\alpha}\mathrm{d}\tau\end{aligned} \tag{46}$$

for the system within the combined Riemann-Liouville fractional derivative.

Proof Using Eqs. (27), (36), and (45), we have

$$\begin{aligned}\frac{\mathrm{d}C_{\mathrm{RL}}}{\mathrm{d}t}=&\left(p_{\mathrm{RL}k}\,{}^{\mathrm{RL}}D_{\gamma}^{\alpha,\beta}q_{\mathrm{RL}k}-H_{\mathrm{RL}}\right)\dot{\xi}_{\mathrm{RL}0}+\xi_{\mathrm{RL}0}\left(\dot{p}_{\mathrm{RL}k}\,{}^{\mathrm{RL}}D_{\gamma}^{\alpha,\beta}q_{\mathrm{RL}k}+p_{\mathrm{RL}k}\frac{\mathrm{d}}{\mathrm{d}t}{}^{\mathrm{RL}}D_{\gamma}^{\alpha,\beta}q_{\mathrm{RL}k}-\frac{\partial H_{\mathrm{RL}}}{\partial t}-\frac{\partial H_{\mathrm{RL}}}{\partial q_{\mathrm{RL}k}}\cdot\dot{q}_{\mathrm{RL}k}-\frac{\partial H_{\mathrm{RL}}}{\partial p_{\mathrm{RL}k}}\cdot\dot{p}_{\mathrm{RL}k}\right)\\&+p_{\mathrm{RL}k}\,{}^{\mathrm{RL}}D_{\gamma}^{\alpha,\beta}\left(\xi_{\mathrm{RL}k}-\dot{q}_{\mathrm{RL}k}\xi_{\mathrm{RL}0}\right)+\left(\xi_{\mathrm{RL}k}-\dot{q}_{\mathrm{RL}k}\xi_{\mathrm{RL}0}\right)\left[{}^{C}D_{1-\gamma}^{\beta,\alpha}p_{\mathrm{RL}k}-\frac{\gamma\left(t_2-t\right)^{-\alpha}}{\Gamma\left(1-\alpha\right)}p_{\mathrm{RL}k}\left(t_2\right)+\frac{\left(1-\gamma\right)\left(t-t_1\right)^{-\beta}}{\Gamma(1-\beta)}p_{\mathrm{RL}k}\left(t_1\right)\right]\\&+q_{\mathrm{RL}k}\left(t_2\right)\xi_{\mathrm{RL}0}\left(t_2\right)\frac{1-\gamma}{\Gamma\left(1-\beta\right)}p_{\mathrm{RL}k}\frac{\mathrm{d}}{\mathrm{d}t}\left(t_2-t\right)^{-\beta}-q_{\mathrm{RL}k}\left(t_1\right)\xi_{\mathrm{RL}0}\left(t_1\right)\frac{\gamma}{\Gamma\left(1-\alpha\right)}p_{\mathrm{RL}k}\frac{\mathrm{d}}{\mathrm{d}t}\left(t-t_1\right)^{-\alpha}\end{aligned}$$

$$= \frac{\partial H_{\mathrm{RL}}}{\partial q_{\mathrm{RL}k}}\xi_{\mathrm{RL}k} - \lambda_{\mathrm{RL}a}\frac{\partial \phi_{\mathrm{RL}a}}{\partial p_{\mathrm{RL}k}}\eta_{\mathrm{RL}k} + \xi_{\mathrm{RL}0}\left(\dot{p}_{\mathrm{RL}k}\cdot\lambda_{\mathrm{RL}a}\frac{\partial\phi_{\mathrm{RL}a}}{\partial p_{\mathrm{RL}k}} - \frac{\partial H_{\mathrm{RL}}}{\partial q_{\mathrm{RL}k}}\dot{q}_{\mathrm{RL}k}\right) + \left(\xi_{\mathrm{RL}k} - \dot{q}_{\mathrm{RL}k}\xi_{\mathrm{RL}0}\right)\left(-\frac{\partial H_{\mathrm{RL}}}{\partial q_{\mathrm{RL}k}} - \lambda_{\mathrm{RL}a}\frac{\partial\phi_{\mathrm{RL}a}}{\partial q_{\mathrm{RL}k}}\right)$$

$$= -\lambda_{\mathrm{RL}a}\frac{\partial\phi_{\mathrm{RL}a}}{\partial p_{\mathrm{RL}k}}\left(\eta_{\mathrm{RL}k} - \dot{p}_{\mathrm{RL}k}\xi_{\mathrm{RL}0}\right) - \lambda_{\mathrm{RL}a}\frac{\partial\phi_{\mathrm{RL}a}}{\partial q_{\mathrm{RL}k}}\left(\xi_{\mathrm{RL}k} - \dot{q}_{\mathrm{RL}k}\xi_{\mathrm{RL}0}\right) = -\lambda_{\mathrm{RL}a}\cdot\delta\phi_{\mathrm{RL}a} = 0.$$

The proof is completed.

