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Complex-Valued Valuations on L^p Spaces

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Abstract: All continuous translation invariant complex-valued valuations on Lebesgue measurable functions are completely classified. And all continuous rotation invariant complex-valued valuations on spherical Lebesgue measurable functions are also completely classified.
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0 Introduction

A function z defined on a lattice $(L,\vee,\wedge)$ and taking values in an Aabelian semigroup is called a valuation if

$$z(f\vee g)+z(f\wedge g)=z(f)+z(g) \qquad (1)$$

for all $f,g\in L$. A function z defined on some subset L_0 of L is called a valuation on L_0 if (1) holds whenever $f,g,f\vee g,f\wedge g\in L_0$. For L_0 the set of convex bodies, K^n, in $\mathbf{R}^n$ with $\vee$ denoting union and $\wedge$ intersection. Valuation on convex bodies is a classical concept. Probably the most famous result on valuations is Hadwiger's classification theorem of continuous rigid motion invariant valuations[1]. For the more recent contributions on valuations on convex bodies readers can refer to Refs. [2-33].

Valuations on convex bodies can be considered as valuations on suitable function spaces. Recently, valuations on functions have been rapidly growing (see Refs. [34-55]). For a space of real-valued functions, the operations $\vee$ and $\wedge$ are defined as pointwise maximum and minimum, respectively. A complete classification of valuations intertwining with the SL(n) on Sobolev space [46-49] and L^p space [45,50-52,54] were established, respectively. Valuations on convex functions [34,36,39-42], quasi-concave functions [37,38], Lipschitz functions [43,44], and functions of Bounded variations [53] were introduced and classified.

Recently, Wang and Liu [55] showed that the Fourier transform is the only valuation which is a continuous, positive GL(n) covariant and logarithmic translation covariant complex-valued valuation on integral functions. This motivates the study of complex-valued valuations on functions.

Let L be a lattice of complex-valued functions. For $f\in L$, let $\Re f$ and $\Im f$ denote the real and imaginary parts of *f*, respectively. The pointwise maximum of *f* and *g*, $f\vee g$ and the pointwise minimum of *f* and *g* $f\wedge g$ are defined by

$$f\vee g=\Re f\vee\Re g+\mathrm{i}(\Im f\vee\Im g) \tag{2}$$

and

$$f\wedge g=\Re f\wedge\Re g+\mathrm{i}(\Im f\wedge\Im g) \tag{3}$$

If *f*, *g* are real-valued functions, then (2) and (3) coincide with the real cases. A function $\varPhi:L\to\mathbf{C}$ is called a valuation if

$$\varPhi(f\vee g)+\varPhi(f\wedge g)=\varPhi(f)+\varPhi(g)$$

for all $f,g\in L$ and $\varPhi(0)=0$ if $0\in L$. It is called continuous if

$$\varPhi(f_i)\to\varPhi(f),\text{ as } f_i\to f \text{ in } L.$$

It is called translation invariant if

$$\varPhi(f(\cdot-t))=\varPhi(f)$$

for every $t\in\mathbf{R}^n$. It is called rotation invariant if

$$\varPhi(f\circ\theta^{-1})=\varPhi(f)$$

for every $\theta\in O(n)$, where θ^{-1} denotes the inverse of θ.

Let $p\geqslant 1$. If $(X,\aleph,\mu)$ is a measure space, then the L^p-space, $L^p(\mathbf{C},\mu)$ is the collection of μ-measurable complex-valued functions $f:X\to\mathbf{C}$ that satisfies

$$\int_X|f|^p\,\mathrm{d}\mu<\infty$$

A measure space $(X,\aleph,\mu)$ is called non-atomic if for every $E\in\aleph$ with $\mu(E)>0$, there exists $F\in\aleph$ with $F\subseteq E$ and $0<\mu(F)<\mu(E)$. Let χ_E denote the characteristic function of the measurable set *E*, i.e.

