Complex-Valued Valuations on $L^p$ Spaces

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Abstract: All continuous translation invariant complex-valued valuations on Lebesgue measurable functions are completely classified. And all continuous rotation invariant complex-valued valuations on spherical Lebesgue measurable functions are also completely classified.

Key words: convex body; valuation; translation invariance; rotation invariance

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0 Introduction

A function $z$ defined on a lattice $(L, \vee, \wedge)$ and taking values in an Abelian semigroup is called a valuation if

$$z(f \vee g) + z(f \wedge g) = z(f) + z(g) \tag{1}$$

for all $f, g \in L$. A function $z$ defined on some subset $L_0$ of $L$ is called a valuation on $L_0$ if (1) holds whenever $f, g, f \vee g, f \wedge g \in L_0$. For $L_0$ the set of convex bodies, $K^*$, in $\mathbb{R}^n$ with $\vee$ denoting union and $\wedge$ intersection. Valuation on convex bodies is a classical concept. Probably the most famous result on valuations is Hadwiger’s classification theorem of continuous rigid motion invariant valuations\cite{1}. For the more recent contributions on valuations on convex bodies readers can refer to Refs. [2-33].

Valuations on convex bodies can be considered as valuations on suitable function spaces. Recently, valuations on functions have been rapidly growing (see Refs. [34-55]). For a space of real-valued functions, the operations $\vee$ and $\wedge$ are defined as pointwise maximum and minimum, respectively. A complete classification of valuations intertwining with the $\text{SL}(n)$ on Sobolev space $[46,48]$ and $L^p$ space $[46,50,52,54]$ were established, respectively. Valuations on convex functions $[34,36,39,42]$, quasi-concave functions $[37,38]$, Lipschitz functions $[43,44]$, and functions of Bounded variations $[45]$ were introduced and classified.

Recently, Wang and Liu $[55]$ showed that the Fourier transform is the only valuation which is a continuous, positive $\text{GL}(n)$ covariant and logarithmic translation covariant complex-valued valuation on integral functions. This motivates the study of complex-valued valuations on

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functions.

Let $L$ be a lattice of complex-valued functions. For $f \in L$, let $\Re f$ and $\Im f$ denote the real and imaginary parts of $f$, respectively. The pointwise maximum of $f$ and $g$, $f \vee g$, and the pointwise minimum of $f$ and $g$, $f \wedge g$, are defined by

\[ f \vee g = \Re f \vee \Re g + i(\Im f \vee \Im g) \]
and

\[ f \wedge g = \Re f \wedge \Re g + i(\Im f \wedge \Im g) \]

(2) and (3) coincide with the real cases. A function $\Phi : L \to \mathbb{C}$ is called a valuation if

\[ \Phi(f \vee g) + \Phi(f \wedge g) = \Phi(f) + \Phi(g) \]
for all $f, g \in L$ and $\Phi(0) = 0$ if $0 \in L$. It is called continuous if $\Phi(f) \to \Phi(f)$, as $f_i \to f$ in $L$.

It is called translation invariant if

\[ \Phi(f(-t)) = \Phi(f) \]
for every $t \in \mathbb{R}^n$. It is called rotation invariant if

\[ \Phi(f(\theta^{-1})) = \Phi(f) \]
for every $\theta \in O(n)$, where $\theta^{-1}$ denotes the inverse of $\theta$.

Let $p \geq 1$. If $(X, \mathcal{R}, \mu)$ is a measure space, then the $L^p$-space, $L^p(X, \mathcal{R}, \mu)$ is the collection of $\mu$-measurable complex-valued functions $f : X \to \mathbb{C}$ that satisfies

\[ \int |f|^p \, d\mu < \infty \]
A measure space $(X, \mathcal{R}, \mu)$ is called non-atomic if for every $E \in \mathcal{R}$ with $\mu(E) > 0$, there exists $F \in \mathcal{R}$ with $F \subseteq E$ and $0 < \mu(F) < \mu(E)$. Let $\chi_c$ denote the characteristic function of the measurable set $E$, i.e.

\[ \chi_c = \begin{cases} 1, & x \in E \\ 0, & x \notin E \end{cases} \]