5.2 Noether Theorem within the Combined Caputo Fractional Derivative

The Hamilton action within the combined Caputo fractional derivative is

$$I_C = \int_{t_1}^{t_2}\left[p_{Ck}\cdot {}^{C}D_{\gamma}^{\alpha,\beta}q_{Ck} - H_C\left(t,\boldsymbol{q}_C,\boldsymbol{p}_C\right)\right]\mathrm{d}t. \tag{47}$$

The infinitesimal transformations are

$$\bar{t} = t + \Delta t,\ \bar{q}_{Ck}(\bar{t}) = q_{Ck}(t) + \Delta q_{Ck}, \bar{p}_{Ck}(\bar{t}) = p_{Ck}(t) + \Delta p_{Ck},\ k = 1, 2, \cdots, n. \tag{48}$$

Namely,

$$\bar{t} = t + \theta_C\xi_{C0}\left(t,\boldsymbol{q}_C,\boldsymbol{p}_C\right) + o\left(\theta_C\right),\ \bar{q}_{Ck}(\bar{t}) = q_{Ck}(t) + \theta_C\xi_{Ck}\left(t,\boldsymbol{q}_C,\boldsymbol{p}_C\right) + o\left(\theta_C\right),\ \bar{p}_{Ck}(\bar{t}) = p_{Ck}(t) + \theta_C\eta_{Ck}\left(t,\boldsymbol{q}_C,\boldsymbol{p}_C\right) + o\left(\theta_C\right), \tag{49}$$

where θ_C is a small parameter, ξ_{C0}, ξ_{Ck}, and η_{Ck} are infinitesimal generators. Letting $\Delta I_C = \bar{I}_C - I_C = 0$, we have

$$p_{Ck}\,{}^{C}D_{\gamma}^{\alpha,\beta}\left(\xi_{Ck} - \dot{q}_{Ck}\xi_{C0}\right) + \left(p_{Ck}\,{}^{C}D_{\gamma}^{\alpha,\beta}q_{Ck} - H_C\right)\dot{\xi}_{C0} + \left(p_{Ck}\frac{\mathrm{d}}{\mathrm{d}t}\,{}^{C}D_{\gamma}^{\alpha,\beta}q_{Ck} - \frac{\partial H_C}{\partial t}\right)\xi_{C0} - \frac{\partial H_C}{\partial q_{Ck}}\xi_{Ck} + \lambda_{Ca}\frac{\partial\phi_{Ca}}{\partial p_{Ck}}\eta_{Ck}$$

$$+\dot{q}_{Ck}\left(t_2\right)\xi_{C0}\left(t_2\right)\cdot\frac{\left(1-\gamma\right)p_{Ck}}{\Gamma\left(1-\beta\right)}\left(t_2 - t\right)^{-\beta} - \dot{q}_{Ck}\left(t_1\right)\xi_{C0}\left(t_1\right)\cdot\frac{\gamma\cdot p_{Ck}}{\Gamma\left(1-\alpha\right)}\left(t - t_1\right)^{-\alpha} = 0. \tag{50}$$

Equation (50) is called fractional Noether identity within a combined Caputo fractional derivative.

Theorem 2 If ξ_{C0}, ξ_{Ck}, and η_{Ck} satisfy Eq. (50), then there exists a conserved quantity

$$C_C = \left(p_{Ck}\,{}^{C}D_{\gamma}^{\alpha,\beta}q_{Ck} - H_C\right)\xi_{C0} + \int_{t_1}^{t}\left[p_{Ck}\,{}^{C}D_{\gamma}^{\alpha,\beta}\left(\xi_{Ck} - q_{Ck}\xi_{C0}\right) + \left(\xi_{Ck} - \dot{q}_{Ck}\xi_{C0}\right)\cdot\,{}^{\mathrm{RL}}D_{1-\gamma}^{\beta,\alpha}p_{Ck}\right]\mathrm{d}\tau$$

$$+\dot{q}_{Ck}\left(t_2\right)\xi_{C0}\left(t_2\right)\frac{1-\gamma}{\Gamma\left(1-\beta\right)}\int_{t_1}^{t}p_{Ck}\left(t_2 - \tau\right)^{-\beta}\mathrm{d}\tau - \dot{q}_{Ck}\left(t_1\right)\xi_{C0}\left(t_1\right)\frac{\gamma}{\Gamma\left(1-\alpha\right)}\int_{t_1}^{t}p_{Ck}\left(\tau - t_1\right)^{-\alpha}\mathrm{d}\tau, \tag{51}$$

for the system within a combined Caputo fractional derivative.