$$\chi_E=\begin{cases}1, & x\in E\\ 0, & x\notin E\end{cases}$$

Theorem 1 Let $(X,\aleph,\mu)$ be a non-atomic measure space and let $\varPhi:L^p(\mathbf{C},\mu)\to\mathbf{C}$ be a continuous valuation. If there exist continuous functions $h_k:\mathbf{R}\to\mathbf{R}$ with $h_k(0)=0$ (k=1,2,3,4) such that $\varPhi(c\chi_E)=(h_1(\Re c)+h_3(\Im c))\mu(E)+\mathrm{i}(h_2(\Re c)+h_4(\Im c))\mu(E)$ for all $c\in\mathbf{C}$ and all $E\in\aleph$ with $\mu(E)<\infty$, then there exist constants $\gamma_k,\delta_k\geqslant 0$ such that $|h_k(a)|\leqslant\gamma_k|a|^p+\delta_k$ for $a\in\mathbf{R}$, and

$$\varPhi(f)=\int_X(h_1\circ\Re f+h_3\circ\Im f)\mathrm{d}\mu$$
$$+\mathrm{i}\int_X(h_2\circ\Re f+h_4\circ\Im f)\mathrm{d}\mu$$

for all $f\in L^p(\mathbf{C},\mu)$. In addition, $\delta_k=0$ (k = 1,2,3,4) if $\mu(X)=\infty$.

Let $L^p(\mathbf{C},\mathbf{R}^n)$ denote the L^p-space of Lebesgue measurable complex-valued functions on $\mathbf{R}^n$.

Theorem 2 A function $\varPhi:L^p(\mathbf{C},\mathbf{R}^n)\to\mathbf{C}$ is a continuous translation invariant valuation if and only if there exist continuous functions $h_k:\mathbf{R}\to\mathbf{R}$ with the property that there exist constants $\gamma_k\geqslant 0$ such that $|h_k(a)|\leqslant\gamma_k|a|^p$ for all $a\in\mathbf{R}$ (k = 1,2,3,4), and

$$\varPhi(f)=\int_{\mathbf{R}^n}(h_1\circ\Re f+h_3\circ\Im f)(x)\mathrm{d}x$$
$$+\mathrm{i}\int_{\mathbf{R}^n}(h_2\circ\Re f+h_4\circ\Im f)(x)\,\mathrm{d}x \tag{4}$$

for all $f\in L^p(\mathbf{C},\mathbf{R}^n)$.

Let S^{n-1} be the unit sphere in $\mathbf{R}^n$ and let $L^p(\mathbf{C},S^{n-1})$ denote the L^p-space of spherical Lebesgue measurable complex-valued functions on S^{n-1}.

Theorem 3 A function $\varPhi:L^p(\mathbf{C},S^{n-1})\to\mathbf{C}$ is a continuous rotation invariant valuation if and only if there exist continuous functions $h_k:\mathbf{R}\to\mathbf{R}$ with the properties that $h_k(0)=0$ and there exist constants $\gamma_k,\delta_k\geqslant 0$ such that $|h_k(a)|\leqslant\gamma_k|a|^p+\delta_k$ for all $a\in\mathbf{R}$ (k = 1,2,3,4), and

$$\varPhi(f)=\int_{S^{n-1}}(h_1\circ\Re f+h_3\circ\Im f)(u)\mathrm{d}u$$
$$+\mathrm{i}\int_{S^{n-1}}(h_2\circ\Re f+h_4\circ\Im f)(u)\mathrm{d}u \tag{5}$$

for all $f\in L^p(\mathbf{C},S^{n-1})$.

1 Notation and Preliminary Results

We collect some properties of complex-valued functions. If f is a complex-valued function on $\mathbf{R}^n$, then

$$f(x)=\Re f+\mathrm{i}\Im f$$

where $\Re f$ and $\Im f$ denote the real part and imaginary part of *f*, respectively. The absolute value of *f* which is also called modulus is defined by

$$|f|=\sqrt{(\Re f)^2+(\Im f)^2}$$

Let $p\geqslant 1$. For a measure space $(X,\aleph,\mu)$, define $L^p(\mathbf{C},\mu)$ as the space of μ-measurable complex-valued functions $f:X\to\mathbf{C}$ that satisfies

$$\|f\|_p=\left(\int_X|f|^p\,\mathrm{d}\mu\right)^{\frac{1}{p}}<\infty$$

Let $f,g\in L^p(\mathbf{C},\mu)$, then $f\vee g, f\wedge g\in L^p(\mathbf{C},\mu)$. The functional $\|\cdot\|:L^p(\mathbf{C},\mu)\to\mathbf{R}$ is a semi-norm. If functions in $L^p(\mathbf{C},\mu)$ that are equal almost everywhere with respect to μ(a.e.$[\mu]$) are identified, then $\|\cdot\|:L^p(\mathbf{C},\mu)\to\mathbf{R}$ becomes a norm. Obviously, $L^p(\mathbf{C},\mu)$ is a lattice of complex-valued functions. Let