Theorem 1 Let $(X, \mathcal{R}, \mu)$ be a non-atomic measure space and let $\Phi : L^p(X, \mathcal{R}, \mu) \to \mathbb{C}$ be a continuous valuation. If there exist continuous functions $h_k : \mathbb{R} \to \mathbb{R}$ with $h_k(0) = 0$ (k=1,2,3,4) such that $\Phi(c \chi_c) = (h_k(\Re c) + h_k(\Im c)) \mu(E) + i(h_k(\Re c) + h_k(\Im c)) \mu(E)$ for all $c \in \mathbb{C}$ and all $E \in \mathcal{R}$ with $\mu(E) < \infty$, then there exist constants $\gamma_k, \delta_k \geq 0$ such that $|h_k(a)| \leq \gamma_k |a|^\delta_k + \delta_k$ for $a \in \mathbb{R}$, and

\[ \Phi(f) = \int_X (h_k \circ \Re f + h_k \circ \Im f) \mu + i \int_X (h_k \circ \Re f + h_k \circ \Im f) \mu \]
for all $f \in L^p(X, \mathcal{R}, \mu)$. In addition, $\delta_k = 0$ (k=1,2,3,4) if $\mu(X) = \infty$.

Let $L^p(\mathbb{C}, \mathbb{R}^n)$ denote the $L^p$-space of Lebesgue measurable complex-valued functions on $\mathbb{R}^n$.

Theorem 2 A function $\Phi : L^p(\mathbb{C}, \mathbb{R}^n) \to \mathbb{C}$ is a continuous translation invariant valuation if and only if there exist continuous functions $h_k : \mathbb{R} \to \mathbb{R}$ with the property that there exist constants $\gamma_k, \delta_k \geq 0$ such that $|h_k(a)| \leq \gamma_k |a|^\delta_k + \delta_k$ for all $a \in \mathbb{R}$ (k=1,2,3,4), and

\[ \Phi(f) = \int_{\mathbb{R}^n} (h_k \circ \Re f + h_k \circ \Im f) \mu + i \int_{\mathbb{R}^n} (h_k \circ \Re f + h_k \circ \Im f) \mu \]
for all $f \in L^p(\mathbb{C}, \mathbb{R}^n)$.

Let $S^{n-1}$ be the unit sphere in $\mathbb{R}^n$ and let $L^p(\mathbb{C}, S^{n-1})$ denote the $L^p$-space of spherical Lebesgue measurable complex-valued functions on $S^{n-1}$.

Theorem 3 A function $\Phi : L^p(\mathbb{C}, S^{n-1}) \to \mathbb{C}$ is a continuous rotation invariant valuation if and only if there exist continuous functions $h_k : \mathbb{R} \to \mathbb{R}$ with the properties that $h_k(0) = 0$ and there exist constants $\gamma_k, \delta_k \geq 0$ such that $|h_k(a)| \leq \gamma_k |a|^\delta_k + \delta_k$ for all $a \in \mathbb{R}$ (k=1,2,3,4), and

\[ \Phi(f) = \int_{S^{n-1}} (h_k \circ \Re f + h_k \circ \Im f) \mu + i \int_{S^{n-1}} (h_k \circ \Re f + h_k \circ \Im f) \mu \]
for all $f \in L^p(\mathbb{C}, S^{n-1})$.

1 Notation and Preliminary Results

We collect some properties of complex-valued functions. If $f$ is a complex-valued function on $\mathbb{R}^n$, then

\[ f(x) = \Re f + i\Im f \]
where $\Re f$ and $\Im f$ denote the real and imaginary part of $f$, respectively. The absolute value of $f$ which is also called modulus is defined by

\[ |f| = \sqrt{\Re f^2 + \Im f^2} \]

Let $p \geq 1$. For a measure space $(X, \mathcal{R}, \mu)$, define $L^p(X, \mathcal{R}, \mu)$ as the space of $\mu$-measurable complex-valued functions $f : X \to \mathbb{C}$ that satisfies