Proof Using Eqs. (30), (37), and (50), it is easy to obtain

$$\frac{\mathrm{d}C_C}{\mathrm{d}t} = \left(p_{Ck}\,{}^{C}D_{\gamma}^{\alpha,\beta}q_{Ck} - H_C\right)\dot{\xi}_{C0} + \xi_{C0}\left(\dot{p}_{Ck}\,{}^{C}D_{\gamma}^{\alpha,\beta}q_{Ck} + p_{Ck}\frac{\mathrm{d}}{\mathrm{d}t}\,{}^{C}D_{\gamma}^{\alpha,\beta}q_{Ck} - \frac{\partial H_C}{\partial t} - \frac{\partial H_C}{\partial q_{Ck}}\cdot\dot{q}_{Ck} - \frac{\partial H_C}{\partial p_{Ck}}\cdot\dot{p}_{Ck}\right)$$

$$+p_{Ck}\cdot\,{}^{C}D_{\gamma}^{\alpha,\beta}\left(\xi_{Ck} - \dot{q}_{Ck}\xi_{C0}\right) + \left(\xi_{Ck} - \dot{q}_{Ck}\xi_{C0}\right){}^{\mathrm{RL}}D_{1-\gamma}^{\beta,\alpha}p_{Ck} + \dot{q}_{Ck}\left(t_2\right)\xi_{C0}\left(t_2\right)\frac{\left(1-\gamma\right)p_{Ck}}{\Gamma\left(1-\beta\right)}\left(t_2 - t\right)^{-\beta}$$

$$-\dot{q}_{Ck}\left(t_1\right)\xi_{C0}\left(t_1\right)\frac{\gamma\cdot p_{Ck}}{\Gamma\left(1-\alpha\right)}\left(t - t_1\right)^{-\alpha} + \dot{q}_{Ck}\left(t_2\right)\xi_{C0}\left(t_2\right)\frac{\left(1-\gamma\right)p_{Ck}}{\Gamma\left(1-\beta\right)}\left(t_2 - t\right)^{-\beta} - \dot{q}_{Ck}\left(t_1\right)\xi_{C0}\left(t_1\right)\frac{\gamma\cdot p_{Ck}}{\Gamma\left(1-\alpha\right)}\left(t - t_1\right)^{-\alpha}$$

$$= \frac{\partial H_C}{\partial q_{Ck}}\xi_{Ck} - \lambda_{Ca}\frac{\partial\phi_{Ca}}{\partial p_{Ck}}\eta_{Ck} + \xi_{C0}\left(\dot{p}_{Ck}\cdot\lambda_{Ca}\frac{\partial\phi_{Ca}}{\partial p_{Ck}} - \frac{\partial H_C}{\partial q_{Ck}}\dot{q}_{Ck}\right) - \left(\xi_{Ck} - q_{Ck}\xi_{C0}\right)\left(\frac{\partial H_C}{\partial q_{Ck}} + \lambda_{Ca}\frac{\partial\phi_{Ca}}{\partial q_{Ck}}\right)$$

$$= -\lambda_{Ca}\frac{\partial\phi_{Ca}}{\partial p_{Ck}}\left(\eta_{Ck} - \xi_{C0}\dot{p}_{Ck}\right) - \lambda_{Ca}\frac{\partial\phi_{Ca}}{\partial q_{Ck}}\left(\xi_{Ck} - \dot{q}_{Ck}\xi_{C0}\right) = -\lambda_{Ca}\cdot\delta\phi_{Ca} = 0.$$

The proof is completed.

6 Lie Symmetry Conserved Quantity

6.1 Lie Symmetry Conserved Quantity within the Combined Riemann-Liouville Fractional Derivative

Eq. (36) can be expressed as

$${}^{\mathrm{RL}}D_{\gamma}^{\alpha,\beta}q_{\mathrm{RL}k} = h_{\mathrm{RL}k}\left(t,\boldsymbol{q}_{\mathrm{RL}},\boldsymbol{p}_{\mathrm{RL}}\right),\ k = 1, 2, \cdots, n, \tag{52}$$

$${}^{C}D_{1-\gamma}^{\beta,\alpha}p_{\mathrm{RL}k} = f_{\mathrm{RL}k}\left(t,\boldsymbol{q}_{\mathrm{RL}},\boldsymbol{p}_{\mathrm{RL}}\right),\ k = 1, 2, \cdots, n. \tag{53}$$

We then study Eqs. (52) and (53) under the infinitesimal transformations (Eq. (43)). For Eq. (52), we obtain

