The Number of Solutions of Certain Equations over Finite Fields

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Abstract: Let $s$ be a positive integer, $p$ be an odd prime, $s=p^r$, and let $F_q$ be a finite field of $q$ elements. Let $N_q$ be the number of solutions of the following equations: $(x_1^n + x_2^n + \cdots + x_s^n)=x_1x_2\cdots x_s$ over the finite field $F_q$, with $n \geq t>0, n,k$, and $k(n+1 \leq j \leq q), m_i(1 \leq i \leq n)$ are positive integers.

In this paper, we find formulas for $N_q$ when there is a positive integer $l$ such that $dD|(p^r+1)$, where $D=\text{lcm} \{d_1,\cdots,d_s\}$.

$N_q=b$\#$(\{x_1,\cdots,x_s\}\in F_q^n|f(x_1,\cdots,x_s)=0)$

Studying the value of $N(f=b)$ is one of the main topics in finite fields. Generally speaking, it is nontrivial to give the formula for $N(f=b)$. Finding the explicit formula for $N(f=b)$ under certain condition has attracted lots of authors for many years.

Markoff-Hurwitz-type equations belong to the following type of the Diophantine equations

$\sum_{j=1}^{n} m_j x_j^2 = a_1 x_1 \cdots x_s$

where $n, a$ are positive integers and $n \geq 3$. This type of equations was first studied by Markoff [1] for the case $n=3, a=3$. More generally, these equations were studied by Hurwitz [2].

Recently, Baoulina [3-5] studied the generalized Markoff-Hurwitz-type equations

$a_1 x_1^m + a_2 x_2^m + \cdots + a_s x_s^m = a x_1 \cdots x_s$

where $a, a \in F_q^*$ and $m_i$ are positive integers satisfying $m_i|(q-1)$ for $i=1,\cdots, n$ and $n \geq 2$. Baoulina [4-6] and Pan et al [7] considered the further generalized Markoff-Hurwitz-type equations of the form:

0 Introduction

Let $p$ be an odd prime. Let $F_q$ be a finite field of $q$ elements with $q=p^r$, $s \geq 1$ and $F_q'$ denote the set of all the nonzero elements of $F_q$. Let $N(f=b)$ denote the number of solutions of the equation $f(x_1,\cdots,x_s)=b$ in $F_q^n=F_q'\times\cdots\times F_q'$, where $f(x_1,\cdots,x_s)$ is a polynomial in $F_q[x_1,\cdots,x_s]$ and $b\in F_q$. That is,

$N(f=b)=\#(\{x_1,\cdots,x_s\}\in F_q^n|f(x_1,\cdots,x_s)=0)$

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Markoff-Hurwitz-type equations belong to the following type of the Diophantine equations

$x_1^2 + x_2^2 + \cdots + x_n^2 = ax_1 \cdots x_n$

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where $a, a \in F_q^*$ and $m_i$ are positive integers satisfying $m_i|(q-1)$ for $i=1,\cdots, n$ and $n \geq 2$. Baoulina [4-6] and Pan et al [7] considered the further generalized Markoff-Hurwitz-type equations of the form:
where \( n \geq 2 \), \( m, k, h \) are positive integers, \( a, a_i \in F_q^* \), for \( i = 1, \ldots, n \). The special case \((1) \) of \( k = 1 \) is investigated by Cao\[8\]. Song and Chen \[9\] presented the formulas for the number of solutions of the following equations

\[
x_1^n + x_2^n + \cdots + x_n^n = a_1 x_1 + a_2 x_2 + \cdots + a_n x_n \tag{1}
\]

over the finite field \( F_q \) under some certain restrictions, where \( n \geq 2, m_i, k_i, j_i, k \) and \( t_i > n \) are positive integers, \( a_i, a \in F_q^* \), for \( 1 \leq i \leq n, 1 \leq j \leq t_i \).