\[ \|f\|_p = \left( \int_X |f|^p \, d\mu \right)^{\frac{1}{p}} \leq \infty \]
Let $f, g \in L^p(X, \mathcal{R}, \mu)$, then $f \vee g, f \wedge g \in L^p(X, \mathcal{R}, \mu)$. The functional $\| \cdot \| : L^p(X, \mathcal{R}, \mu) \to \mathbb{R}$ is a semi-norm. If functions in $L^p(X, \mathcal{R}, \mu)$ that are equal almost everywhere with respect to $\mu$ (a.e. $\mu$) are identified, then $\| \cdot \| : L^p(X, \mathcal{R}, \mu) \to \mathbb{R}$ becomes a norm. Obviously, $L^0(X, \mathcal{R}, \mu)$ is a lattice of complex-valued functions. Let
L'(R, μ) denote the subset of L'(C, μ), where the functions take real values. For f, g ∈ L'(C, μ), if \( |f - g|_p \to 0 \), then \( f_n \to f \) in L'(C, μ). Moreover, \( f_n \to f \in L'(C, μ) \iff R f_n \to R f, 3 f \to 3 f \in L'(R, μ) \).

The following characterizations of real-valued valuations on functions which were established by Tsang [51] will play key role in our proof. Let L'(R, R^n) and L'(R, S^{n-1}) denote the L' space of Lebesgue measurable real-valued functions on R^n and the L' space of spherical Lebesgue measurable real-valued functions on S^{n-1} respectively.

**Theorem 4** [51] A function \( \Phi : L'(R, R^n) \to R \) is a continuous translation invariant valuation if and only if there exists a continuous function \( h : R \to R \) with the property that there exists a constant \( \gamma \geq 0 \) such that \( |h(a)| \leq \gamma |a|^p + \delta \) for all \( a \in R \), and

\[
\Phi(f) = \int_{R^n} (h \circ f)(x)dx
\]

for all \( f \in L'(R, R^n) \).

**Theorem 5** [51] A function \( \Phi : L'(R, S^{n-1}) \to R \) is a continuous rotation invariant valuation if and only if there exists a continuous function \( h : R \to R \) with the properties that \( h(0) = 0 \) and there exist constants \( \gamma, \delta \geq 0 \) such that \( |h(a)| \leq \gamma |a|^p + \delta \) for all \( a \in R \), and

\[
\Phi(f) = \int_{S^{n-1}} (h \circ f)(u)du
\]

for all \( f \in L'(R, S^{n-1}) \).

### 2 Main Results

**Lemma 1** Let \( h_k : R \to R \) be continuous functions with the properties that \( h_k(0) = 0 \) and there exist \( \gamma_k, \delta_k \geq 0 \) such that \( |h_k(a)| \leq \gamma_k |a|^p + \delta_k \) for all \( a \in R \) \((k = 1, \ldots, 4)\). If the function \( \Phi : L'(C, \mu) \to C \) is defined by

\[
\Phi(f) = \int_X (h \circ R f + h \circ S f) d\mu
\]

then \( \Phi \) is a continuous valuation provided that \( \delta_k = 0 \) if \( \mu(X) = \infty \).

**Proof** For \( f, g \in L'(C, \mu) \), let

\[
E = \{ x \in X : R f \leq R g, 3 f \leq 3 g \},
\]

\[
F = \{ x \in X : R f \leq R g, 3 f > 3 g \},
\]

\[
G = \{ x \in X : R f > R g, 3 f \leq 3 g \},
\]

\[
H = \{ x \in X : R f > R g, 3 f > 3 g \}.
\]

By (6) and (2), we obtain

\[
\Phi(f \lor g) = \int_X (h \circ R (f \lor g) + h \circ 3 (f \lor g)) d\mu
\]

\[
+ i\int_X (h \circ R (f \lor g) + h \circ 3 (f \lor g)) d\mu
\]

\[
= \int_X (h \circ R g + h \circ 3 g) d\mu + i\int_X (h \circ R g + h \circ 3 g) d\mu
\]

\[
+ \int_X (h \circ R f + h \circ 3 f) d\mu + i\int_X (h \circ R f + h \circ 3 f) d\mu
\]

\[
+ \int_X (h \circ R f + h \circ 3 f) d\mu + i\int_X (h \circ R f + h \circ 3 f) d\mu
\]

\[
+ \int_X (h \circ R g + h \circ 3 g) d\mu + i\int_X (h \circ R g + h \circ 3 g) d\mu
\]

Similarly, by (6) and (3), we have

\[
\Phi(f \land g) = \Phi(f \lor g)
\]

Hence \( \Phi \) is a valuation.

