Inequalities on Complex $L_p$ Centroid Bodies

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Abstract: Based on the notion of the complex $L_p$ centroid body, we establish Brunn-Minkowski type inequalities and monotonicity inequalities for complex $L_p$ centroid bodies in this article. Moreover, we obtain the affirmative form of Shephard type problem for the complex $L_p$ centroid bodies and its negative form.

Key words: complex $L_p$ centroid body; Brunn-Minkowski type inequalities; Shephard type problem

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0 Introduction

Let $\mathcal{K}^n$ denote the set of convex bodies (compact, convex sets with non-empty interiors) in Euclidean space $\mathbb{R}^n$. For the set of convex bodies containing the origin in their interiors, the set of origin-symmetric convex bodies, we write $\mathcal{K}_o^n$ and $\mathcal{K}^{n*}_o$, respectively. Let $V(K)$ denote the volume of $K$ and $S^{n-1}$ the unit sphere.

Centroid bodies are a classical notion from geometry which have attracted increasing attention in recent years $^{[1-11]}$. In 1997, Lutwak and Zhang $^{[12]}$ introduced the concept of $L_p$ centroid body as follows: For each compact star-shaped about the origin $K \in \mathbb{R}^n$ and $p > 1$, the $L_p$ centroid body, $\Gamma_p K$, of $K$ is the origin-symmetric convex body whose support function is defined by

$$h(\Gamma_p K, u) = \frac{1}{(n + p)c_{n,p}V(K)} \int_{S^{n-1}} |u \cdot v|^p \rho(K, v)^{n+p} dS(v)$$

for any $u \in S^{n-1}$. Refs. $^{[13-19]}$ had conducted a series of studies on the $L_p$ centroid body, and many scholars were attracted. The $L_p$ centroid body has got many results from these articles. Particularly, Refs. $^{[20, 21]}$ gave the Brunn-Minkowski inequality and monotonicity inequality for the $L_p$ centroid body. Grinberg and Zhang $^{[22]}$ gave the Shephard problems for the $L_p$ centroid body.

However, complex convex geometry has been studied in many works $^{[23-28]}$. In this paper, we mainly study the complex centroid body. First, we introduce some notations in complex vector space $\mathbb{C}^n$. Let $\mathcal{C}(\mathbb{C}^n)$ denote
the set of compact convex subsets of complex vector space $\mathbb{C}^n$. Let $\mathcal{K}(\mathbb{C}^n)$, $\mathcal{K}_s(\mathbb{C}^n)$ and $\mathcal{K}_{os}(\mathbb{C}^n)$ denote the set of complex convex bodies, the set of complex convex bodies containing the origin in their interiors, and the set of origin symmetric complex convex bodies, respectively. Let $\mathcal{S}(\mathbb{C}^n)$, $\mathcal{S}_s(\mathbb{C}^n)$ and $\mathcal{S}_{os}(\mathbb{C}^n)$ denote the set of complex star bodies, the set of complex star bodies containing the origin in their interiors, and the set of origin symmetric complex star bodies, respectively.

Harberl\cite{29} firstly proposed the complex centroid body of $K$ and established the Busemann-Petty centroid inequality. In 2021, Wu\cite{30} introduced the concept of the $L_p$ complex centroid body $\Gamma_{p,c}K$ as follows: If $p > 1$, $K \in \mathcal{K}_s(\mathbb{C}^n)$ and $C \in \mathcal{K}(\mathbb{C})$, the complex $L_p$ centroid body $\Gamma_{p,c}K$ is the convex body with support function

$$h(\Gamma_{p,c}K,u)^p = \frac{1}{(2n + p) V(K)} \int_{S^{2n-1}} h(Cu,v)^p \rho(K,v)^{2n+p} dS(v)$$

(2)

where the integration is with respect to the push forward of the Lebesgue measure under the canonical isomorphism $\eta$ and for $\eta$, it is the canonical isomorphism between $\mathbb{C}^n$ and $\mathbb{R}^{2n}$, i.e.,

$$\eta(c) = (\Re[c_1], \ldots, \Re[c_n], \Im[c_1], \ldots, \Im[c_n]), c \in \mathbb{C}^n$$

where $\Re, \Im$ are the real part and imaginary part, respectively. It is obvious to get that if $p \geq 1, K \in \mathcal{S}_s(\mathbb{C}^n)$, then

$$\Gamma_{p,c}(\lambda K) = \lambda \Gamma_{p,c}K$$

(3)

In this article, associated with the definition of complex $L_p$ centroid body, we continuously study the complex $L_p$ centroid body. Let $\Gamma_{p,c}K$ denote the polar of $\Gamma_{p,c}K$ and $\Gamma_{p,c}^*K$ denote the polar for complex conjugate of $\Gamma_{p,c}K$. First, we establish the Brunn-Minkowski type inequalities for complex $L_p$ centroid bodies.