$$
\begin{aligned}
{}^{\mathrm{RL}}D_{\bar{\gamma}}^{\alpha,\beta}\bar{q}_{\mathrm{RL}k}-h_{\mathrm{RL}k}\left(\bar{t},\bar{\boldsymbol{q}}_{\mathrm{RL}k},\bar{\boldsymbol{p}}_{\mathrm{RL}k}\right)= {}^{\mathrm{RL}}D_{\gamma}^{\alpha,\beta}q_{\mathrm{RL}k}-h_{\mathrm{RL}k}\left(t,\boldsymbol{q}_{\mathrm{RL}},\boldsymbol{p}_{\mathrm{RL}}\right)+\theta_{\mathrm{RL}}\Bigg[{}^{\mathrm{RL}}D_{\gamma}^{\alpha,\beta}\left(\xi_{\mathrm{RL}k}-\dot{q}_{\mathrm{RL}k}\xi_{\mathrm{RL}0}\right)+\xi_{\mathrm{RL}0}\frac{\mathrm{d}}{\mathrm{d}t}{}^{\mathrm{RL}}D_{\gamma}^{\alpha,\beta}q_{\mathrm{RL}k}-X_{\mathrm{RL}}^{(0)}\left(h_{\mathrm{RL}k}\right)\\
-q_{\mathrm{RL}k}\left(t_1\right)\frac{\gamma\xi_{\mathrm{RL}0}\left(t_1\right)}{\Gamma(1-\alpha)}\frac{\mathrm{d}}{\mathrm{d}t}\left(t-t_1\right)^{-\alpha}+q_{\mathrm{RL}k}\left(t_2\right)\xi_{\mathrm{RL}0}\left(t_2\right)\frac{1-\gamma}{\Gamma\left(1-\beta\right)}\frac{\mathrm{d}}{\mathrm{d}t}\left(t_2-t\right)^{-\beta}\Bigg],
\end{aligned}\tag{54}
$$

where $X_{\mathrm{RL}}^{(0)}=\xi_{\mathrm{RL}0}\frac{\partial}{\partial t}+\xi_{\mathrm{RL}i}\frac{\partial}{\partial q_{\mathrm{RL}i}}+\eta_{\mathrm{RL}i}\frac{\partial}{\partial p_{\mathrm{RL}i}}$, $i=1,2,\cdots,n$. For Eq. (53), we have

$$
\begin{aligned}
{}^{C}D_{1-\bar{\gamma}}^{\beta,\alpha}\bar{p}_{\mathrm{RL}k}=f_{\mathrm{RL}k}\left(\bar{t},\bar{\boldsymbol{q}}_{\mathrm{RL}},\bar{\boldsymbol{p}}_{\mathrm{RL}}\right)= {}^{C}D_{1-\gamma}^{\beta,\alpha}p_{\mathrm{RL}k}-f_{\mathrm{RL}k}\left(t,\boldsymbol{q}_{\mathrm{RL}},\boldsymbol{p}_{\mathrm{RL}}\right)+\theta_{\mathrm{RL}}\Bigg[{}^{C}D_{1-\gamma}^{\beta,\alpha}\left(\eta_{\mathrm{RL}k}-\dot{p}_{\mathrm{RL}k}\xi_{\mathrm{RL}0}\right)+\xi_{\mathrm{RL}0}\frac{\mathrm{d}}{\mathrm{d}t}{}^{C}D_{1-\gamma}^{\beta,\alpha}p_{\mathrm{RL}k}-X_{\mathrm{RL}}^{(0)}\left(f_{\mathrm{RL}k}\right)\\
-\frac{1-\gamma}{\Gamma\left(1-\beta\right)}\left(t-t_1\right)^{-\beta}\dot{p}_{\mathrm{RL}k}\left(t_1\right)\xi_{\mathrm{RL}0}\left(t_1\right)+\frac{\gamma}{\Gamma(1-\alpha)}\left(t_2-t\right)^{-\alpha}\dot{p}_{\mathrm{RL}k}\left(t_2\right)\xi_{\mathrm{RL}0}\left(t_2\right)\Bigg].
\end{aligned}\tag{55}
$$

The fractional primary constraint (Eq. (27)) gives

$$\phi_{\mathrm{RL}a}\left(t,\bar{\boldsymbol{q}}_{\mathrm{RL}},\bar{\boldsymbol{p}}_{\mathrm{RL}}\right)=\phi_{\mathrm{RL}a}\left(t,\boldsymbol{q}_{\mathrm{RL}},\boldsymbol{p}_{\mathrm{RL}}\right)+\theta_{\mathrm{RL}}X_{\mathrm{RL}}^{(0)}\left(\phi_{\mathrm{RL}a}\right).\tag{56}$$

Therefore, we have

$$
\begin{aligned}
&{}^{\mathrm{RL}}D_{\gamma}^{\alpha,\beta}\left(\xi_{\mathrm{RL}k}-\dot{q}_{\mathrm{RL}k}\xi_{\mathrm{RL}0}\right)+\xi_{\mathrm{RL}0}\frac{\mathrm{d}}{\mathrm{d}t}{}^{\mathrm{RL}}D_{\gamma}^{\alpha,\beta}q_{\mathrm{RL}k}-X_{\mathrm{RL}}^{(0)}\left(h_{\mathrm{RL}k}\right)\\
&-q_{\mathrm{RL}k}\left(t_1\right)\frac{\gamma\xi_{\mathrm{RL}0}\left(t_1\right)}{\Gamma(1-\alpha)}\frac{\mathrm{d}}{\mathrm{d}t}\left(t-t_1\right)^{-\alpha}+q_{\mathrm{RL}k}\left(t_2\right)\xi_{\mathrm{RL}0}\left(t_2\right)\frac{1-\gamma}{\Gamma\left(1-\beta\right)}\frac{\mathrm{d}}{\mathrm{d}t}\left(t_2-t\right)^{-\beta}=0,
\end{aligned}\tag{57}
$$