$L^p(\mathbf{R},\mu)$ denote the subset of $L^p(\mathbf{C},\mu)$, where the functions take real values. For $f_i, f\in L^p(\mathbf{C},\mu)$, if $\|f_i-f\|_p\to 0$, then $f_i\to f$ in $L^p(\mathbf{C},\mu)$. Moreover,

$$f_i\to f\in L^p(\mathbf{C},\mu)\Leftrightarrow \Re f_i\to\Re f,\ \Im f_i\to\Im f\in L^p(\mathbf{R},\mu)$$

The following characterizations of real-valued valuations on functions which were established by Tsang [51] will play key role in our proof. Let $L^p(\mathbf{R},\mathbf{R}^n)$ and $L^p(\mathbf{R},S^{n-1})$ denote the L^p space of Lebesgue measurable real-valued functions on $\mathbf{R}^n$ and the L^p space of spherical Lebesgue measurable real-valued functions on S^{n-1} respectively.

Theorem 4 [51] A function $\Phi: L^p(\mathbf{R},\mathbf{R}^n)\to\mathbf{R}$ is a continuous translation invariant valuation if and only if there exists a continuous function $h:\mathbf{R}\to\mathbf{R}$ with the property that there exists a constant $\gamma\geqslant 0$ such that $|h(a)|\leqslant\gamma|a|^p$ for all $a\in\mathbf{R}$, and

$$\Phi(f)=\int_{\mathbf{R}^n}(h\circ f)(x)\mathrm{d}x$$

for all $f\in L^p(\mathbf{R},\mathbf{R}^n)$.

Theorem 5[51] A function $\Phi: L^p(\mathbf{R},S^{n-1})\to\mathbf{R}$ is a continuous rotation invariant valuation if and only if there exists a continuous function $h:\mathbf{R}\to\mathbf{R}$ with the properties that $h(0)=0$ and there exist constants $\gamma,\delta\geqslant 0$ such that $|h(a)|\leqslant\gamma|a|^p+\delta$ for all $a\in\mathbf{R}$, and

$$\Phi(f)=\int_{S^{n-1}}(h\circ f)(u)\mathrm{d}u$$

for all $f\in L^p(\mathbf{R},S^{n-1})$.

2 Main Results

Lemma 1 Let $h_k:\mathbf{R}\to\mathbf{R}$ be continuous functions with the properties that $h_k(0)=0$ and there exist $\gamma_k,\delta_k\geqslant 0$ such that $|h_k(a)|\leqslant\gamma_k|a|^p+\delta_k$ for all $a\in\mathbf{R}$ (k =1,$\cdots$,4). If the function $\Phi: L^p(\mathbf{C},\mu)\to\mathbf{C}$ is defined by

$$\Phi(f)=\int_X(h_1\circ\Re f+h_3\circ\Im f)\mathrm{d}\mu+\mathrm{i}\int_X(h_2\circ\Re f+h_4\circ\Im f)\mathrm{d}\mu \qquad (6)$$

then Φ is a continuous valuation provided that $\delta_k=0$ if $\mu(X)=\infty$.

Proof For $f,g\in L^p(\mathbf{C},\mu)$, let

$$E=\{x\in X:\Re f\leqslant\Re g,\ \Im f\leqslant\Im g\},$$
$$F=\{x\in X:\Re f\leqslant\Re g,\ \Im f>\Im g\},$$
$$G=\{x\in X:\Re f>\Re g,\ \Im f\leqslant\Im g\},$$
$$H=\{x\in X:\Re f>\Re g,\ \Im f>\Im g\}.$$

By (6) and (2), we obtain

$$\begin{aligned}
&\Phi(f\vee g)\\
&=\int_X(h_1\circ\Re(f\vee g)+h_3\circ\Im(f\vee g))\mathrm{d}\mu\\
&+\mathrm{i}\int_X(h_2\circ\Re(f\vee g)+h_4\circ\Im(f\vee g))\mathrm{d}\mu\\
&=\int_E(h_1\circ\Re g+h_3\circ\Im g)\mathrm{d}\mu+\mathrm{i}\int_E(h_2\circ\Re g+h_4\circ\Im g)\mathrm{d}\mu\\
&+\int_F(h_1\circ\Re g+h_3\circ\Im f)\mathrm{d}\mu+\mathrm{i}\int_F(h_2\circ\Re g+h_4\circ\Im f)\mathrm{d}\mu\\
&+\int_G(h_1\circ\Re f+h_3\circ\Im g)\mathrm{d}\mu+\mathrm{i}\int_G(h_2\circ\Re f+h_4\circ\Im g)\mathrm{d}\mu\\
&+\int_H(h_1\circ\Re f+h_3\circ\Im f)\mathrm{d}\mu+\mathrm{i}\int_H(h_2\circ\Re f+h_4\circ\Im f)\mathrm{d}\mu
\end{aligned}$$