**Theorem 1** If $p \geq 1, K,L \in \mathcal{S}_{os}(\mathbb{C}^n)$ and $C \in \mathcal{K}(\mathbb{C})$, then

$$V\left(\Gamma_{p,c} \left( K \oplus_p L \right) \right)^{\frac{p}{2n}} \geq V\left( \Gamma_{p,c}K \right)^{\frac{p}{2n}} + V\left( \Gamma_{p,c}L \right)^{\frac{p}{2n}}$$

with equality if and only if $K$ and $L$ are real dilation.

**Theorem 2** If $p \geq 1, K,L \in \mathcal{S}_{os}(\mathbb{C}^n)$ and $C \in \mathcal{K}(\mathbb{C})$, then

$$V\left( \Gamma_{p,c} \left( K \oplus_p L \right) \right)^{\frac{p}{2n}} \geq V\left( \Gamma_{p,c}K \right)^{\frac{p}{2n}} + V\left( \Gamma_{p,c}L \right)^{\frac{p}{2n}}$$

with equality if and only if $K$ and $L$ are real dilation.

**Theorem 3** For $p \geq 1, K,L \in \mathcal{S}_s(\mathbb{C}^n)$, $C \in \mathcal{K}(\mathbb{C})$, if $\tilde{V}_{-p}(K,Q) \leq \tilde{V}_{-p}(L,Q)$ for any $Q \in \mathcal{S}_s(\mathbb{C}^n)$, then

$$\frac{V(\Gamma_{p,c}K)^\frac{p}{2n}}{V(K)^\frac{p}{2n}} \leq \frac{V(\Gamma_{p,c}L)^\frac{p}{2n}}{V(L)^\frac{p}{2n}}$$

(6)

with equality if and only if $K=\lambda L$.

**Theorem 4** For $p \geq 1, K,L \in \mathcal{S}_s(\mathbb{C}^n)$, $C \in \mathcal{K}(\mathbb{C})$, if $\tilde{V}_{-p}(K,Q) \leq \tilde{V}_{-p}(L,Q)$, for any $Q \in \mathcal{S}_s(\mathbb{C}^n)$, then

$$\frac{V(\Gamma_{p,c}K)^\frac{p}{2n}}{V(K)^\frac{p}{2n}} \geq \frac{V(\Gamma_{p,c}L)^\frac{p}{2n}}{V(L)^\frac{p}{2n}}$$

(7)

with equality if and only if $K=\lambda L$.

Finally, we study the $L_p$ Shephard type problem of complex $L_p$ centroid bodies and give the negative form.

**Theorem 5** Let $\mathcal{Z}_{p,c}$ denote the set of polar for complex conjugate of $\Gamma_{p,c}K$. For $K \in \mathcal{S}_s(\mathbb{C}^n)$, $L \in \mathcal{Z}_{p,c}$, $p \geq 1$, if $\Gamma_{p,c}K \subset \Gamma_{p,c}L$, then $V(K) \leq V(L)$ with equality if and only if $K=\lambda L$.

**Theorem 6** For $p \geq 1, L \in \mathcal{S}_s(\mathbb{C}^n)$, if $L$ is not origin symmetric star body, then there exists $K \in \mathcal{S}_{os}(\mathbb{C}^n)$ such that $\Gamma_{p,c}K \subset \Gamma_{p,c}L$, but $V(K) > V(L)$.

Throughout this paper, we assume that $\dim C > 0$.

### 1 Preliminaries

In this section, we collect complex reformulations of well-known results from convex geometry. These complex versions can be directly deduced from their real counterparts by an appropriate application of $\eta$. For standard reference, the readers may consult the books of Gardner\cite{31} and Schneider\cite{32}.