$$
\begin{aligned}
&{}^{C}D_{1-\gamma}^{\beta,\alpha}\left(\eta_{\mathrm{RL}k}-\dot{p}_{\mathrm{RL}k}\xi_{\mathrm{RL}0}\right)+\xi_{\mathrm{RL}0}\frac{\mathrm{d}}{\mathrm{d}t}{}^{C}D_{1-\gamma}^{\beta,\alpha}p_{\mathrm{RL}k}-X_{\mathrm{RL}}^{(0)}\left(f_{\mathrm{RL}k}\right)\\
&-\frac{1-\gamma}{\Gamma\left(1-\beta\right)}\left(t-t_1\right)^{-\beta}\dot{p}_{\mathrm{RL}k}\left(t_1\right)\xi_{\mathrm{RL}0}\left(t_1\right)+\frac{\gamma}{\Gamma(1-\alpha)}\left(t_2-t\right)^{-\alpha}\dot{p}_{\mathrm{RL}k}\left(t_2\right)\xi_{\mathrm{RL}0}\left(t_2\right)=0,
\end{aligned}\tag{58}
$$

and

$$X_{\mathrm{RL}}^{(0)}\left(\phi_{\mathrm{RL}a}\right)=0.\tag{59}$$

Equations (57) and (58) are called the determined equations within the combined Riemann-Liouville fractional derivative, and Eq. (59) is called the limited equation within the combined Riemann-Liouville fractional derivative.

Besides, if we consider the deduction process of the system (Eq. (36)), an extra additional limited equation

$$\frac{\partial\phi_{\mathrm{RL}a}}{\partial q_{\mathrm{RL}i}}\left(\xi_{\mathrm{RL}i}-\dot{q}_{\mathrm{RL}i}\xi_{\mathrm{RL}0}\right)+\frac{\partial\phi_{\mathrm{RL}a}}{\partial p_{\mathrm{RL}i}}\left(\eta_{\mathrm{RL}i}-\dot{p}_{\mathrm{RL}i}\xi_{\mathrm{RL}0}\right)=0\tag{60}$$

can be obtained. Then we have

Theorem 3 For the system within the combined Riemann-Liouville fractional derivative (Eq. (36)), when Eqs. (45), (57) and (58) are satisfied, Eq. (46) gives a Lie symmetry conserved quantity; when Eqs. (45), (57)-(59) are satisfied, Eq. (46) gives a weak Lie symmetry conserved quantity; when Eqs. (45), (57)-(60) are satisfied, Eq. (46) gives a strong Lie symmetry conserved quantity.

6.2 Lie Symmetry Conserved Quantity within the Combined Caputo Fractional Derivative

Eq. (37) can be expressed as

$${}^{C}D_{\gamma}^{\alpha,\beta}q_{Ck}=h_{Ck}\left(t,\boldsymbol{q}_C,\boldsymbol{p}_C\right),\ k=1,2,\cdots,n,\tag{61}$$

$${}^{\mathrm{RL}}D_{1-\gamma}^{\beta,\alpha}p_{Ck}=f_{Ck}\left(t,\boldsymbol{q}_C,\boldsymbol{p}_C\right),\ k=1,2,\cdots,n.\tag{62}$$

Using the similar way, we can obtain the determined equations within the combined Caputo fractional derivative

$${}^{C}D_{\gamma}^{\alpha,\beta}\left(\xi_{Ck}-\dot{q}_{Ck}\xi_{C0}\right)+\xi_{C0}\frac{\mathrm{d}}{\mathrm{d}t}{}^{C}D_{\gamma}^{\alpha,\beta}q_{Ck}-X_C^{(0)}\left(h_{Ck}\right)-\dot{q}_{Ck}\left(t_1\right)\frac{\gamma\xi_{C0}\left(t_1\right)}{\Gamma(1-\alpha)}\left(t-t_1\right)^{-\alpha}+\dot{q}_{Ck}\left(t_2\right)\xi_{C0}\left(t_2\right)\frac{1-\gamma}{\Gamma(1-\beta)}\left(t_2-t\right)^{-\beta}=0,\tag{63}$$

$$
\begin{aligned}
&{}^{\mathrm{RL}}D_{1-\gamma}^{\beta,\alpha}\left(\eta_{Ck}-\dot{p}_{Ck}\xi_{C0}\right)+\xi_{C0}\frac{\mathrm{d}}{\mathrm{d}t}{}^{\mathrm{RL}}D_{1-\gamma}^{\beta,\alpha}p_{Ck}-X_C^{(0)}-\frac{1-\gamma}{\Gamma\left(1-\beta\right)}p_{Ck}\left(t_1\right)\xi_{C0}\left(t_1\right)\frac{\mathrm{d}}{\mathrm{d}t}\left(t-t_1\right)^{-\beta}\\
&+\frac{\gamma}{\Gamma(1-\alpha)}p_{Ck}\left(t_2\right)\xi_{C0}\left(t_2\right)\frac{\mathrm{d}}{\mathrm{d}t}\left(t_2-t\right)^{-\alpha}=0,
\end{aligned}\tag{64}
$$