Similarly, by (6) and (3), we have

$$\begin{aligned}
&\Phi(f\wedge g)\\
&=\int_E(h_1\circ\Re f+h_3\circ\Im f)\mathrm{d}\mu+\mathrm{i}\int_E(h_2\circ\Re f+h_4\circ\Im f)\mathrm{d}\mu\\
&+\int_F(h_1\circ\Re f+h_3\circ\Im g)\mathrm{d}\mu+\mathrm{i}\int_F(h_2\circ\Re f+h_4\circ\Im g)\mathrm{d}\mu\\
&+\int_G(h_1\circ\Re g+h_3\circ\Im f)\mathrm{d}\mu+\mathrm{i}\int_G(h_2\circ\Re g+h_4\circ\Im f)\mathrm{d}\mu\\
&+\int_H(h_1\circ\Re g+h_3\circ\Im g)\mathrm{d}\mu+\mathrm{i}\int_H(h_2\circ\Re g+h_4\circ\Im g)\mathrm{d}\mu
\end{aligned}$$

Note that $E\cup F\cup G\cup H=X$ and that E, F, G, H are pairwise disjoints. Thus,

$$\Phi(f\vee g)+\Phi(f\wedge g)=\Phi(f)+\Phi(g)$$

Hence Φ is a valuation.

It remains to show that Φ is continuous. Let $f\in L^p(\mathbf{C},\mu)$ and let $\{f_k\}$ be a sequence in $L^p(\mathbf{C},\mu)$ with $f_k\to f$ in $L^p(\mathbf{C},\mu)$. Next, we will show that $\Phi(f_k)$ converges to $\Phi(f)$ by showing that every subsequence $\Phi(f_{k_l})$ of $\Phi(f_{k_l})$ has a subsequence, $\Phi(f_{k_{l_m}})$ which converges to $\Phi(f)$. Set $f=\alpha+\mathrm{i}\beta$ and $f=\alpha_k+\mathrm{i}\beta_k$ with $\alpha,\beta,\alpha_k,\ \beta_k\in L^p(\mathbf{R},\mu)$ such that $\alpha_k\to\alpha$ and $\beta_k\to\beta$ in $L^p(\mathbf{R},\mu)$. Let $\{f_{k_l}\}$ be a subsequence of $\{f_k\}$, then $\{f_{k_l}\}$ converges to f in $L^p(\mathbf{C},\mu)$. Then there exists a subsequence $\{f_{k_{l_m}}\}$ of $\{f_{k_l}\}$ which converges to f in $L^p(\mathbf{C},\mu)$, where $f_{k_{l_m}}=\alpha_{k_{l_m}}+\mathrm{i}\beta_{k_{l_m}}$ with $\alpha_{k_{l_m}},\beta_{k_{l_m}}\in L^p(\mathbf{R},\mu)$ such that $\alpha_{k_{l_m}}\to\alpha$ and $\beta_{k_{l_m}}\to\beta$ in $L^p(\mathbf{R},\mu)$. Since h_1 is continuous, we have

$$(h_1\circ\alpha_{k_{l_m}})(x)\to(h_1\circ\alpha)(x),\ \text{a.e.}[\mu]$$

Since $|h_1(a)|\leqslant\gamma_1|a|^p+\delta_1$ for all $a\in\mathbf{R}$, we get

$$|(h_1\circ\alpha_{k_{l_m}})(x)|\leqslant\gamma_1|\alpha_{k_{l_m}}|^p+\delta_1,\ \text{a.e.}[\mu].$$

If $\mu(x) < \infty$, apply $\alpha_{k_{l_m}} \to \alpha$ in $L^p(\mathbf{R},\mu)$ to get

$$\lim_{m\to\infty}\int_X \gamma_1 \left|\alpha_{k_{l_m}}\right|^p + \delta_1 \mathrm{d}\mu = \int_X \gamma_1 |\alpha|^p \mathrm{d}\mu + \delta_1\mu(X)$$

And we take $\delta_1 = 0$ in the above equation if $\mu(x) = \infty$. By a modification of Lebesgue's Dominated Convergence Theorem (see Ref.[51], Proposition 2.2), we have $h_1 \circ (\alpha) \in L^1(\mathbf{R},\mu)$ and

$$\lim_{m\to\infty}\int_X (h_1 \circ \alpha_{k_{l_m}})\mathrm{d}\mu = \int_X (h_1 \circ \alpha)\mathrm{d}\mu \tag{7}$$