In this paper, we consider the number of solutions of the following equations

\[
(x_1^{n_1} + x_2^{n_2} + \cdots + x_l^{n_l})^f = a_1 x_1 + a_2 x_2 + \cdots + a_l x_l \tag{2}
\]

over the finite field \( F_q \) under some other restrictions, where \( n \geq 2, t > n, k, j, k (n + 1 \leq j \leq t), m_i (1 \leq i \leq n) \) are positive integers. In what follows, we always let

\[
D = \text{lcm}[m_1, \ldots, m_n],
\]

\[
d_0 = \gcd(d, k),
\]

\[
d = \gcd(\sum_{i=1}^{m} m \div m_i - km_i, (q - 1) / D).
\]

For any positive integers \( v_i, v_2, \ldots, v_r \), we let \( I(v_1, v_2, \ldots, v_r) \) denote the number of \( r \)-tuples \((j_1, j_2, \ldots, j_r)\) of integers with \( 1 \leq j_i \leq v_i - 1 \) \((1 \leq i \leq r)\), such that \( j_1 / v_1 + j_2 / v_2 + \cdots + j_r / v_r \) is an integer. Denote by \( N_q \) the number of solutions of (2) in \( F_q^* \). Our main result is the following theorem.

**Theorem 1** Suppose that \( \gcd(k_{1}, \ldots, k_{r}) = d \), \( dD > 2 \) and there is a positive integer \( l \) such that \( dD \mid (p^l + 1) \), with \( l \) chosen minimal. Then \( 2 \mid s \) and

\[
N_q = q^d - 1 + (-1)^{l/2}((1 + d)q^{d/2} - 1)(q - 1)I(d_1, \ldots, d_n)
\]

\[
+ q^{d - a}(1 - 1)^{l/2} + \sum_{r=2}^{\infty} \left((-1)^{r/2} q^{r/2}ight) \sum_{j_1, \ldots, j_r \in \mathbb{Z}} I(d_{j_1}, \ldots, d_{j_r})
\]

\[
= \sum_{j_1, \ldots, j_r \in \mathbb{Z}} I(d_{j_1}, \ldots, d_{j_r}) + (-1)^{l/2} \sum_{j_1, \ldots, j_r \in \mathbb{Z}} I(d_{j_1}, \ldots, d_{j_r})
\]

\[
= d_1 \cdots d_n \frac{q^{(n-1)/2} (d - d_0)}{D} - (-1)^{(l/2) \div \text{a power in } q^{(n-2)/2}} \frac{d_1 \cdots d_n \frac{q^{(n-1)/2}}{D}}{D}
\]

(this is \((d_0 - 1)\))

This paper is organized as follows. In Section 1, we review some useful known lemmas which will be needed later. Subsequently, in Section 2, we prove Theorem 1. Some interesting applications of Theorem 1 will be provided as corollaries at the end of this paper.

**1 Preliminary Lemmas**

In this section, we present some useful lemmas that are needed in the proof of Theorem 1 as follows.

**Lemma 1**\[6,11\] For any positive integer \( m \), the number of elements of \( m \)-th power in \( F_q^* \) is \( q^{1 - m} \).

**Lemma 2**\[6\] Let \( t_1, t_2, \ldots, t \) be positive integers and \( l' = \gcd(t_1, t_2, \ldots, t, q - 1) \). For any elements \( a, a \in F_q^* \), we have

\[
N(ax_1^{d_1} \cdots x_n^{d_n}) = a
\]

\[
N(ax_1^{d_1} \cdots x_n^{d_n}) = a
\]

\[
= \begin{cases} (t(q - 1))^{-1}, & \text{if } a^{-1} \alpha \text{ is a } t'-\text{th power in } F_q^* \\ 0, & \text{otherwise} \end{cases}
\]

The following two lemmas are the main results in Ref.[6] and fundamental for our results.