the limited equation within the combined Caputo fractional derivative

$$X_C^{(0)}\left(\phi_{Ca}\right)=0, \tag{65}$$

and the additional limited equation within the combined Caputo fractional derivative

$$\frac{\partial \phi_{Ca}}{\partial q_{Ci}}\left(\xi_{Ci}-\dot{q}_{Ci}\xi_{C0}\right)+\frac{\partial \phi_{Ca}}{\partial p_{Ci}}\left(\eta_{Ci}-\dot{p}_{Ci}\xi_{C0}\right)=0, \tag{66}$$

where $X_C^{(0)}=\xi_{C0}\frac{\partial}{\partial t}+\xi_{Ci}\frac{\partial}{\partial q_{Ci}}+\eta_{Ci}\frac{\partial}{\partial p_{Ci}}$, $i=1,2,\cdots,n$. Then we have

Theorem 4 For the system within the combined Caputo fractional derivative (Eq. (37)), when Eqs. (50), (63) and (64) are satisfied, Eq. (51) gives a Lie symmetry conserved quantity; when Eqs. (50), (63)-(65) are satisfied, Eq. (51) gives a weak Lie symmetry conserved quantity; when Eqs. (50), (63)-(66) are satisfied, Eq. (51) gives a strong Lie symmetry conserved quantity.

7 An Example

The singular Lagrangian is

$$L_{\mathrm{RL}}=-\frac{c}{2}\left(q_{\mathrm{RL1}}\right)^2+\frac{b}{2}\left(q_{\mathrm{RL2}}\right)^2+\frac{1}{2}\left(q_{\mathrm{RL2}}\,{}^{\mathrm{RL}}D_\gamma^{\alpha,\beta}q_{\mathrm{RL1}}-q_{\mathrm{RL1}}\,{}^{\mathrm{RL}}D_\gamma^{\alpha,\beta}q_{\mathrm{RL2}}\right). \tag{67}$$

Try to study its Noether symmetry and conserved quantity.

From Eqs. (19) and (20), we can obtain

$$p_{\mathrm{RL1}}=\frac{\partial L_{\mathrm{RL}}}{\partial\,{}^{\mathrm{RL}}D_\gamma^{\alpha,\beta}q_{\mathrm{RL1}}}=\frac{1}{2}q_{\mathrm{RL2}},\; p_{\mathrm{RL2}}=\frac{\partial L_{\mathrm{RL}}}{\partial\,{}^{\mathrm{RL}}D_\gamma^{\alpha,\beta}q_{\mathrm{RL2}}}=-\frac{1}{2}q_{\mathrm{RL1}}, \tag{68}$$

$$H_{\mathrm{RL}}=p_{\mathrm{RL1}}\,{}^{\mathrm{RL}}D_\gamma^{\alpha,\beta}q_{\mathrm{RL1}}+p_{\mathrm{RL2}}\,{}^{\mathrm{RL}}D_\gamma^{\alpha,\beta}q_{\mathrm{RL2}}-L_{\mathrm{RL}}=\frac{c}{2}\left(q_{\mathrm{RL1}}\right)^2-\frac{b}{2}\left(q_{\mathrm{RL2}}\right)^2. \tag{69}$$

Then Eq. (27) gives two fractional primary constraints

$$\phi_{\mathrm{RL1}}=p_{\mathrm{RL1}}-\frac{1}{2}q_{\mathrm{RL2}}=0,\; \phi_{\mathrm{RL2}}=p_{\mathrm{RL2}}+\frac{1}{2}q_{\mathrm{RL1}}=0. \tag{70}$$

From Eq. (39), we obtain

$$b\dot{q}_{\mathrm{RL1}}q_{\mathrm{RL2}}+\lambda_{\mathrm{RL1}}\dot{q}_{\mathrm{RL1}}+\dot{p}_{\mathrm{RL2}}\,{}^{\mathrm{RL}}D_\gamma^{\alpha,\beta}q_{\mathrm{RL1}}-\dot{q}_{\mathrm{RL1}}\left[{}^{C}D_{1-\gamma}^{\beta,\alpha}p_{\mathrm{RL2}}-\frac{\left(t_2-t\right)^{-\alpha}}{\Gamma(1-\alpha)}p_{\mathrm{RL2}}\left(t_2\right)+\frac{(1-\gamma)\left(t-t_1\right)^{-\beta}}{\Gamma(1-\beta)}p_{\mathrm{RL2}}\left(t_1\right)\right]=0, \tag{71}$$

$$cq_{\mathrm{RL1}}\dot{q}_{\mathrm{RL2}}+\lambda_{\mathrm{RL2}}\dot{q}_{\mathrm{RL2}}-\dot{p}_{\mathrm{RL1}}\,{}^{\mathrm{RL}}D_\gamma^{\alpha,\beta}q_{\mathrm{RL2}}+\dot{q}_{\mathrm{RL2}}\left[{}^{C}D_{1-\gamma}^{\beta,\alpha}p_{\mathrm{RL1}}-\frac{\gamma\left(t_2-t\right)^{-\alpha}}{\Gamma(1-\alpha)}p_{\mathrm{RL1}}\left(t_2\right)+\frac{(1-\gamma)\left(t-t_1\right)^{-\beta}}{\Gamma(1-\beta)}p_{\mathrm{RL1}}\left(t_1\right)\right]=0. \tag{72}$$