#### 1.1 Complex Support Functions and Radial Functions

For a complex number $c \in \mathbb{C}^n$, we write $\tau$ for its conjugate and $|c|$ for its norm. If $\phi \in \mathcal{C}^{\alpha,\alpha}$, then $\phi^*$ denotes the conjugate transpose of $\phi$ and if $\phi$ is in-
A complex convex body \( K \in \mathcal{K}(\mathbb{C}^n) \) is uniquely determined by its support function \( h(K,x) : \mathbb{C}^n \to \mathbb{R} \),

\[
h(K,x) = \max \{ \Re[x \cdot y] : y \in K \}
\]

where \( \Re \) means the standard Hermitian inner product in \( \mathbb{C}^n \) and \( \Re[x \cdot y] \) is the real part of \( x \cdot y \). It is easy to see that \( h_{\lambda K} = \lambda h_K \) for all \( \lambda > 0 \) and \( h_{\varphi K} = \varphi^* \circ h_K \) for all \( \varphi \in \text{GL}(n, \mathbb{C}) \). The complex radial function \( \rho_K(x) = h(K,x) : \mathbb{C}^n \setminus \{0\} \to [0, \infty) \) of a compact star-shaped (about the origin) \( K \) is defined, for \( x \in \mathbb{C}^n \setminus \{0\} \), by

\[
\rho_K(x) = \max \{ \lambda \geq 0 : \lambda x \in K \}.
\]

It is easy to see that \( \rho_{\lambda K} = \lambda \rho_K \) for all \( \lambda > 0 \) and \( \rho_{\varphi K} = \varphi^* \circ \rho_K \) for all \( \varphi \in \text{GL}(n, \mathbb{C}) \). If \( \rho_K \) is positive and continuous, \( K \) will be called a star body. Moreover, if \( K \in \mathcal{K}_0(\mathbb{C}^n) \) , it is easy to certify that

\[
h_K = \frac{1}{\rho_K}, \quad \rho_K = \frac{1}{h_K}
\]

An application of polar coordinates to the volume of a complex star body \( K \in \mathcal{S}_0(\mathbb{C}^n) \) gives that

\[
V(K) = \frac{1}{2n} \int_{S^{2n-1}} \rho(K,u)^{2n} dS(K,u)
\]

### 1.2 Complex \( L_p \) Mixed Volume and Dual \( L_p \) Mixed Volume

For \( p \geq 1 \), \( K, L \in \mathcal{K}_0(\mathbb{C}^n) \) and \( \alpha, \beta \geq 0 \) (not both zero), the complex \( L_p \) Minkowski combination \( \alpha \cdot K +_p \beta \cdot L \) is defined by

\[
h(\alpha \cdot K +_p \beta \cdot L, u)^p = \alpha h(K,u)^p + \beta h(L,u)^p.
\]

The complex \( L_p \) mixed volume, \( V_p(K,L) \) of \( K, L \in \mathcal{K}_0(\mathbb{C}^n) \) is defined by (see Ref.[33])

\[
\frac{2n}{p} V_p(K,L) = \lim_{\epsilon \to 0^+} \frac{V(\epsilon \cdot K +_p \epsilon \cdot L) - V(K)}{\epsilon}
\]

By (9) we have \( V_p(K,L) = V_p(\eta K, \eta L) \) for \( \eta \in \text{GL}(n, \mathbb{C}) \),

\[
V_p(\eta K, \eta L) = |\eta|^2 V_p(K,L)
\]

For every Borel set \( \sigma \subset S^{2n-1} \), the complex surface area measure \( S_K \) of \( K \in \mathcal{K}(\mathbb{C}^n) \) is defined by

\[
S_K(\sigma) = h^{-1} \{ x \in K, \exists u \in \sigma, \Re[x \cdot y] = h_K(u) \}
\]

where \( h^{-1} \) stands for \((2n-1)\)-dimensional Hausdorff measure on \( \mathbb{R}^{2n} \).