Similarly,

$$\lim_{m\to\infty}\int_X (h_2 \circ \alpha_{k_{l_m}})\mathrm{d}\mu = \int_X (h_2 \circ \alpha)\mathrm{d}\mu \tag{8}$$

$$\lim_{m\to\infty}\int_X (h_3 \circ \beta_{k_{l_m}})\mathrm{d}\mu = \int_X (h_3 \circ \beta)\mathrm{d}\mu \tag{9}$$

$$\lim_{m\to\infty}\int_X (h_4 \circ \beta_{k_{l_m}})\mathrm{d}\mu = \int_X (h_4 \circ \beta)\mathrm{d}\mu \tag{10}$$

Thus,

$$\begin{aligned}
&\left|\varPhi(f_{k_{l_m}}) - \varPhi(f)\right| \\
&= \Big|\int_X (h_1 \circ \alpha_{k_{l_m}} - h_1 \circ \alpha)\mathrm{d}\mu + \int_X (h_3 \circ \beta_{k_{l_m}} - h_3 \circ \beta)\mathrm{d}\mu \\
&+ \mathrm{i}\big(\int_X (h_2 \circ \alpha_{k_{l_m}} - h_2 \circ \alpha)\mathrm{d}\mu + \int_X (h_4 \circ \beta_{k_{l_m}} - h_4 \circ \beta)\mathrm{d}\mu\big)\Big| \\
&\leqslant \left|\int_X (h_1 \circ \alpha_{k_{l_m}} - h_1 \circ \alpha)\mathrm{d}\mu\right| + \left|\int_X (h_3 \circ \beta_{k_{l_m}} - h_3 \circ \beta)\mathrm{d}\mu\right| \\
&+ \left|\int_X (h_2 \circ \alpha_{k_{l_m}} - h_2 \circ \alpha)\mathrm{d}\mu\right| + \left|\int_X (h_4 \circ \beta_{k_{l_m}} - h_4 \circ \beta)\mathrm{d}\mu\right|
\end{aligned}$$

From (7)-(10), we conclude $\varPhi(f_{k_{l_m}}) \to \varPhi(f)$. Hence, $\varPhi$ is continuous.

Lemma 2 If the function $\varPhi : L^p(\mathbf{C},\mu) \to \mathbf{C}$ is a valuation, then

$$\varPhi(f) = \varPhi(\Re f) + \varPhi(\mathrm{i}\Im f)$$

for all $f \in L^p(\mathbf{C},\mu)$.

Proof If $\Re f, \Im f \geqslant 0$ or $\Re f, \Im f \leqslant 0$, then, by (2) and (3), we have

$$\varPhi(\Re f) + \varPhi(\mathrm{i}\Im f) = \varPhi(f) + \varPhi(0) \tag{11}$$

If $\Re f \geqslant 0, \Im f \leqslant 0$ or $\Re f \leqslant 0, \Im f \geqslant 0$, then, by (2) and (3), we have

$$\varPhi(f) + \varPhi(0) = \varPhi(\Re f) + \varPhi(\mathrm{i}\Im f) \tag{12}$$

Note that $\varPhi(0) = 0$, apply (11) and (12), then get

$$\varPhi(f) = \varPhi(\Re f) + \varPhi(\mathrm{i}\Im f)$$

for all $f \in L^p(\mathbf{C},\mu)$.

If we restrict f to $L^p(\mathbf{R},\mu)$, then it is obvious that $\varPhi$ is a valuation on $L^p(\mathbf{R},\mu)$. Also, we can construct another valuation on $L^p(\mathbf{R},\mu)$ which is related to $\varPhi$.

Lemma 3 Let $\varPhi : L^p(\mathbf{C},\mu) \to \mathbf{C}$ be a valuation. If the functions $\varPhi' : L^p(\mathbf{R},\mu) \to \mathbf{C}$ is defined by

$$\varPhi'(f) = \varPhi(\mathrm{i}f)$$

for all $f \in L^p(\mathbf{R},\mu)$, then $\varPhi'$ is a valuation on $L^p(\mathbf{R},\mu)$.

Proof For $f, g \in L^p(\mathbf{R},\mu)$, by (2) and (3), we have

$$\mathrm{i}(f \vee g) = \mathrm{i}f \vee \mathrm{i}g \text{ and } \mathrm{i}(f \wedge g) = \mathrm{i}f \wedge \mathrm{i}g \tag{13}$$

By (13) and the valuation property of $\varPhi$, it follows that

$$\begin{aligned}
&\varPhi'(f \vee g) + \varPhi'(f \wedge g) \\
&= \varPhi(\mathrm{i}f \vee \mathrm{i}g) + \varPhi(\mathrm{i}f \wedge \mathrm{i}g) \\
&= \varPhi(\mathrm{i}f) + \varPhi(\mathrm{i}g) = \varPhi'(f) + \varPhi'(g)
\end{aligned}$$

for all $f, g \in L^p(\mathbf{R},\mu)$. Thus, $\varPhi'$ is a valuation on $L^p(\mathbf{R},\mu)$.

