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Complex-Valued Valuations on L^p Spaces

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Abstract: All continuous translation invariant complex-valued valuations on Lebesgue measurable functions are completely classified. And all continuous rotation invariant complex-valued valuations on spherical Lebesgue measurable functions are also completely classified.

Key words: convex body; valuation; translation invariance; rotation invariance

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0 Introduction

A function z defined on a lattice $(L,\vee,\wedge)$ and taking values in an Aabelian semigroup is called a valuation if

$$z(f\vee g)+z(f\wedge g)=z(f)+z(g) \tag{1}$$

for all $f,g\in L$. A function z defined on some subset L_0 of L is called a valuation on L_0 if (1) holds whenever $f,g,f\vee g,f\wedge g\in L_0$. For L_0 the set of convex bodies, K^n, in $\mathbf{R}^n$ with $\vee$ denoting union and $\wedge$ intersection. Valuation on convex bodies is a classical concept. Probably the most famous result on valuations is Hadwiger's classification theorem of continuous rigid motion invariant valuations[1]. For the more recent contributions on valuations on convex bodies readers can refer to Refs. [2-33].

Valuations on convex bodies can be considered as valuations on suitable function spaces. Recently, valuations on functions have been rapidly growing (see Refs. [34-55]). For a space of real-valued functions, the operations $\vee$ and $\wedge$ are defined as pointwise maximum and minimum, respectively. A complete classification of valuations intertwining with the SL(n) on Sobolev space [46-49] and L^p space [45,50-52,54] were established, respectively. Valuations on convex functions [34,36,39-42], quasi-concave functions [37,38], Lipschitz functions [43,44], and functions of Bounded variations [53] were introduced and classified.

Recently, Wang and Liu [55] showed that the Fourier transform is the only valuation which is a continuous, positive GL(n) covariant and logarithmic translation covariant complex-valued valuation on integral functions. This motivates the study of complex-valued valuations on

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functions.

Let L be a lattice of complex-valued functions. For $f \in L$, let $\Re f$ and $\Im f$ denote the real and imaginary parts of f, respectively. The pointwise maximum of f and g, $f \vee g$ and the pointwise minimum of f and g $f \wedge g$ are defined by

$$f \vee g = \Re f \vee \Re g + \mathrm{i}(\Im f \vee \Im g) \tag{2}$$

and

$$f \wedge g = \Re f \wedge \Re g + \mathrm{i}(\Im f \wedge \Im g) \tag{3}$$

If f, g are real-valued functions, then (2) and (3) coincide with the real cases. A function $\Phi : L \to \mathbf{C}$ is called a valuation if

$$\Phi(f \vee g) + \Phi(f \wedge g) = \Phi(f) + \Phi(g)$$

for all $f, g \in L$ and $\Phi(0) = 0$ if $0 \in L$. It is called continuous if

$$\Phi(f_i) \to \Phi(f), \text{ as } f_i \to f \text{ in } L.$$

It is called translation invariant if

$$\Phi(f(\cdot - t)) = \Phi(f)$$

for every $t \in \mathbf{R}^n$. It is called rotation invariant if

$$\Phi(f \circ \theta^{-1}) = \Phi(f)$$

for every $\theta \in O(n)$, where θ^{-1} denotes the inverse of θ.

Let $p \geqslant 1$. If $(X, \aleph, \mu)$ is a measure space, then the L^p-space, $L^p(\mathbf{C}, \mu)$ is the collection of μ-measurable complex-valued functions $f : X \to \mathbf{C}$ that satisfies

$$\int_X |f|^p \, \mathrm{d}\mu < \infty$$

A measure space $(X, \aleph, \mu)$ is called non-atomic if for every $E \in \aleph$ with $\mu(E) > 0$, there exists $F \in \aleph$ with $F \subseteq E$ and $0 < \mu(F) < \mu(E)$. Let χ_E denote the characteristic function of the measurable set E, i.e.