In addition, the complex surface area measures are translation invariant and \( S_K(\sigma) = S_K(\tau \sigma) \) for all \( \tau \in S^{2n-1} \) and each Borel set \( \sigma \subset S^{2n-1} \). If \( p \geq 1 \), we define the complex \( L_p \) surface area measure \( S_p(K,\cdot) \) of \( K \in \mathcal{K}(\mathbb{C}^n) \) as

\[
S_p(K,\sigma) = \int h(K,v)^{-p} dS(K,v).
\]

For \( K, L \in \mathcal{K}_0(\mathbb{C}^n) \), there is the \( L_p \) surface area measure \( S_{p,K} \) of \( K \) on \( S^{2n-1} \) such that

\[
V_p(K,L) = \frac{1}{2n} \int_{S^{2n-1}} h(L,u)^p dS_p(K,u)
\]

It turns out that the measure \( S_{p,K} \) is absolutely continuous with respect to \( S_K \) and has Radon Nikodym derivative \( dS_{p,K}/dS_K = h_{h_K}^p \). There is the complex \( L_p \) Minkowski inequality for complex convex body: If \( p \geq 1, K, L \in \mathcal{K}_0(\mathbb{C}^n) \), then

\[
V_p(K,L)^{2n} \geq V(K)^{2n-p} V(L)^p
\]

with equality if and only if \( K \) and \( L \) are real dilation. The real \( L_p \) Minkowski inequality and its proof are shown in Ref.[32].

For \( p \geq 1, K, L \in \mathcal{S}_0(\mathbb{C}^n) \) and \( \alpha, \beta \geq 0 \) (not both zero), the complex \( L_p \) harmonic radial combination \( \alpha \cdot K +_p \beta \cdot L \) is defined by

\[
\rho(\alpha \cdot K +_p \beta \cdot L, u)^p = \alpha \rho(K,u)^p + \beta \rho(L,u)^p
\]

Then the dual complex \( L_p \) mixed volume \( \tilde{V}_{-p}(K,L) \) is defined by (see Ref.[33])

\[
\tilde{V}_{-p}(K,L) = -\frac{p}{2n} \lim_{\epsilon \to 0^+} \frac{\tilde{V}(\epsilon \cdot K +_p \epsilon \cdot L) - \tilde{V}(K)}{\epsilon}
\]

The polar coordinate formula for volume yields

\[
\tilde{V}_{-p}(K,L) = \frac{1}{2n} \int_{S^{2n-1}} \rho(K,u)^{2n+p} \rho(L,u)^{-p} dS(u)
\]

Particularly, \( \tilde{V}_{-p}(K,K) = V(K) \).

The integral representation (12), together with the Hölder inequality[34] immediately gives that

\[
\tilde{V}_{-p}(K,L)^{2n} \geq V(K)^{2n+p} V(L)^{-p}
\]

with equality if and only if \( K \) and \( L \) are real dilation. For the real \( L_p \) harmonic radial combination and real \( L_p \) dual Minkowski inequality, we refer to Ref.[35].
1.3 The Complex $L_p$ Harmonic Blaschke Combination

The notion of real $L_p$ harmonic Blaschke combination was given by Lu and Leng [36]. Then, we extend real $L_p$ harmonic Blaschke combination to the complex case.

For $p \geq 1, K, L \in \mathcal{S}_0(C^n)$ and $\lambda, \mu \geq 0$ (not both zero), the complex $L_p$ harmonic Blaschke combination $\lambda * K +_p \mu * L$ of $K$ and $L$ is defined by

$$\frac{\rho(\lambda * K +_p \mu * L)}{V(\lambda * K +_p \mu * L)} = \lambda \frac{\rho(K)}{V(K)} + \mu \frac{\rho(L)}{V(L)}$$

(14)

where $\lambda * K$ is $L_p$ harmonic Blaschke scalar multiplication and $\lambda + K = \lambda^\frac{1}{p} K$. Taking $\frac{\lambda}{\mu} = \frac{1}{2}, K = L$ in $\lambda * K +_p \mu * L$, then the complex $L_p$ harmonic Blaschke body $\hat{V}_{pL}K$ is introduced by

$$\hat{V}_{pL}K = \frac{1}{2} * K +_p \frac{1}{2} * (-K)$$

(15)

Obviously, $\hat{V}_{pL}K$ is origin symmetric.

2 Proofs of Theorems

In this section, we will prove Theorem 1-Theorem 6.