Lemma 4 Let $\varPhi : L^p(\mathbf{R},\mu) \to \mathbf{C}$ be a valuation. If the functions $\varPhi_1, \varPhi_2 : L^p(\mathbf{R},\mu) \to \mathbf{R}$ are defined by

$$\varPhi(f) = \varPhi_1(f) + \mathrm{i}\varPhi_2(f)$$

for all $f \in L^p(\mathbf{R},\mu)$, then both $\varPhi_1, \varPhi_2$ are real-valued valuations on $L^p(\mathbf{R},\mu)$.

Proof Since $\varPhi$ is a valuation, we have

$$\begin{aligned}
&\varPhi(f \vee g) + \varPhi(f \wedge g) \\
&= \varPhi_1(f \vee g) + \mathrm{i}\varPhi_2(f \vee g) \\
&+ \varPhi_1(f \wedge g) + \mathrm{i}\varPhi_2(f \wedge g) \\
&= \varPhi(f) + \varPhi(g) \\
&= \varPhi_1(f) + \mathrm{i}\varPhi_2(f) + \varPhi_1(g) + \mathrm{i}\varPhi_2(g)
\end{aligned}$$

for all $f, g \in L^p(\mathbf{R},\mu)$. Thus,

$$\varPhi_1(f \vee g) + \varPhi_1(f \wedge g) = \varPhi_1(f) + \varPhi_1(g)$$

and

$$\varPhi_2(f \vee g) + \varPhi_2(f \wedge g) = \varPhi_2(f) + \varPhi_2(g)$$

for all $f, g \in L^p(\mathbf{R},\mu)$.

Therefore, both $\varPhi_1, \varPhi_2$ are real-valued valuations on $L^p(\mathbf{R},\mu)$.

In order to establish a representation theorem for continuous complex-valued valuations on $L^p(\mathbf{C},\mu)$, we will use the corresponding representation theorem for real case which was obtained by Tsang [51].

Theorem 6[51] Let $(X, \aleph, \mu)$ be a non-atomic measure space and let $\varPhi : L^p(\mathbf{R},\mu) \to \mathbf{R}$ be a continuous translation invariant valuation. If there exists a continuous function $h : \mathbf{R} \to \mathbf{R}$ with $h(0) = 0$ such that $\varPhi(b\chi_E) = h(b)\mu(E)$ for all $b \in \mathbf{R}$ and all $E \in \aleph$ with $\mu(E) < \infty$, then there exist constants $\gamma_1, \delta \geqslant 0$ such that $|h(a)| \leqslant \gamma|a|^p + \delta$ for all $a \in \mathbf{R}$, and

$$\varPhi(f) = \int_X \left(h \circ f\right)\mathrm{d}\mu$$

for all $f \in L^p(\mathbf{R},\mu)$. In addition, $\delta = 0$ if $\mu(X) = \infty$.

Proof of Theorem 1

Let $\Phi : L^p(\mathbf{C},\mu) \to \mathbf{C}$ be a continuous valuation. For $f \in L^p(\mathbf{C},\mu)$, by Lemma 2, Lemma 3 and Lemma 4, we have

$$\begin{aligned}\Phi(f) &= \Phi(\Re f) + \Phi(\mathrm{i}\Im f)\\ &= \Phi_1(\Re f) + \mathrm{i}\Phi_2(\Re f) + \Phi_1(\mathrm{i}\Im f) + \mathrm{i}\Phi_2(\mathrm{i}\Im f)\\ &= \Phi_1(\Re f) + \mathrm{i}\Phi_2(\Re f) + \Phi_1'(\Im f) + \mathrm{i}\Phi_2'(\Im f)\end{aligned}$$

where $\Phi_1'(\Im f) = \Phi_1(\mathrm{i}\Im f)$ and $\Phi_2'(\Im f) = \mathrm{i}\Phi_2(\mathrm{i}\Im f)$.

Since $\Re(f \vee g) = \Re f \vee \Re g$, $\Re(f \wedge g) = \Re f \wedge \Re g$, $\Im(f \vee g) = \Im f \vee \Im g$, and $\Im(f \wedge g) = \Im f \wedge \Im g$. Moreover, Lemma 3 and Lemma 4 imply that $\Phi_1, \Phi_2, \Phi_1', \Phi_2'$ are real-valued valuations on $L^p(\mathbf{R},\mu)$.