It remains to show that \( \Phi \) is continuous. Let \( f \in L'(C, \mu) \) and let \( \{ f_k \} \) be a sequence in \( L'(C, \mu) \) with \( f_k \to f \) in \( L'(C, \mu) \). Next, we will show that \( \Phi(f_k) \) converges to \( \Phi(f) \) by showing that every subsequence \( \Phi(f_{k_n}) \) of \( \Phi(f_k) \) has a subsequence, \( \Phi(f_{k_n}) \) which converges to \( \Phi(f) \). Set \( f = \alpha + i\beta \) and \( f = \alpha + i\beta \) with \( \alpha, \beta, \alpha_n, \beta_n \in L'(R, \mu) \) such that \( \alpha_n \to \alpha \) and \( \beta_n \to \beta \) in \( L'(R, \mu) \). Let \( \{ f_{k_n} \} \) be a subsequence of \( \{ f_k \} \), then \( \{ f_{k_n} \} \) converges to \( f \) in \( L'(C, \mu) \). Then there exists a subsequence \( \{ f_{k_{n_m}} \} \) of \( \{ f_{k_n} \} \) which converges to \( f \) in \( L'(C, \mu) \), where \( f_{k_{n_m}} = \alpha_{k_{n_m}} + i\beta_{k_{n_m}} \) with \( \alpha_{k_{n_m}}, \beta_{k_{n_m}} \in L'(R, \mu) \) such that \( \alpha_{k_{n_m}} \to \alpha \) and \( \beta_{k_{n_m}} \to \beta \) in \( L'(R, \mu) \). Since \( h \) is continuous, we have

\[
(h \circ \alpha_{k_{n_m}})(x) \to (h \circ \alpha)(x), \text{ a.e. } [\mu]
\]

Since \( |h(a)| \leq \gamma |a|^p + \delta \) for all \( a \in R \), we get

\[
|h \circ \alpha_{k_{n_m}}(x)| \leq \gamma |\alpha_{k_{n_m}}| + \delta, \text{ a.e. } [\mu].
\]
If \( \mu(x) < \infty \), apply \( \alpha_{\delta_0} \rightarrow \alpha \) in \( L^p(\mathbb{R}, \mu) \) to get
\[
\lim_{x \to \infty} \int_\mathbb{R} \left| g_{\delta_0}(\alpha) \right| d\mu = \int_\mathbb{R} \left| g_{\alpha} \right|^p d\mu + \delta \mu(X)
\]
And we take \( \delta_0 = 0 \) in the above equation if \( \mu(x) = \infty \).

By a modification of Lebesgue’s Dominated Convergence Theorem (see Ref. [51], Proposition 2.2), we have
\[
\lim_{n \to \infty} \int_X (h \circ \alpha_{\delta_n}) d\mu = \int_X (h \circ \alpha) d\mu
\]
Similarly,
\[
\lim_{n \to \infty} \int_X (h \circ \beta_{\delta_n}) d\mu = \int_X (h \circ \beta) d\mu
\]
Thus,
\[
\left| \phi(f_{\delta_n}) - \phi(f) \right| = \left| \int_X (h \circ \alpha_{\delta_n} - h \circ \alpha) d\mu + \int_X (h \circ \beta_{\delta_n} - h \circ \beta) d\mu \right|
\]
for all \( f \in L^p(\mathbb{C}, \mu) \).

Proof For \( f, g \in L^p(\mathbb{R}, \mu) \), by (2) and (3), we have
\[
i(f \lor g) = i(f \lor g) \quad \text{and} \quad i(f \land g) = i(f \land g) \quad \text{(13)}
\]
By (13) and the valuation property of \( \Phi \), it follows that
\[
\Phi'(f \lor g) + \Phi'(f \land g) = \Phi(i(f \lor g)) + \Phi(i(f \land g))
\]
for all \( f, g \in L^p(\mathbb{R}, \mu) \). Thus, \( \Phi' \) is a valuation on \( L^p(\mathbb{R}, \mu) \).

Lemma 4 Let \( \Phi: L^p(\mathbb{R}, \mu) \to \mathbb{C} \) be a valuation. If the functions \( \Phi_1, \Phi_2: L^p(\mathbb{R}, \mu) \to \mathbb{R} \) are defined by
\[
\Phi(f) = \Phi_1(f) + i\Phi_2(f)
\]
for all \( f \in L^p(\mathbb{R}, \mu) \), then both \( \Phi_1, \Phi_2 \) are real-valued valuations on \( L^p(\mathbb{R}, \mu) \).