Proof of Theorem 1 For $p \geq 1$ and $C \in \mathcal{K}(C)$, the $L_p$ harmonic Blaschke combination (14) together with (2) yields

$$h(\Gamma_{pL}(\lambda * K +_p \mu * L), u)^p = \lambda h(\Gamma_{pL}(K), u)^p + \mu h(\Gamma_{pL}(L), u)^p$$

(16)

From (10) and for any $Q \in \mathcal{S}_0(C^n)$, we obtain

$$V_p(\hat{V}_{pL}K) = \frac{1}{2n} \int_{S^{n-1}} h(\Gamma_{pL}(K), u)^p \, ds_p(u)$$

$$= \frac{1}{2n} \int_{S^{n-1}} (h(\Gamma_{pL}(K), u)^p + h(\Gamma_{pL}(L), u)^p) \, ds_p(u)$$

$$= V_p(\hat{V}_{pL}K) + V_p(\hat{V}_{pL}L)$$

Therefore, by (11), we get

$$V_p(\hat{V}_{pL}K) = \frac{1}{(2n + p)V(L)} \hat{V}_{pL}(L, \Pi_{pL}K)$$

(20)

with equality if and only if $Q, \Gamma_{pL}K$ and $\Gamma_{pL}L$ are real dilation. Taking $Q = \Gamma_{pL}(K +_p L)$ in (17), one has

$$V(\Gamma_{pL}(K +_p L))^{\frac{2n}{p}} \geq \frac{p}{2n} V(\Gamma_{pL}K)^{\frac{2n}{p}} + V(\Gamma_{pL}L)^{\frac{2n}{p}}$$

Together (16) with the equality condition of (17), we know that the equality holds if and only if $K$ and $L$ are real dilation.

Proof of Theorem 2 From (8) and (16), one has

$$\rho(\Gamma_{pL}(\lambda * K +_p \mu * L), u)^p = \lambda \rho(\Gamma_{pL}(K), u)^p + \mu \rho(\Gamma_{pL}(L), u)^p$$

Then by (12) and the inverse Minkowski’s integral inequality [34], we obtain

$$V_p(\hat{V}_{pL}K)^{\frac{2n}{p}} = \left[\frac{1}{2n} \int_{S^{n-1}} \left(\rho(\Gamma_{pL}(K), u)^p + \rho(\Gamma_{pL}(L), u)^p\right) \, ds_p(u)ight]^{\frac{p}{2n}}$$

$$\geq \frac{1}{2n} \int_{S^{n-1}} \rho(\Gamma_{pL}(K), u)^p \, ds_p(u)$$

Taking $Q = \Gamma_{pL}(K +_p L)$ in (19) and by (13), one yields the inequality (6). According to the equality conditions of Minkowski’s integral inequalities, we see that equality holds in (19) if and only if $K$ and $L$ are real dilation.

Next, we turn to prove Theorem 3 and Theorem 4. Lemma 1 provides a connection of $\Gamma_{pL}K$ and $\Pi_{pL}K$ in terms of mixed volumes and their dual.

Lemma 1 If $C \in \mathcal{K}(C)$ and $K, L \in \mathcal{S}_0(C^n)$, then,

$$V_p(K, \Gamma_{pL}L) = \frac{1}{(2n + p)V(L)} V_{pL}(L, \Pi_{pL}K)$$

(20)
Proof From (3), (8), (10) and definition of $L_p$ projection body\cite{29}, we have
\[
V_p(K, \Gamma_{p,c}L) = \frac{1}{2n} \int_{S^{n-1}} h(\Gamma_{p,c}L, u)^p dS_p(K, u)
= \frac{1}{2n(2n+p)V(L)} \int_{S^{2n-1}} \rho(L, u)^{2n+p} \times h(Cu, v)^p dS(v)dS_p(K, u)
= \frac{1}{2n(2n+p)V(L)} \int_{S^{2n-1}} \rho(L, u)^{2n+p} \times h(\Pi_{p,c}K, u)^p dS(v)
= \frac{1}{(2n+p)V(L)} \bar{V}_{p}(L, \Pi_{p,c}K)
\]
which ends the proof of Lemma 1.