If we restrict to $f \in L^p(\mathbf{R},\mu)$, then the continuity of Φ implies that Φ_1, Φ_2, are continuous on $L^p(\mathbf{R},\mu)$.

If we consider $f \in L^p(\mathbf{C},\mu)$ with $\Re f = 0$, then the continuity of Φ implies that Φ_1', Φ_2' are continuous on $L^p(\mathbf{R},\mu)$. Thus, $\Phi_1, \Phi_2, \Phi_1', \Phi_2'$ are continuous real-valued valuations on $L^p(\mathbf{R},\mu)$. It follows from Theorem 6 that there exist continuous functions $h_k : \mathbf{R} \to \mathbf{R}$ with the properties that $h_k(0) = 0$ and there exist constants $\gamma_k, \delta_k \geqslant 0$ such that $|h_k(a)| \leqslant \gamma_k |a|^p + \delta_k$ for all $a \in \mathbf{R}$ (k = 1,2,3,4), and

$$\begin{aligned}\Phi(f) &= \int_X \left(h_1 \circ \Re f + h_3 \circ \Im f\right)\mathrm{d}\mu\\ &\quad + \mathrm{i}\int_X \left(h \circ \Re f + h_4 \circ \Im f\right)\mathrm{d}\mu\end{aligned}$$

for all $f \in L^p(\mathbf{C},\mu)$. In addition, $\delta_k = 0$ (k = 1,2,3,4) if $\mu(X) = \infty$.

If μ is Lebesgue measure, then $L^p(\mathbf{C},\mu)$ becomes the space of Lebesgue measurable complex-valued functions. We usually write as $L^p(\mathbf{C},\mathbf{R}^n)$.

Lemma 5 Let $h_k : \mathbf{R} \to \mathbf{R}$ be continuous functions with the property that there exists $\gamma_k \geqslant 0$ such that $|h_k(a)| \leqslant \gamma_k |a|^p$ for all $a \in \mathbf{R}$ (k = 1,⋯,4). If the function $\Phi : L^p(\mathbf{C},\mathbf{R}^n) \to \mathbf{C}$ is defined by

$$\begin{aligned}\Phi(f) &= \int_{\mathbf{R}^n} \left(h_1 \circ \Re f + h_3 \circ \Im f\right)\mathrm{d}x\\ &\quad + \mathrm{i}\int_X \left(h_2 \circ \Re f + h_4 \circ \Im f\right)\mathrm{d}x\end{aligned}$$

Then Φ is a continuous translation invariant valuation.

Proof Let M enote the collection of Lebesgue measurable sets in $\mathbf{R}^n$. Take $X = \mathbf{R}^n, \aleph = M$ and μ Lebesgue measure in Lemma 1 to conclude that Φ is a continuous valuation on $L^p(\mathbf{C},\mathbf{R}^n)$.

For every $t \in \mathbf{R}^n$ and every $f \in L^p(\mathbf{C},\mathbf{R}^n)$, we have

$$\begin{aligned}\Phi(f(x-t)) &= \int_{\mathbf{R}^n} \left(h_1 \circ \Re f + h_3 \circ \Im f\right)(x-t)\mathrm{d}x\\ &\quad + \mathrm{i}\int_X \left(h_2 \circ \Re f + h_4 \circ \Im f\right)(x-t)\mathrm{d}x\\ &= \Phi(f)\end{aligned}$$

which means that Φ is translation invariant.

Proof of Theorem 2

It follows from Lemma 5 that (4) determines a continuous translation invariant valuation on $L^p(\mathbf{C},\mathbf{R}^n)$. Conversely, let $\Phi : L^p(\mathbf{C},\mathbf{R}^n) \to \mathbf{C}$ be a continuous translation invariant valuation. Taking $X = \mathbf{R}^n$, $\aleph = M$ and μ Lebesgue measure in the proof of Theorem 1, we obtain

$$\Phi(f) = \Phi_1(\Re f) + \mathrm{i}\Phi_2(\Re f) + \Phi_1'(\Im f) + \mathrm{i}\Phi_2'(\Im f)$$

where $\Phi_1, \Phi_2, \Phi_1', \Phi_2'$ are real-valued valuations on $L^p(\mathbf{R},\mu)$. Theorem 4 implies that there exist continuous functions $h_k : \mathbf{R} \to \mathbf{R}$ with the property that there exist constants $\gamma_k \geqslant 0$ such that $|h_k(a)| \leqslant \gamma_k |a|^p$ for all $a \in \mathbf{R}$ (k = 1,2,3,4), and

$$\begin{aligned}\Phi(f) &= \int_{\mathbf{R}^n} \left(h_1 \circ \Re f + h_3 \circ \Im f\right)\mathrm{d}x\\ &\quad + \mathrm{i}\int_{\mathbf{R}^n} \left(h_2 \circ \Re f + h_4 \circ \Im f\right)\mathrm{d}x\end{aligned}$$

for all $f \in L^p(\mathbf{C},\mathbf{R}^n)$.