$$\chi_E = \begin{cases} 1, & x \in E \\ 0, & x \notin E \end{cases}$$

Theorem 1 Let $(X, \aleph, \mu)$ be a non-atomic measure space and let $\Phi : L^p(\mathbf{C}, \mu) \to \mathbf{C}$ be a continuous valuation. If there exist continuous functions $h_k : \mathbf{R} \to \mathbf{R}$ with $h_k(0) = 0$ (k=1,2,3,4) such that $\Phi(c\chi_E) = (h_1(\Re c) + h_3(\Im c))\mu(E) + \mathrm{i}(h_2(\Re c) + h_4(\Im c))\mu(E)$ for all $c \in \mathbf{C}$ and all $E \in \aleph$ with $\mu(E) < \infty$, then there exist constants $\gamma_k, \delta_k \geqslant 0$ such that $|h_k(a)| \leqslant \gamma_k |a|^p + \delta_k$ for $a \in \mathbf{R}$, and

$$\Phi(f) = \int_X (h_1 \circ \Re f + h_3 \circ \Im f) \mathrm{d}\mu + \mathrm{i}\int_X (h_2 \circ \Re f + h_4 \circ \Im f) \mathrm{d}\mu$$

for all $f \in L^p(\mathbf{C}, \mu)$. In addition, $\delta_k = 0$ (k = 1,2,3,4) if $\mu(X) = \infty$.

Let $L^p(\mathbf{C}, \mathbf{R}^n)$ denote the L^p-space of Lebesgue measurable complex-valued functions on $\mathbf{R}^n$.

Theorem 2 A function $\Phi : L^p(\mathbf{C}, \mathbf{R}^n) \to \mathbf{C}$ is a continuous translation invariant valuation if and only if there exist continuous functions $h_k : \mathbf{R} \to \mathbf{R}$ with the property that there exist constants $\gamma_k \geqslant 0$ such that $|h_k(a)| \leqslant \gamma_k |a|^p$ for all $a \in \mathbf{R}$ (k = 1,2,3,4), and

$$\Phi(f) = \int_{\mathbf{R}^n} (h_1 \circ \Re f + h_3 \circ \Im f)(x) \mathrm{d}x + \mathrm{i}\int_{\mathbf{R}^n} (h_2 \circ \Re f + h_4 \circ \Im f)(x) \, \mathrm{d}x \tag{4}$$

for all $f \in L^p(\mathbf{C}, \mathbf{R}^n)$.

Let S^{n-1} be the unit sphere in $\mathbf{R}^n$ and let $L^p(\mathbf{C}, S^{n-1})$ denote the L^p-space of spherical Lebesgue measurable complex-valued functions on S^{n-1}.

Theorem 3 A function $\Phi : L^p(\mathbf{C}, S^{n-1}) \to \mathbf{C}$ is a continuous rotation invariant valuation if and only if there exist continuous functions $h_k : \mathbf{R} \to \mathbf{R}$ with the properties that $h_k(0) = 0$ and there exist constants $\gamma_k, \delta_k \geqslant 0$ such that $|h_k(a)| \leqslant \gamma_k |a|^p + \delta_k$ for all $a \in \mathbf{R}$ (k = 1,2,3,4), and

$$\Phi(f) = \int_{S^{n-1}} (h_1 \circ \Re f + h_3 \circ \Im f)(u) \mathrm{d}u + \mathrm{i}\int_{S^{n-1}} (h_2 \circ \Re f + h_4 \circ \Im f)(u) \mathrm{d}u \tag{5}$$

for all $f \in L^p(\mathbf{C}, S^{n-1})$.

1 Notation and Preliminary Results

We collect some properties of complex-valued functions. If f is a complex-valued function on $\mathbf{R}^n$, then

$$f(x) = \Re f + \mathrm{i}\Im f$$

where $\Re f$ and $\Im f$ denote the real part and imaginary part of f, respectively. The absolute value of f which is also called modulus is defined by

$$|f| = \sqrt{(\Re f)^2 + (\Im f)^2}$$

Let $p \geqslant 1$. For a measure space $(X, \aleph, \mu)$, define $L^p(\mathbf{C}, \mu)$ as the space of μ-measurable complex-valued functions $f : X \to \mathbf{C}$ that satisfies

$$\|f\|_p = \left(\int_X |f|^p \, \mathrm{d}\mu\right)^{\frac{1}{p}} < \infty$$

Let $f, g \in L^p(\mathbf{C}, \mu)$, then $f \vee g, f \wedge g \in L^p(\mathbf{C}, \mu)$. The functional $\|\cdot\| : L^p(\mathbf{C}, \mu) \to \mathbf{R}$ is a semi-norm. If functions in $L^p(\mathbf{C}, \mu)$ that are equal almost everywhere with respect to μ(a.e.$[\mu]$) are identified, then $\|\cdot\| : L^p(\mathbf{C}, \mu) \to \mathbf{R}$ becomes a norm. Obviously, $L^p(\mathbf{C}, \mu)$ is a lattice of complex-valued functions. Let