Lemma 2 If $p \geq 1, K, L \in S_o(C^n)$, then
\[
\frac{\bar{V}_{p}(K, \Gamma_{p,c}K)}{v(K)} = \frac{\bar{V}_{p}(L, \Gamma_{p,c}L)}{v(L)}
\tag{21}
\]
Proof From (3), (8) and (12), it easily gets
\[
\bar{V}_{p}(L, \Gamma_{p,c}K) = \frac{1}{2n} \int_{S^{2n-1}} \rho(L, u)^{2n+p} \rho(\Gamma_{p,c}K, u)^p dS(u)
= \frac{1}{2n} \int_{S^{2n-1}} \rho(L, u)^{2n+p} h(\Gamma_{p,c}K, u)^p dS(u)
= \frac{1}{2n(2n+p)V(L)} \int_{S^{2n-1}} \rho(L, u)^{2n+p} \times \rho(\Gamma_{p,c}K, u)^p h(Cu, v)^p dS(v)dS(u)
= \frac{V(L)}{2nV(K)} \int_{S^{2n-1}} \rho(K, v)^{2n+p} \rho(\Gamma_{p,c}K, u)^p dS(v)
= \frac{V(L)}{V(K)} \bar{V}_{p}(L, \Pi_{p,c}K)
\]
That is to say,
\[
\frac{\bar{V}_{p}(L, \Gamma_{p,c}K)}{V(L)} = \frac{\bar{V}_{p}(K, \Gamma_{p,c}K)}{v(K)}
\]
which yields (21).

Remark 1 If $p \geq 1, K \in S_o(C^n)$, then $\bar{V}_{p}(K, \Gamma_{p,c}K) \geq v(K)$. If $K$ is a central ellipsoid or an Hermitian ellipsoid, then the equality holds.

Now we are in a position to prove Theorem 3 and Theorem 4.

Proof of Theorem 3 Since $K, L \in S_o(C^n)$ and $\bar{V}_{p}(K, Q) \leq \bar{V}_{p}(L, Q)$ for any $Q \in S_o(C^n)$, then taking $Q=\Gamma_{p,c}M$ for any $M \in S_o(C^n)$, we have
\[
\bar{V}_{p}(K, \Gamma_{p,c}M) \leq \bar{V}_{p}(L, \Gamma_{p,c}M)
\tag{22}
\]
with equality if and only if $K = L$. By Lemma 1, we obtain
\[
V(K)\bar{V}_{p}(M, \Gamma_{p,c}K) \leq V(L)\bar{V}_{p}(M, \Gamma_{p,c}L)
\tag{23}
\]
Taking $M = \Gamma_{p,c}L$ in (23) and by (11), one has
\[
V(L)\bar{V}_{p}(\Gamma_{p,c}L) \geq V(K)\bar{V}_{p}(\Gamma_{p,c}K)
\tag{24}
\]
i.e.,
\[
\frac{V(\Gamma_{p,c}K)}{V(K)} \leq \frac{V(\Gamma_{p,c}L)}{V(L)}
\tag{25}
\]

From Lemma 1, we see that inequalities (22) and (23) are equivalent. Thus, equality holds in (25) if and only if $K = L$.

Proof of Theorem 4 Since $\bar{V}_{p}(K, Q) \leq \bar{V}_{p}(L, Q)$ for any $Q \in S_o(C^n)$, then, taking $Q=\Gamma_{p,c}M$ for any $M \in S_o(C^n)$, we get
\[
\bar{V}_{p}(K, \Gamma_{p,c}M) \leq \bar{V}_{p}(L, \Gamma_{p,c}M)
\tag{26}
\]
with equality if and only if $K = L$. Combining (21) and (26), we obtain
\[
\frac{V(K)\bar{V}_{p}(M, \Gamma_{p,c}K)}{V(M)} \leq \frac{V(L)\bar{V}_{p}(M, \Gamma_{p,c}L)}{V(M)}
\tag{27}
\]
Taking $M = \Gamma_{p,c}L$ and by (13), it yields
\[
V(L)\bar{V}_{p}(\Gamma_{p,c}L)
\]

\[ \geq V(K)\tilde{V}_p\left(\Gamma_{p,C}^*L, \Gamma_{p,C}^*K\right) \]
\[ \geq V(K)\tilde{V}\left(\Gamma_{p,C}^*L\right)^{\frac{2n+p}{2n}} V\left(\Gamma_{p,C}^*K\right)^{\frac{p}{2n}} \] (28)

with equality in the second inequality of (30) if and only if \( \Gamma_{p,C}^*L \) and \( \Gamma_{p,C}^*K \) are real dilation. Thus, it follows from (28) that we have
\[ \frac{V\left(\Gamma_{p,C}^*L\right)^{\frac{p}{2n}}}{V(K)} \geq \frac{V\left(\Gamma_{p,C}^*K\right)^{\frac{p}{2n}}}{V(L)} \] (29)

with equality if and only if \( K = L \).