Let W denote the σ-algebra defined as

$$W = \left\{E : E \subseteq S^{n-1}, \{\lambda x : x \in E, 0 \leqslant \lambda \leqslant 1\} \in M\right\}$$

Also denote by σ the spherical Lebesgue measure. If μ is the spherical Lebesgue measure, then $L^p(\mathbf{C},\sigma)$ denotes the space of spherical Lebesgue measurable complex valued functions. We usually write as $L^p(\mathbf{C},S^{n-1})$.

Lemma 6 Let $h_k : \mathbf{R} \to \mathbf{R}$ be continuous functions with the properties that $h_k(0) = 0$ and there exists $\gamma_k, \delta_k \geqslant 0$ such that $|h_k(a)| \leqslant \gamma_k |a|^p + \delta_k$ for all $a \in \mathbf{R}$ (k=1,⋯,4). If the function $\Phi : L^p(\mathbf{C},S^{n-1}) \to \mathbf{C}$ is defined by

$$\begin{aligned}\Phi(f) &= \int_{S^{n-1}} \left(h_1 \circ \Re f + h_3 \circ \Im f\right)\mathrm{d}u\\ &\quad + \mathrm{i}\int_{S^{n-1}} \left(h_2 \circ \Re f + h_4 \circ \Im f\right)\mathrm{d}u\end{aligned}$$

then Φ is a continuous rotation invariant valuation.

Proof Take $X = S^{n-1}, \aleph = W$ and $\mu = \sigma$ in Lemma 1 to conclude that Φ is a continuous valuation on $L^p(\mathbf{C},S^{n-1})$.

Note that $\theta u \in S^{n-1}$ for every $\theta \in O(n)$ and every $u \in S^{n-1}$. Since the spherical Lebesgue measure is rotation invariant, we have

$$\begin{aligned}\Phi(f\circ\theta^{-1}) &= \int_{S^{n-1}}\left(h_1\circ\Re f+h_3\circ\Im f\right)(\theta u)\mathrm{d}u\\ &+\mathrm{i}\int_{S^{n-1}}\left(h_2\circ\Re f+h_4\circ\Im f\right)(\theta u)\mathrm{d}u\\ &=\Phi(f)\end{aligned}$$

for all $f\in L^p(\mathbf{C},S^{n-1})$, which completes the proof.

Proof of Theorem 3

It follows from Lemma 6 that (5) determines a continuous rotation invariant valuation on $L^p(\mathbf{C},S^{n-1})$.

Conversely, let $\Phi: L^p(\mathbf{C},S^{n-1})\to\mathbf{C}$ be a continuous rotation invariant valuation. Taking $X=S^{n-1}$, $\aleph=W$ and $\mu=\sigma$ in the proof of Theorem 1, we obtain

$$\Phi(f)=\Phi_1(\Re f)+\mathrm{i}\Phi_2(\Re f)+\Phi_1'(\Im f)+\mathrm{i}\Phi_2'(\Im f)$$

where $\Phi_1,\Phi_2,\Phi_1',\Phi_2'$ are real-valued valuations on $L^p(\mathbf{R},S^{n-1})$.

Theorem 5 implies that there exist continuous functions $h_k:\mathbf{R}\to\mathbf{R}$ with the properties that $h_k(0)=0$ and there exist constants $\gamma_k,\delta_k\geqslant 0$ such that $|h_k(a)|\leqslant\gamma_k|a|^p+\delta_k$ for all $a\in\mathbf{R}$ (k = 1,2,3,4), and

$$\begin{aligned}\Phi(f) &= \int_{S^{n-1}}\left(h_1\circ\Re f+h_3\circ\Im f\right)\mathrm{d}u\\ &+\mathrm{i}\int_{S^{n-1}}\left(h_2\circ\Re f+h_4\circ\Im f\right)\mathrm{d}u\end{aligned}$$

for all $f\in L^p(\mathbf{C},S^{n-1})$.

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