$L^p(\mathbf{R},\mu)$ denote the subset of $L^p(\mathbf{C},\mu)$, where the functions take real values. For $f_i, f \in L^p(\mathbf{C},\mu)$, if $\|f_i - f\|_p \to 0$, then $f_i \to f$ in $L^p(\mathbf{C},\mu)$. Moreover,

$$f_i \to f \in L^p(\mathbf{C},\mu) \Leftrightarrow \Re f_i \to \Re f, \Im f_i \to \Im f \in L^p(\mathbf{R},\mu)$$

The following characterizations of real-valued valuations on functions which were established by Tsang [51] will play key role in our proof. Let $L^p(\mathbf{R},\mathbf{R}^n)$ and $L^p(\mathbf{R},S^{n-1})$ denote the L^p space of Lebesgue measurable real-valued functions on $\mathbf{R}^n$ and the L^p space of spherical Lebesgue measurable real-valued functions on S^{n-1} respectively.

Theorem 4 [51] A function $\Phi: L^p(\mathbf{R},\mathbf{R}^n) \to \mathbf{R}$ is a continuous translation invariant valuation if and only if there exists a continuous function $h:\mathbf{R}\to\mathbf{R}$ with the property that there exists a constant $\gamma \geqslant 0$ such that $|h(a)| \leqslant \gamma |a|^p$ for all $a \in \mathbf{R}$, and

$$\Phi(f) = \int_{\mathbf{R}^n} (h \circ f)(x)\mathrm{d}x$$

for all $f \in L^p(\mathbf{R},\mathbf{R}^n)$.

Theorem 5[51] A function $\Phi: L^p(\mathbf{R},S^{n-1}) \to \mathbf{R}$ is a continuous rotation invariant valuation if and only if there exists a continuous function $h:\mathbf{R}\to\mathbf{R}$ with the properties that $h(0)=0$ and there exist constants $\gamma,\delta \geqslant 0$ such that $|h(a)| \leqslant \gamma|a|^p + \delta$ for all $a \in \mathbf{R}$, and

$$\Phi(f) = \int_{S^{n-1}} (h \circ f)(u)\mathrm{d}u$$

for all $f \in L^p(\mathbf{R},S^{n-1})$.

2 Main Results

Lemma 1 Let $h_k:\mathbf{R}\to\mathbf{R}$ be continuous functions with the properties that $h_k(0)=0$ and there exist $\gamma_k,\delta_k \geqslant 0$ such that $|h_k(a)| \leqslant \gamma_k |a|^p + \delta_k$ for all $a \in \mathbf{R}$ (k =1,⋯,4). If the function $\Phi: L^p(\mathbf{C},\mu) \to \mathbf{C}$ is defined by

$$\Phi(f) = \int_X (h_1 \circ \Re f + h_3 \circ \Im f)\mathrm{d}\mu + \mathrm{i}\int_X (h_2 \circ \Re f + h_4 \circ \Im f)\mathrm{d}\mu \tag{6}$$

then Φ is a continuous valuation provided that $\delta_k = 0$ if $\mu(X) = \infty$.

Proof For $f, g \in L^p(\mathbf{C},\mu)$, let

$$E = \{x \in X : \Re f \leqslant \Re g,\ \Im f \leqslant \Im g\},$$
$$F = \{x \in X : \Re f \leqslant \Re g,\ \Im f > \Im g\},$$
$$G = \{x \in X : \Re f > \Re g,\ \Im f \leqslant \Im g\},$$
$$H = \{x \in X : \Re f > \Re g,\ \Im f > \Im g\}.$$

By (6) and (2), we obtain

$$\begin{aligned}
&\Phi(f \vee g)\\
&= \int_X (h_1 \circ \Re(f \vee g) + h_3 \circ \Im(f \vee g))\mathrm{d}\mu\\
&+\mathrm{i}\int_X (h_2 \circ \Re(f \vee g) + h_4 \circ \Im(f \vee g))\mathrm{d}\mu\\
&= \int_E (h_1 \circ \Re g + h_3 \circ \Im g)\mathrm{d}\mu + \mathrm{i}\int_E (h_2 \circ \Re g + h_4 \circ \Im g)\mathrm{d}\mu\\
&+\int_F (h_1 \circ \Re g + h_3 \circ \Im f)\mathrm{d}\mu + \mathrm{i}\int_F (h_2 \circ \Re g + h_4 \circ \Im f)\mathrm{d}\mu\\
&+\int_G (h_1 \circ \Re f + h_3 \circ \Im g)\mathrm{d}\mu + \mathrm{i}\int_G (h_2 \circ \Re f + h_4 \circ \Im g)\mathrm{d}\mu\\
&+\int_H (h_1 \circ \Re f + h_3 \circ \Im f)\mathrm{d}\mu + \mathrm{i}\int_H (h_2 \circ \Re f + h_4 \circ \Im f)\mathrm{d}\mu
\end{aligned}$$