Now, we are dedicated to proving Theorem 5 and Theorem 6.

**Proof of Theorem 5** For \( p \geq 1 \) and \( M \in S'_a(C^n) \), it follows from the Lemma 2,
\[ \frac{\tilde{V}_p\left(K, \Gamma_{p,C}^*M\right)}{V(K)} = \frac{\tilde{V}_p\left(M, \Gamma_{p,C}^*K\right)}{V(M)}, \]
\[ \frac{\tilde{V}_p\left(L, \Gamma_{p,C}^*M\right)}{V(L)} = \frac{\tilde{V}_p\left(M, \Gamma_{p,C}^*L\right)}{V(M)} \] (30)

Since \( \Gamma_{p,C}^*K \subseteq \Gamma_{p,C}^*L \), then \( \Gamma_{p,C}^*L \subseteq \Gamma_{p,C}^*K \), hence for all \( u \in S^{2n-1} \), we have
\[ p\left(\Gamma_{p,C}^*L\right)^{-p} \leq p\left(\Gamma_{p,C}^*K\right)^{-p} \] (31)

Combining (30) and (31), we get
\[ \frac{\tilde{V}_p\left(K, \Gamma_{p,C}^*M\right)}{V(K)} \leq \frac{\tilde{V}_p\left(L, \Gamma_{p,C}^*M\right)}{V(L)} \] (32)

For \( L \in S_{p,C}^* \) and taking \( \Gamma_{p,C}^*M \) for \( L \) in (32), then from (13), we get \( V(K) \leq V(L) \) with equality if and only if \( K = L \).

**Proof of Theorem 6** By (3), (15) and (16), we have
\[ h\left(\Gamma_{p,C}\left(\hat{\nu}_{p,C}K\right), u\right)^p = h\left(\Gamma_{p,C}\left(\frac{1}{2} K + \frac{1}{2} (-K)\right), u\right)^p \]
\[ = \frac{1}{2} h\left(\Gamma_{p,C}K, u\right)^p + \frac{1}{2} h\left(\Gamma_{p,C}(-K), u\right)^p \]
\[ = h\left(\Gamma_{p,C}K, u\right)^p \] (33)

Meanwhile, according to (12), (13) and (14), it yields
\[ \frac{\tilde{V}_p\left(\lambda \ast K_{p,C}^*L, Q\right)}{V(\lambda \ast K_{p,C}^*L)} \]
\[ = \lambda \frac{\tilde{V}_p\left(K, Q\right)}{V(K)} + \mu \frac{\tilde{V}_p\left(L, Q\right)}{V(L)} \]
\[ \geq V(Q)^{\frac{p}{2n}} \left[ \lambda V(K)^{\frac{p}{2n}} + \mu V(L)^{\frac{p}{2n}} \right] \] (34)

Taking \( Q = \lambda \ast K_{p,C}^*L, \) and \( \lambda = \mu = \frac{1}{2} K = L \) in (34), then \( V(\hat{\nu}_{p,C}L) \geq V(L) \) with equality if and only if \( L \) is an origin symmetric body.

Since \( L \) is not an origin symmetric, we get \( V(\hat{\nu}_{p,C}L) > V(L) \). Choose \( \epsilon > 0 \) such that
\[ V\left((1-\epsilon)\hat{\nu}_{p,C}L\right) > V(L) \]

Let \( K = (1-\epsilon)\hat{\nu}_{p,C}L \), then \( V(L) < V(K) \). According to (3), we see that
\[ \Gamma_{p,C}^*K = \Gamma_{p,C}\left((1-\epsilon)\hat{\nu}_{p,C}L\right) \]
\[ = (1-\epsilon)\Gamma_{p,C}\left(\hat{\nu}_{p,C}L\right) \]
\[ = (1-\epsilon)\Gamma_{p,C}L \subseteq \Gamma_{p,C}L \]
which ends the proof of Theorem 6.

**References**