Proof Since \( \Phi \) is a valuation, we have
\[
\Phi(f \lor g) + \Phi(f \land g) = \Phi_1(f \lor g) + i\Phi_2(f \lor g) + \Phi_1(f \land g) + i\Phi_2(f \land g)
\]
for all \( f, g \in L^p(\mathbb{R}, \mu) \). Thus,
\[
\Phi_1(f \lor g) + i\Phi_2(f \lor g) = \Phi_1(f) + \Phi_2(g)
\]
for all \( f, g \in L^p(\mathbb{R}, \mu) \).

Therefore, both \( \Phi_1, \Phi_2 \) are real-valued valuations on \( L^p(\mathbb{R}, \mu) \).

In order to establish a representation theorem for continuous complex-valued valuations on \( L^p(\mathbb{C}, \mu) \), we will use the corresponding representation theorem for real case which was obtained by Tsang [51].

Theorem 6 [51] Let \( (X, \mathcal{F}, \mu) \) be a non-atomic measure space and let \( \Phi: L^p(\mathbb{R}, \mu) \to \mathbb{R} \) be a continuous translation invariant valuation. If there exists a continuous function \( h: \mathbb{R} \to \mathbb{R} \) with \( h(0) = 0 \) such that
\[
\Phi(h(X)) = h(b) \mu(E)
\]
for all \( b \in \mathbb{R} \) and all \( E \in \mathcal{F} \) with \( \mu(E) < \infty \), then there exist constants \( \gamma, \delta \geq 0 \) such that
\[
|h(a)| \leq \gamma |a|^\mu + \delta \quad \text{for all} \ a \in \mathbb{R}, \text{and}
\]
\[
\Phi(f) = \int_X (h \circ f) d\mu
\]
for all \( f \in L^p(\mathbb{R}, \mu) \). In addition, \( \delta = 0 \) if \( \mu(X) = \infty \).
Proof of Theorem 1

Let \( \Phi : L'(\mathbb{C}, \mu) \rightarrow \mathbb{C} \) be a continuous valuation. For \( f \in L'(\mathbb{C}, \mu) \), by Lemma 2, Lemma 3 and Lemma 4, we have

\[
\Phi(f) = \Phi(\mathbb{R}f) + \Phi(i\mathbb{R}f)
\]

\[
= \Phi(\mathbb{R}f) + i\Phi(\mathbb{R}f) + \Phi(i\mathbb{R}f) + i\Phi(i\mathbb{R}f)
\]

\[
= \Phi(\mathbb{R}f) + i\Phi(\mathbb{R}f) + \Phi(\mathbb{I}f) + i\Phi(\mathbb{I}f)
\]

where \( \Phi(\mathbb{R}f) = \Phi(i\mathbb{R}f) \) and \( \Phi(\mathbb{I}f) = i\Phi(i\mathbb{R}f) \).

Since \( \mathbb{R}(f \vee g) = \mathbb{R}f \vee \mathbb{R}g \), \( \mathbb{R}(f \wedge g) = \mathbb{R}f \wedge \mathbb{R}g \), \( \mathbb{I}(f \vee g) = \mathbb{I}f \vee \mathbb{I}g \), and \( \mathbb{I}(f \wedge g) = \mathbb{I}f \wedge \mathbb{I}g \). Moreover, Lemma 3 and Lemma 4 imply that \( \Phi, \Phi_1, \Phi'_1, \Phi_2, \Phi'_2 \) are real-valued valuations on \( L'(\mathbb{R}, \mu) \).

If we restrict to \( f \in L'(\mathbb{R}, \mu) \), then the continuity of \( \Phi \) implies that \( \Phi, \Phi_1, \Phi_2 \) are continuous on \( L'(\mathbb{R}, \mu) \).