Similarly, by (6) and (3), we have

$$\begin{aligned}
&\Phi(f \wedge g)\\
&= \int_E (h_1 \circ \Re f + h_3 \circ \Im f)\mathrm{d}\mu + \mathrm{i}\int_E (h_2 \circ \Re f + h_4 \circ \Im f)\mathrm{d}\mu\\
&+\int_F (h_1 \circ \Re f + h_3 \circ \Im g)\mathrm{d}\mu + \mathrm{i}\int_F (h_2 \circ \Re f + h_4 \circ \Im g)\mathrm{d}\mu\\
&+\int_G (h_1 \circ \Re g + h_3 \circ \Im f)\mathrm{d}\mu + \mathrm{i}\int_G (h_2 \circ \Re g + h_4 \circ \Im f)\mathrm{d}\mu\\
&+\int_H (h_1 \circ \Re g + h_3 \circ \Im g)\mathrm{d}\mu + \mathrm{i}\int_H (h_2 \circ \Re g + h_4 \circ \Im g)\mathrm{d}\mu
\end{aligned}$$

Note that $E \cup F \cup G \cup H = X$ and that E, F, G, H are pairwise disjoints. Thus,

$$\Phi(f \vee g) + \Phi(f \wedge g) = \Phi(f) + \Phi(g)$$

Hence Φ is a valuation.

It remains to show that Φ is continuous. Let $f \in L^p(\mathbf{C},\mu)$ and let $\{f_k\}$ be a sequence in $L^p(\mathbf{C},\mu)$ with $f_k \to f$ in $L^p(\mathbf{C},\mu)$. Next, we will show that $\Phi(f_k)$ converges to $\Phi(f)$ by showing that every subsequence $\Phi(f_{k_l})$ of $\Phi(f_{k_l})$ has a subsequence, $\Phi(f_{k_{l_m}})$ which converges to $\Phi(f)$. Set $f = \alpha + \mathrm{i}\beta$ and $f = \alpha_k + \mathrm{i}\beta_k$ with $\alpha, \beta, \alpha_k, \beta_k \in L^p(\mathbf{R},\mu)$ such that $\alpha_k \to \alpha$ and $\beta_k \to \beta$ in $L^p(\mathbf{R},\mu)$. Let $\{f_{k_l}\}$ be a subsequence of $\{f_k\}$, then $\{f_{k_l}\}$ converges to f in $L^p(\mathbf{C},\mu)$. Then there exists a subsequence $\{f_{k_{l_m}}\}$ of $\{f_{k_l}\}$ which converges to f in $L^p(\mathbf{C},\mu)$, where $f_{k_{l_m}} = \alpha_{k_{l_m}} + \mathrm{i}\beta_{k_{l_m}}$ with $\alpha_{k_{l_m}}, \beta_{k_{l_m}} \in L^p(\mathbf{R},\mu)$ such that $\alpha_{k_{l_m}} \to \alpha$ and $\beta_{k_{l_m}} \to \beta$ in $L^p(\mathbf{R},\mu)$. Since h_1 is continuous, we have

$$(h_1 \circ \alpha_{k_{l_m}})(x) \to (h_1 \circ \alpha)(x), \text{a.e.}[\mu]$$

Since $|h_1(a)| \leqslant \gamma_1 |a|^p + \delta_1$ for all $a \in \mathbf{R}$, we get

$$\left|(h_1 \circ \alpha_{k_{l_m}})(x)\right| \leqslant \gamma_1 \left|\alpha_{k_{l_m}}\right|^p + \delta_1, \text{a.e.}[\mu].$$

If $\mu(x)<\infty$, apply $\alpha_{k_{l_m}}\to\alpha$ in $L^p(\mathbf{R},\mu)$ to get

$$\lim_{m\to\infty}\int_X \gamma_1\left|\alpha_{k_{l_m}}\right|^p+\delta_1\mathrm{d}\mu=\int_X \gamma_1|\alpha|^p\,\mathrm{d}\mu+\delta_1\mu(X)$$

And we take $\delta_1=0$ in the above equation if $\mu(x)=\infty$. By a modification of Lebesgue's Dominated Convergence Theorem (see Ref.[51], Proposition 2.2), we have $h_1\circ(\alpha)\in L^1(\mathbf{R},\mu)$ and

$$\lim_{m\to\infty}\int_X(h_1\circ\alpha_{k_{l_m}})\mathrm{d}\mu=\int_X(h_1\circ\alpha)\mathrm{d}\mu \tag{7}$$