**Lemma 3**\[6\] Let \( n > 2 \). Suppose that there is a positive integer \( l \) such that \( 2 l \mid s \) and \( dD \mid (p^l + 1) \). Then

\[
N(x_1^n + \cdots + x_n^n) = 0
\]

\[
= q^n + 1 + (-1)\left((-1)^{(l/2)} q^{l/2} - 1\right)(q - 1)I(d_1, \ldots, d_n)
\]

\[
= q^n + 1 + (-1)^{(l/2)} q^{l/2} - 1\right)(q - 1)I(d_1, \ldots, d_n)
\]

\[
+ (-1)^{(l/2)} \sum_{j_1, \ldots, j_r \in \mathbb{Z}} I(d_{j_1}, \ldots, d_{j_r})
\]

\[
+ (-1)^{l/2} \sum_{j_1, \ldots, j_r \in \mathbb{Z}} I(d_{j_1}, \ldots, d_{j_r})
\]

\[
= \frac{d_1 \cdots d_n}{D} q^{(n-2)/2} T_2
\]

\[
= \frac{d_1 \cdots d_n}{D} q^{(n-2)/2} T_2
\]

where

\[
T_i = \begin{cases} d - d_0, & \text{if } a \text{ is a } d'-\text{th power in } F_q^* \\ d_0, & \text{if } a \text{ is } d \text{th power but not a } d'-\text{th power in } F_q^* \\ 0, & \text{if } a \text{ is not a } d \text{th power in } F_q \end{cases}
\]
and
\[ T_2 = \begin{cases} d_n - 1, & \text{if } a \text{ is a } d_n \text{-th power in } F_q \\ -1, & \text{if } a \text{ is not a } d_n \text{-th power in } F_q \end{cases} \]

2 Proof of Theorem 1

In this section, we give the proof of Theorem 1.

Proof of Theorem 1 Let \( \mathcal{N}_q \) (resp. \( \bar{\mathcal{N}}_q \)) denote the number of the solutions of the equations
\[ \sum_{i=1}^{k} x_i^{n_i} + \sum_{i=1}^{q} x_i^{n_i} = x_1 x_2 \cdots x_{m+1} x_{m+2} \cdots x_n \]
with \( x_i^{n_i} \cdots x_{n_i} = 0 \) (resp. \( x_i^{n_i} \cdots x_{n_i} \neq 0 \)). Clearly, one has
\[ \mathcal{N}_q = \bar{\mathcal{N}}_q + \mathcal{N}_q \]  
(3)

Then we can solve the problem in two cases. One is \( x_i^{n_i} \cdots x_{n_i} = 0 \) and the other one is \( x_i^{n_i} \cdots x_{n_i} \neq 0 \).

Case (i) \( x_i^{n_i} \cdots x_{n_i} = 0 \). Then one has
\[ \mathcal{N}_q(x_i^{n_i} \cdots x_{n_i} = 0) = \mathcal{N}(x_i^{n_i} \cdots x_{n_i} = 0) = (q-1)^{n-a} \]
(4)

Using the assumption there is a positive integer \( l \) such that \( 2l/s \) and \( \text{dD}(p') + 1 \). Thus, by (4) and Lemma 3,
\[ \bar{\mathcal{N}}_q = (q^{-a} - (q-1)^{-a})N((x_1^{n_1} + \cdots + x_n^{n_n})^t = 0) \]
\[ = (q^{-a} - (q-1)^{-a})N(x_1^{n_1} + \cdots + x_n^{n_n} = 0) \]
\[ = (q^{-a} - (q-1)^{-a})(q^{-a} + (-1)^{(r/2)\cdot a} q^{-a}) \times(q-1)^{l/2}I(d_1, \cdots, d_n) \]
(5)

Case (ii) If \( x_i^{n_i} \cdots x_{n_i} \neq 0 \), we let \( \delta = x_i^{n_i} \cdots x_{n_i} \).