If we consider \( f \in L'(\mathbb{C}, \mu) \) with \( \Phi(0) = 0 \), then the continuity of \( \Phi \) implies that \( \Phi_1, \Phi_2 \) are continuous on \( L'(\mathbb{R}, \mu) \). Thus, \( \Phi, \Phi_1, \Phi_2, \Phi_1', \Phi_2' \) are continuous real-valued valuations on \( L'(\mathbb{R}, \mu) \). It follows from Theorem 6 that there exist continuous functions \( h_k : \mathbb{R} \rightarrow \mathbb{R} \) with the properties that \( h_k(0) = 0 \) and there exist constants \( \gamma_k, \delta_k \geq 0 \) such that \( |h_k(a)| \leq \gamma_k |a|^\delta_k \) for all \( a \in \mathbb{R} \) (k = 1, 2, 3, 4), and

\[
\Phi(f) = \int_{\mathbb{R}} (h_1 \circ \mathbb{R}f + h_2 \circ \mathbb{R}f)d\mu
\]

\[
+ \int_{\mathbb{R}} (h_3 \circ \mathbb{R}f + h_4 \circ \mathbb{R}f)d\mu
\]

for all \( f \in L'(\mathbb{C}, \mu) \). In addition, \( \delta_k = 0 \) (k = 1, 2, 3, 4) if \( \mu(X) = \infty \).

If \( \mu \) is Lebesgue measure, then \( L'(\mathbb{C}, \mu) \) becomes the space of Lebesgue measurable complex-valued functions. We usually write as \( L'(\mathbb{R}^n) \).

Lemma 5 Let \( h_k : \mathbb{R} \rightarrow \mathbb{R} \) be continuous functions with the property that there exists \( \gamma_k, \delta_k \geq 0 \) such that \( |h_k(a)| \leq \gamma_k |a|^\delta_k \) for all \( a \in \mathbb{R} \) (k = 1, 2, 3, 4). If the function \( \Phi : L'(\mathbb{R}^n) \rightarrow \mathbb{C} \) is defined by

\[
\Phi(f) = \int_{\mathbb{R}^n} (h_1 \circ \mathbb{R}f + h_2 \circ \mathbb{R}f)d\mu
\]

\[
+ \int_{\mathbb{R}^n} (h_3 \circ \mathbb{R}f + h_4 \circ \mathbb{R}f)d\mu
\]

Then \( \Phi \) is a continuous translation invariant valuation.

Proof Let \( M \) denote the collection of Lebesgue measurable sets in \( \mathbb{R}^n \). Take \( X = \mathbb{R}^n, \mathbb{R} = M \) and \( \mu \) Lebesgue measure in Lemma 1 to conclude that \( \Phi \) is a continuous valuation on \( L'(\mathbb{C}, \mathbb{R}^n) \).

For every \( t \in \mathbb{R}^n \) and every \( f \in L'(\mathbb{C}, \mathbb{R}^n) \), we have

\[
\Phi(f(x-t)) = \int_{\mathbb{R}^n} (h_1 \circ \mathbb{R}f + h_2 \circ \mathbb{R}f)(x-t)d\mu
\]

\[
+ i\int_{\mathbb{R}^n} (h_3 \circ \mathbb{R}f + h_4 \circ \mathbb{R}f)(x-t)d\mu
\]

\[
= \Phi(f)
\]

which means that \( \Phi \) is translation invariant.

Proof of Theorem 2

It follows from Lemma 5 that \( (4) \) determines a continuous translation invariant valuation on \( L'(\mathbb{C}, \mathbb{R}^+ \mathbb{R}^+) \). Conversely, let \( \Phi : L'(\mathbb{C}, \mathbb{R}^+ \mathbb{R}^+) \rightarrow \mathbb{C} \) be a continuous translation invariant valuation. Taking \( X = \mathbb{R}^+, \mathbb{R} = M \) and \( \mu \) Lebesgue measure in the proof of Theorem 1, we obtain

\[
\Phi(f) = \Phi(\mathbb{R}f) + i\Phi(\mathbb{R}f) + \Phi(\mathbb{I}f) + i\Phi(\mathbb{I}f)
\]

where \( \Phi, \Phi_1, \Phi_2, \Phi_3, \Phi_4 \) are real-valued valuations on \( L'(\mathbb{R}, \mu) \). Theorem 4 implies that there exist continuous functions \( h_k : \mathbb{R} \rightarrow \mathbb{R} \) with the property that there exist constants \( \gamma_k, \delta_k \geq 0 \) such that \( |h_k(a)| \leq \gamma_k |a|^\delta_k \) for all \( a \in \mathbb{R} \) (k = 1, 2, 3, 4), and

\[
\Phi(f) = \int_{\mathbb{R}} (h_1 \circ \mathbb{R}f + h_2 \circ \mathbb{R}f)d\mu
\]

\[
+ i\int_{\mathbb{R}} (h_3 \circ \mathbb{R}f + h_4 \circ \mathbb{R}f)d\mu
\]

for all \( f \in L'(\mathbb{C}, \mathbb{R}^+) \).