Then, using Eq. (36), the fractional constrained Hamilton equation within the combined Riemann-Liouville fractional derivative is

$$\begin{aligned}
{}^{\mathrm{RL}}D_\gamma^{\alpha,\beta}q_{\mathrm{RL1}}&=-2bq_{\mathrm{RL2}}+2\left[{}^{C}D_{1-\gamma}^{\beta,\alpha}p_{\mathrm{RL2}}-\frac{\gamma\left(t_2-t\right)^{-\alpha}}{\Gamma(1-\alpha)}p_{\mathrm{RL2}}\left(t_2\right)+\frac{(1-\gamma)\left(t-t_1\right)^{-\beta}}{\Gamma(1-\beta)}p_{\mathrm{RL2}}\left(t_1\right)\right],\\
{}^{\mathrm{RL}}D_\gamma^{\alpha,\beta}q_{\mathrm{RL2}}&=-2cq_{\mathrm{RL1}}-2\left[{}^{C}D_{1-\gamma}^{\beta,\alpha}p_{\mathrm{RL1}}-\frac{\gamma\left(t_2-t\right)^{-\alpha}}{\Gamma(1-\alpha)}p_{\mathrm{RL1}}\left(t_2\right)+\frac{(1-\gamma)\left(t-t_1\right)^{-\beta}}{\Gamma(1-\beta)}p_{\mathrm{RL1}}\left(t_1\right)\right].
\end{aligned} \tag{73}$$

From Noether identity (Eq. (45)), we have

$$\begin{aligned}
&p_{\mathrm{RL1}}\,{}^{\mathrm{RL}}D_\gamma^{\alpha,\beta}\left(\xi_{\mathrm{RL1}}-\dot{q}_{\mathrm{RL1}}\xi_{\mathrm{RL0}}\right)+p_{\mathrm{RL2}}\,{}^{\mathrm{RL}}D_\gamma^{\alpha,\beta}\left(\xi_{\mathrm{RL2}}-\dot{q}_{\mathrm{RL2}}\xi_{\mathrm{RL0}}\right)-cq_{\mathrm{RL1}}\xi_{\mathrm{RL1}}+bq_{\mathrm{RL2}}\xi_{\mathrm{RL2}}\\
&+\left(p_{\mathrm{RL1}}\frac{\mathrm{d}}{\mathrm{d}t}\,{}^{\mathrm{RL}}D_\gamma^{\alpha,\beta}q_{\mathrm{RL1}}+p_{\mathrm{RL2}}\frac{\mathrm{d}}{\mathrm{d}t}\,{}^{\mathrm{RL}}D_\gamma^{\alpha,\beta}q_{\mathrm{RL2}}\right)\xi_{\mathrm{RL0}}+\lambda_{\mathrm{RL1}}\eta_{\mathrm{RL1}}+\lambda_{\mathrm{RL2}}\eta_{\mathrm{RL2}}+\left(p_{\mathrm{RL1}}\,{}^{\mathrm{RL}}D_\gamma^{\alpha,\beta}q_{\mathrm{RL1}}+p_{\mathrm{RL2}}\,{}^{\mathrm{RL}}D_\gamma^{\alpha,\beta}q_{\mathrm{RL2}}-H_{\mathrm{RL}}\right)\dot{\xi}_{\mathrm{RL0}}\\
&+q_{\mathrm{RL1}}\left(t_2\right)\xi_{\mathrm{RL0}}\left(t_2\right)\times\frac{(1-\gamma)p_{\mathrm{RL1}}}{\Gamma(1-\beta)}\frac{\mathrm{d}}{\mathrm{d}t}\left(t_2-t\right)^{-\beta}-q_{\mathrm{RL1}}\left(t_1\right)\xi_{\mathrm{RL0}}\left(t_1\right)\frac{\gamma p_{\mathrm{RL1}}}{\Gamma(1-\alpha)}\frac{\mathrm{d}}{\mathrm{d}t}\left(t-t_1\right)^{-\alpha}\\
&+q_{\mathrm{RL2}}\left(t_2\right)\xi_{\mathrm{RL0}}\left(t_2\right)\times\frac{(1-\gamma)p_{\mathrm{RL2}}}{\Gamma(1-\beta)}\frac{\mathrm{d}}{\mathrm{d}t}\left(t_2-t\right)^{-\beta}-q_{\mathrm{RL2}}\left(t_1\right)\xi_{\mathrm{RL0}}\left(t_1\right)\frac{\gamma p_{\mathrm{RL2}}}{\Gamma(1-\alpha)}\frac{\mathrm{d}}{\mathrm{d}t}\left(t-t_1\right)^{-\alpha}=0.
\end{aligned} \tag{74}$$