Similarly,

$$\lim_{m\to\infty}\int_X(h_2\circ\alpha_{k_{l_m}})\mathrm{d}\mu=\int_X(h_2\circ\alpha)\mathrm{d}\mu \tag{8}$$

$$\lim_{m\to\infty}\int_X(h_3\circ\beta_{k_{l_m}})\mathrm{d}\mu=\int_X(h_3\circ\beta)\mathrm{d}\mu \tag{9}$$

$$\lim_{m\to\infty}\int_X(h_4\circ\beta_{k_{l_m}})\mathrm{d}\mu=\int_X(h_4\circ\beta)\mathrm{d}\mu \tag{10}$$

Thus,

$$\begin{aligned}&\left|\Phi(f_{k_{l_m}})-\Phi(f)\right|\\&=\left|\int_X(h_1\circ\alpha_{k_{l_m}}-h_1\circ\alpha)\mathrm{d}\mu+\int_X(h_3\circ\beta_{k_{l_m}}-h_3\circ\beta)\mathrm{d}\mu\right.\\&\left.+\mathrm{i}(\int_X(h_2\circ\alpha_{k_{l_m}}-h_2\circ\alpha)\mathrm{d}\mu+\int_X(h_4\circ\beta_{k_{l_m}}-h_4\circ\beta)\mathrm{d}\mu)\right|\\&\leqslant\left|\int_X(h_1\circ\alpha_{k_{l_m}}-h_1\circ\alpha)\mathrm{d}\mu\right|+\left|\int_X(h_3\circ\beta_{k_{l_m}}-h_3\circ\beta)\mathrm{d}\mu\right|\\&+\left|\int_X(h_2\circ\alpha_{k_{l_m}}-h_2\circ\alpha)\mathrm{d}\mu\right|+\left|\int_X(h_4\circ\beta_{k_{l_m}}-h_4\circ\beta)\mathrm{d}\mu\right|\end{aligned}$$

From (7)-(10), we conclude $\Phi(f_{k_{l_m}})\to\Phi(f)$. Hence, Φ is continuous.

Lemma 2 If the function $\Phi:L^p(\mathbf{C},\mu)\to\mathbf{C}$ is a valuation, then

$$\Phi(f)=\Phi(\Re f)+\Phi(\mathrm{i}\Im f)$$

for all $f\in L^p(\mathbf{C},\mu)$.

Proof If $\Re f,\Im f\geqslant 0$ or $\Re f,\Im f\leqslant 0$, then, by (2) and (3), we have

$$\Phi(\Re f)+\Phi(\mathrm{i}\Im f)=\Phi(f)+\Phi(0) \tag{11}$$

If $\Re f\geqslant 0,\Im f\leqslant 0$ or $\Re f\leqslant 0,\Im f\geqslant 0$, then, by (2) and (3), we have

$$\Phi(f)+\Phi(0)=\Phi(\Re f)+\Phi(\mathrm{i}\Im f) \tag{12}$$

Note that $\Phi(0)=0$, apply (11) and (12), then get

$$\Phi(f)=\Phi(\Re f)+\Phi(\mathrm{i}\Im f)$$

for all $f\in L^p(\mathbf{C},\mu)$.

If we restrict f to $L^p(\mathbf{R},\mu)$, then it is obvious that Φ is a valuation on $L^p(\mathbf{R},\mu)$. Also, we can construct another valuation on $L^p(\mathbf{R},\mu)$ which is related to Φ.

Lemma 3 Let $\Phi:L^p(\mathbf{C},\mu)\to\mathbf{C}$ be a valuation. If the functions $\Phi':L^p(\mathbf{R},\mu)\to\mathbf{C}$ is defined by

$$\Phi'(f)=\Phi(\mathrm{i}f)$$

for all $f\in L^p(\mathbf{R},\mu)$, then Φ' is a valuation on $L^p(\mathbf{R},\mu)$.