Let \( W \) denote the \( \sigma \)-algebra defined as

\( W = \{ E : E \subseteq S^{n-1}, \{M \in M \mid \lambda \leq \lambda \leq M \} \} \)

Also denote by \( \sigma \) the spherical Lebesgue measure. If \( \mu \) is the spherical Lebesgue measure, then \( L'(\mathbb{C}, \sigma) \) denotes the space of spherical Lebesgue measurable complex valued functions. We usually write this as \( L'(\mathbb{C}, S^{n-1}) \).

Lemma 6 Let \( h_k : \mathbb{R} \rightarrow \mathbb{R} \) be continuous functions with the properties that \( h_k(0) = 0 \) and there exist constants \( \gamma_k, \delta_k \geq 0 \) such that \( |h_k(a)| \leq \gamma_k |a|^\delta_k \) for all \( a \in \mathbb{R} \) (k = 1, 2, 3, 4). If the function \( \Phi : L'(\mathbb{C}, S^{n-1}) \rightarrow \mathbb{C} \) is defined by

\[
\Phi(f) = \int_{S^{n-1}} (h_1 \circ \mathbb{R}f + h_2 \circ \mathbb{R}f)d\mu
\]

\[
+ i\int_{S^{n-1}} (h_3 \circ \mathbb{R}f + h_4 \circ \mathbb{R}f)d\mu
\]

then \( \Phi \) is a continuous rotation invariant valuation.

Proof Take \( X = S^{n-1}, \mathbb{R} = W \) and \( \mu = \sigma \) in Lemma 1 to conclude that \( \Phi \) is a continuous valuation on \( L'(\mathbb{C}, S^{n-1}) \).

Note that \( \theta u \in S^{n-1} \) for every \( \theta \in O(n) \) and every \( u \in S^{n-1} \). Since the spherical Lebesgue measure is
rotation invariant, we have
\[
\Phi(f \circ \theta^{-1}) = \int_{S^{n-1}} \left( h_2 \circ Rf + h_4 \circ \mathbb{S}f \right)(\theta u) du
\]
\[
+ i \int_{S^{n-1}} \left( h_2 \circ Rf + h_4 \circ \mathbb{S}f \right)(\theta u) du
\]
\[
= \Phi(f)
\]
for all \( f \in L^r(C, S^{n-1}) \), which completes the proof.

**Proof of Theorem 3**

It follows from Lemma 6 that (5) determines a continuous rotation invariant valuation on \( L^r(C, S^{n-1}) \).

Conversely, let \( \Phi : L^r(C, S^{n-1}) \to \mathbb{C} \) be a continuous rotation invariant valuation. Taking \( X = S^{n-1}, \mathcal{R} = W \) and \( \mu = \sigma \) in the proof of Theorem 1, we obtain
\[
\Phi(f) = \Phi_1(Rf) + i \Phi_2(Rf) + \Phi(\mathbb{S}f) + i \Phi_4(\mathbb{S}f)
\]
where \( \Phi_1, \Phi_2, \Phi_3, \Phi_4 \) are real-valued valuations on \( L^r(R, S^{n-1}) \).

Theorem 5 implies that there exist continuous functions \( h_i : R \to R \) with the properties that \( h_i(0) = 0 \) and there exist constants \( \gamma_i, \delta_i \geq 0 \) such that
\[
|h_i(a)| \leq \gamma_i |a|^\alpha + \delta_i \quad \text{for all } a \in R \quad (k = 1, 2, 3, 4),
\]
\[
\Phi(f) = \int_{S^{n-1}} \left( h_2 \circ Rf + h_4 \circ \mathbb{S}f \right) du
\]
\[
+ i \int_{S^{n-1}} \left( h_2 \circ Rf + h_4 \circ \mathbb{S}f \right) du
\]
for all \( f \in L^r(C, S^{n-1}) \).

**References**


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