Then we can verify that

$$\xi_{\mathrm{RL0}}=-1,\ \xi_{\mathrm{RL1}}=\xi_{\mathrm{RL2}}=0,\ \eta_{\mathrm{RL1}}=\eta_{\mathrm{RL2}}=0 \tag{75}$$

satisfy Eq. (74). Therefore, we have

$$\begin{aligned}C_{\mathrm{RL}}=&-\left(p_{\mathrm{RL1}}\,{}^{\mathrm{RL}}D_{\gamma}^{\alpha,\beta}q_{\mathrm{RL1}}+p_{\mathrm{RL2}}\,{}^{\mathrm{RL}}D_{\gamma}^{\alpha,\beta}q_{\mathrm{RL2}}-\frac{c}{2}q_{\mathrm{RL1}}^{2}+\frac{b}{2}q_{\mathrm{RL2}}^{2}\right)+\int_{t_1}^{t}\left\{p_{\mathrm{RL1}}\frac{\mathrm{d}}{\mathrm{d}\tau}\,{}^{\mathrm{RL}}D_{\gamma}^{\alpha,\beta}q_{\mathrm{RL1}}+p_{\mathrm{RL2}}\frac{\mathrm{d}}{\mathrm{d}\tau}\,{}^{\mathrm{RL}}D_{\gamma}^{\alpha,\beta}q_{\mathrm{RL2}}\right.\\&+\dot{q}_{\mathrm{RL1}}\left[{}^{C}D_{1-\gamma}^{\beta,\alpha}p_{\mathrm{RL1}}-\frac{\gamma(t_2-\tau)^{-\alpha}}{\Gamma(1-\alpha)}p_{\mathrm{RL1}}(t_2)+\frac{(1-\gamma)(\tau-t_1)^{-\beta}}{\Gamma(1-\beta)}p_{\mathrm{RL1}}(t_1)\right]\\&\left.+\dot{q}_{\mathrm{RL2}}\left[{}^{C}D_{1-\gamma}^{\beta,\alpha}p_{\mathrm{RL2}}-\frac{\gamma(t_2-\tau)^{-\alpha}}{\Gamma(1-\alpha)}p_{\mathrm{RL2}}(t_2)+\frac{(1-\gamma)(\tau-t_1)^{-\beta}}{\Gamma(1-\beta)}p_{\mathrm{RL2}}(t_1)\right]\right\}\mathrm{d}\tau.\end{aligned} \tag{76}$$

Specially, we degenerate the above fractional order to the integer order, namely $\alpha,\beta\to 1$, so we can get

$$C_{\mathrm{RL0}}=-\frac{c}{2}q_{\mathrm{RL1}}^{2}+\frac{b}{2}q_{\mathrm{RL2}}^{2}. \tag{77}$$

If $t\in[1,10]$, and let $b=c=1, q_{\mathrm{RL1}}(1)=0, q_{\mathrm{RL2}}(1)=1,$ the trajectory of C_{RL0} can be drawn as follows.

It follows from Fig. 1 that C_{RL0} is a constant.

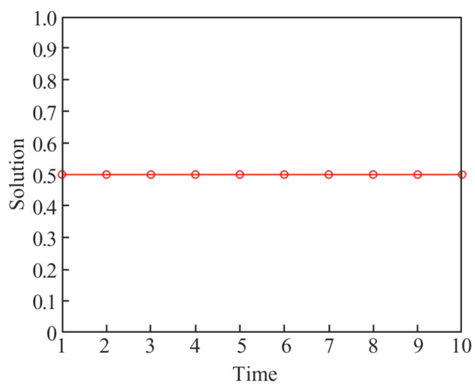

Fig. 1 The trajectory of C_{RL0}

8 Conclusion

Fractional calculus has important applications in various fields, and it is also a research hotspot. In this paper, based on the combined Riemann-Liouville fractional derivative and combined Caputo fractional derivative, we establish Noether theorem and Lie theorem for the fractional constrained Hamiltonian system. Besides, Noether identity, determined equation, limited equation and additional limited equation are presented.

In this paper, Noether-type conserved quantity is obtained from Lie symmetry, whether the Lie symmetry of the system can lead to another conserved quantity, called Hojman conserved quantity, remains to be studied. At the same time, Mei symmetry is also an effective method to solve the differential equation. Therefore, Mei symmetry is also a future research direction.

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由奇异Lagrange函数描述的约束Hamilton系统的对称性

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摘要：基于联合分数阶导数建立了奇异系统。首先，分析了分数阶Lagrange方程；其次，给出了分数阶初级约束；第三，研究了分数阶约束Hamilton系统的Noether对称性和Lie对称性方法；最后，举例对所得结果进行阐述。

关键词：联合分数阶导数；约束Hamilton方程；守恒量；Noether对称性；Lie对称性

□