Proof For $f,g\in L^p(\mathbf{R},\mu)$, by (2) and (3), we have

$$\mathrm{i}(f\vee g)=\mathrm{i}f\vee\mathrm{i}g \quad\text{and}\quad \mathrm{i}(f\wedge g)=\mathrm{i}f\wedge\mathrm{i}g \tag{13}$$

By (13) and the valuation property of Φ, it follows that

$$\begin{aligned}&\Phi'(f\vee g)+\Phi'(f\wedge g)\\&=\Phi(\mathrm{i}f\vee\mathrm{i}g)+\Phi(\mathrm{i}f\wedge\mathrm{i}g)\\&=\Phi(\mathrm{i}f)+\Phi(\mathrm{i}g)=\Phi'(f)+\Phi'(g)\end{aligned}$$

for all $f,g\in L^p(\mathbf{R},\mu)$. Thus, Φ' is a valuation on $L^p(\mathbf{R},\mu)$.

Lemma 4 Let $\Phi:L^p(\mathbf{R},\mu)\to\mathbf{C}$ be a valuation. If the functions $\Phi_1,\Phi_2:L^p(\mathbf{R},\mu)\to\mathbf{R}$ are defined by

$$\Phi(f)=\Phi_1(f)+\mathrm{i}\Phi_2(f)$$

for all $f\in L^p(\mathbf{R},\mu)$, then both Φ_1,Φ_2 are real-valued valuations on $L^p(\mathbf{R},\mu)$.

Proof Since Φ is a valuation, we have

$$\begin{aligned}&\Phi(f\vee g)+\Phi(f\wedge g)\\&=\Phi_1(f\vee g)+\mathrm{i}\Phi_2(f\vee g)\\&+\Phi_1(f\wedge g)+\mathrm{i}\Phi_2(f\wedge g)\\&=\Phi(f)+\Phi(g)\\&=\Phi_1(f)+\mathrm{i}\Phi_2(f)+\Phi_1(g)+\mathrm{i}\Phi_2(g)\end{aligned}$$

for all $f,g\in L^p(\mathbf{R},\mu)$. Thus,

$$\Phi_1(f\vee g)+\Phi_1(f\wedge g)=\Phi_1(f)+\Phi_1(g)$$

and

$$\Phi_2(f\vee g)+\Phi_2(f\wedge g)=\Phi_2(f)+\Phi_2(g)$$

for all $f,g\in L^p(\mathbf{R},\mu)$.

Therefore, both Φ_1,Φ_2 are real-valued valuations on $L^p(\mathbf{R},\mu)$.

In order to establish a representation theorem for continuous complex-valued valuations on $L^p(\mathbf{C},\mu)$, we will use the corresponding representation theorem for real case which was obtained by Tsang [51].

Theorem 6[51] Let $(X,\aleph,\mu)$ be a non-atomic measure space and let $\Phi:L^p(\mathbf{R},\mu)\to\mathbf{R}$ be a continuous translation invariant valuation. If there exists a continuous function $h:\mathbf{R}\to\mathbf{R}$ with $h(0)=0$ such that $\Phi(b\chi_E)=h(b)\mu(E)$ for all $b\in\mathbf{R}$ and all $E\in\aleph$ with $\mu(E)<\infty$, then there exist constants $\gamma_1,\delta\geqslant 0$ such that $|h(a)|\leqslant\gamma|a|^p+\delta$ for all $a\in\mathbf{R}$, and

$$\Phi(f)=\int_X(h\circ f)\mathrm{d}\mu$$

for all $f\in L^p(\mathbf{R},\mu)$. In addition, $\delta=0$ if $\mu(X)=\infty$.

Proof of Theorem 1

Let $\Phi : L^p(\mathbf{C},\mu) \to \mathbf{C}$ be a continuous valuation. For $f \in L^p(\mathbf{C},\mu)$, by Lemma 2, Lemma 3 and Lemma 4, we have

$$\begin{aligned}\Phi(f) &= \Phi(\Re f) + \Phi(\mathrm{i}\Im f)\\ &= \Phi_1(\Re f) + \mathrm{i}\Phi_2(\Re f) + \Phi_1(\mathrm{i}\Im f) + \mathrm{i}\Phi_2(\mathrm{i}\Im f)\\ &= \Phi_1(\Re f) + \mathrm{i}\Phi_2(\Re f) + \Phi_1'(\Im f) + \mathrm{i}\Phi_2'(\Im f)\end{aligned}$$

where $\Phi_1'(\Im f) = \Phi_1(\mathrm{i}\Im f)$ and $\Phi_2'(\Im f) = \mathrm{i}\Phi_2(\mathrm{i}\Im f)$.

Since $\Re(f \vee g) = \Re f \vee \Re g$, $\Re(f \wedge g) = \Re f \wedge \Re g$, $\Im(f \vee g) = \Im f \vee \Im g$, and $\Im(f \wedge g) = \Im f \wedge \Im g$. Moreover, Lemma 3 and Lemma 4 imply that $\Phi_1, \Phi_2, \Phi_1', \Phi_2'$ are real-valued valuations on $L^p(\mathbf{R},\mu)$.

If we restrict to $f \in L^p(\mathbf{R},\mu)$, then the continuity of Φ implies that Φ_1, Φ_2, are continuous on $L^p(\mathbf{R},\mu)$.

If we consider $f \in L^p(\mathbf{C},\mu)$ with $\Re f = 0$, then the continuity of Φ implies that Φ_1', Φ_2' are continuous on $L^p(\mathbf{R},\mu)$. Thus, $\Phi_1, \Phi_2, \Phi_1', \Phi_2'$ are continuous real-valued valuations on $L^p(\mathbf{R},\mu)$. It follows from Theorem 6 that there exist continuous functions $h_k : \mathbf{R} \to \mathbf{R}$ with the properties that $h_k(0) = 0$ and there exist constants $\gamma_k, \delta_k \geqslant 0$ such that $|h_k(a)| \leqslant \gamma_k |a|^p + \delta_k$ for all $a \in \mathbf{R}$ (k = 1,2,3,4), and

$$\begin{aligned}\Phi(f) &= \int_X \left(h_1 \circ \Re f + h_3 \circ \Im f\right) \mathrm{d}\mu\\ &\quad + \mathrm{i}\int_X \left(h \circ \Re f + h_4 \circ \Im f\right) \mathrm{d}\mu\end{aligned}$$

for all $f \in L^p(\mathbf{C},\mu)$. In addition, $\delta_k = 0$ (k = 1,2,3,4) if $\mu(X) = \infty$.

If μ is Lebesgue measure, then $L^p(\mathbf{C},\mu)$ becomes the space of Lebesgue measurable complex-valued functions. We usually write as $L^p(\mathbf{C},\mathbf{R}^n)$.

Lemma 5 Let $h_k : \mathbf{R} \to \mathbf{R}$ be continuous functions with the property that there exists $\gamma_k \geqslant 0$ such that $|h_k(a)| \leqslant \gamma_k |a|^p$ for all $a \in \mathbf{R}$ (k = 1,⋯,4). If the function $\Phi : L^p(\mathbf{C},\mathbf{R}^n) \to \mathbf{C}$ is defined by

$$\begin{aligned}\Phi(f) &= \int_{\mathbf{R}^n} \left(h_1 \circ \Re f + h_3 \circ \Im f\right) \mathrm{d}x\\ &\quad + \mathrm{i}\int_X \left(h_2 \circ \Re f + h_4 \circ \Im f\right) \mathrm{d}x\end{aligned}$$

Then Φ is a continuous translation invariant valuation.

Proof Let M enote the collection of Lebesgue measurable sets in $\mathbf{R}^n$. Take $X = \mathbf{R}^n, \aleph = M$ and μ Lebesgue measure in Lemma 1 to conclude that Φ is a continuous valuation on $L^p(\mathbf{C},\mathbf{R}^n)$.

For every $t \in \mathbf{R}^n$ and every $f \in L^p(\mathbf{C},\mathbf{R}^n)$, we have

$$\begin{aligned}\Phi(f(x-t)) &= \int_{\mathbf{R}^n} \left(h_1 \circ \Re f + h_3 \circ \Im f\right)(x-t)\mathrm{d}x\\ &\quad + \mathrm{i}\int_X \left(h_2 \circ \Re f + h_4 \circ \Im f\right)(x-t)\mathrm{d}x\\ &= \Phi(f)\end{aligned}$$

which means that Φ is translation invariant.

Proof of Theorem 2

It follows from Lemma 5 that (4) determines a continuous translation invariant valuation on $L^p(\mathbf{C},\mathbf{R}^n)$. Conversely, let $\Phi : L^p(\mathbf{C},\mathbf{R}^n) \to \mathbf{C}$ be a continuous translation invariant valuation. Taking $X = \mathbf{R}^n$, $\aleph = M$ and μ Lebesgue measure in the proof of Theorem 1, we obtain

$$\Phi(f) = \Phi_1(\Re f) + \mathrm{i}\Phi_2(\Re f) + \Phi_1'(\Im f) + \mathrm{i}\Phi_2'(\Im f)$$

where $\Phi_1, \Phi_2, \Phi_1', \Phi_2'$ are real-valued valuations on $L^p(\mathbf{R},\mu)$. Theorem 4 implies that there exist continuous functions $h_k : \mathbf{R} \to \mathbf{R}$ with the property that there exist constants $\gamma_k \geqslant 0$ such that $|h_k(a)| \leqslant \gamma_k |a|^p$ for all $a \in \mathbf{R}$ (k = 1,2,3,4), and

$$\begin{aligned}\Phi(f) &= \int_{\mathbf{R}^n} \left(h_1 \circ \Re f + h_3 \circ \Im f\right) \mathrm{d}x\\ &\quad + \mathrm{i}\int_{\mathbf{R}^n} \left(h_2 \circ \Re f + h_4 \circ \Im f\right) \mathrm{d}x\end{aligned}$$

for all $f \in L^p(\mathbf{C},\mathbf{R}^n)$.

Let W denote the σ-algebra defined as

$$W = \left\{E : E \subseteq S^{n-1}, \{\lambda x : x \in E, 0 \leqslant \lambda \leqslant 1\} \in M\right\}$$

Also denote by σ the spherical Lebesgue measure. If μ is the spherical Lebesgue measure, then $L^p(\mathbf{C},\sigma)$ denotes the space of spherical Lebesgue measurable complex valued functions. We usually write as $L^p(\mathbf{C},S^{n-1})$.

Lemma 6 Let $h_k : \mathbf{R} \to \mathbf{R}$ be continuous functions with the properties that $h_k(0) = 0$ and there exists $\gamma_k, \delta_k \geqslant 0$ such that $|h_k(a)| \leqslant \gamma_k |a|^p + \delta_k$ for all $a \in \mathbf{R}$ (k=1,⋯,4). If the function $\Phi : L^p(\mathbf{C},S^{n-1}) \to \mathbf{C}$ is defined by

$$\begin{aligned}\Phi(f) &= \int_{S^{n-1}} \left(h_1 \circ \Re f + h_3 \circ \Im f\right) \mathrm{d}u\\ &\quad + \mathrm{i}\int_{S^{n-1}} \left(h_2 \circ \Re f + h_4 \circ \Im f\right) \mathrm{d}u\end{aligned}$$

then Φ is a continuous rotation invariant valuation.

Proof Take $X = S^{n-1}, \aleph = W$ and $\mu = \sigma$ in Lemma 1 to conclude that Φ is a continuous valuation on $L^p(\mathbf{C},S^{n-1})$.

Note that $\theta u \in S^{n-1}$ for every $\theta \in O(n)$ and every $u \in S^{n-1}$. Since the spherical Lebesgue measure is

rotation invariant, we have

$$\begin{aligned}\Phi(f\circ\theta^{-1}) &= \int_{S^{n-1}}\left(h_1\circ\Re f+h_3\circ\Im f\right)(\theta u)\mathrm{d}u\\ &\quad+\mathrm{i}\int_{S^{n-1}}\left(h_2\circ\Re f+h_4\circ\Im f\right)(\theta u)\mathrm{d}u\\ &=\Phi(f)\end{aligned}$$

for all $f\in L^p(\mathbf{C},S^{n-1})$, which completes the proof.

Proof of Theorem 3

It follows from Lemma 6 that (5) determines a continuous rotation invariant valuation on $L^p(\mathbf{C},S^{n-1})$.

Conversely, let $\Phi:L^p(\mathbf{C},S^{n-1})\to\mathbf{C}$ be a continuous rotation invariant valuation. Taking $X=S^{n-1}$, $\aleph=W$ and $\mu=\sigma$ in the proof of Theorem 1, we obtain

$$\Phi(f)=\Phi_1(\Re f)+\mathrm{i}\Phi_2(\Re f)+\Phi_1'(\Im f)+\mathrm{i}\Phi_2'(\Im f)$$

where $\Phi_1,\Phi_2,\Phi_1',\Phi_2'$ are real-valued valuations on $L^p(\mathbf{R},S^{n-1})$.

Theorem 5 implies that there exist continuous functions $h_k:\mathbf{R}\to\mathbf{R}$ with the properties that $h_k(0)=0$ and there exist constants $\gamma_k,\delta_k\geqslant 0$ such that $|h_k(a)|\leqslant\gamma_k|a|^p+\delta_k$ for all $a\in\mathbf{R}$ (k = 1,2,3,4), and

$$\begin{aligned}\Phi(f)&=\int_{S^{n-1}}\left(h_1\circ\Re f+h_3\circ\Im f\right)\mathrm{d}u\\ &\quad+\mathrm{i}\int_{S^{n-1}}\left(h_2\circ\Re f+h_4\circ\Im f\right)\mathrm{d}u\end{aligned}$$

for all $f\in L^p(\mathbf{C},S^{n-1})$.

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