Define
\[ U := \{ \beta \in F_q^* : \beta \text{ be a d-th power in } F_q^* \} \]

Note that \( \gcd(k_1, \cdots, k_r, q-1) = d \). Then from Lemma 2, we can deduce that
\[ \bar{\mathcal{N}}_q = \mathcal{N}((x_1^{n_1} + \cdots + x_n^{n_n})^t = \delta x_1 x_2 \cdots x_n) \]
\[ = (q-1)^{n-a} \]
\[ \times \sum_{\mu \in U} \mathcal{N}(x_1^{n_1} + \cdots + x_n^{n_n} = \delta x_1 x_2 \cdots x_n) \]
(6)

Noting that integer \( l \) such that \( \text{dD}(p') + 1 \), with \( l \) chosen minimal. Thus for any given \( \delta \in U \), from Lemma 1 and Lemma 3, one has
\[ \sum_{\mu \in U} \mathcal{N}(x_1^{n_1} + \cdots + x_n^{n_n} = \delta x_1 x_2 \cdots x_n) \]
\[ = \frac{q-1}{d}(q^{-a} + (-1)^{l/2} + (q^{-a})^{(a/2)-2}) \times(q-1)^{l/2}I(d_1, \cdots, d_n) \]
(7)

Then by (6) together with (7), we have
\[ \bar{\mathcal{N}}_q = (q-1)^{-a} - (q-1)^{-a} + (1)^{(a/2)\cdot a} q^{-a} \]
\[ \times\sum_{\mu \in U} (q^{-a} + (-1)^{l/2} + (q^{-a})^{(a/2)-2}) \times(q-1)^{l/2}I(d_1, \cdots, d_n) \]
(8)

The desired result can follow immediately from (3), (5) and (8). This ends the proof of Theorem 1.

To conclude this section, we present some corollaries.

Corollary 1 Suppose that \( d_1, \cdots, d_m \) are odd, \( d_1, \cdots, d_m \) are even, \( d_1, \cdots, d_m, d_1/2, \cdots, d_m/2 \) are pairwise coprime, 0 \( m \leq n \). Under the conditions of Theorem 1, we have
\[ \bar{\mathcal{N}}_q = q^{-a} + T_3 + (q-1)^{-a}((-1)^{l/2} + (q^{-a})^{(a/2)-2}) \times(q-1)^{l/2}I(d_1, \cdots, d_n) \]
(9)

where
\[ T_3 = \begin{cases} q^{-a/2} - 1, & \text{if } m = 0 \text{ and } n \text{ is even} \\ 0, & \text{otherwise} \end{cases} \]

and
Let $v$ be a positive integer. It is also known (Ref. [14], Proposition 6.17) that

$$I(v, \ldots, v) = \frac{(v-1)^{y} + (-1)^{y}(v-1)}{v}$$

Then we have the second corollary.

**Corollary 2** Suppose that $d_1 = \cdots = d_s = D$. Under the conditions of Theorem 1, we have

$$N_q = q^{v-1} + (-1)^{(t/2t-1)}q^{v-1} (q-1)\frac{(D-1)^v + (-1)^v(D-1)}{D} + (q-1)^{v-n}$$

$$\cdot \left( (-1)^n \sum_{m=0}^{n} (-1)^{m/2} q^{m/2} \frac{(D-1)^m + (-1)^m(D-1)}{D} \right)$$

Clearly, Corollaries 1-2 are some special cases of Theorem 1. For example, consider the further generalized Markoff-Hurwitz-type equation over $F_2$,

$$(x_1^5 + x_2^5 + x_3^5)^2 = x_1x_2x_3x^4$$  \hspace{1cm} (9)

Clearly $m_1 = 5, m_2 = 2, m_3 = 6$, $k = 2, k_4 = 2, k_5 = 4$. Then we get $d_1 = \gcd(5,8) = 1$, $d_2 = \gcd(2,8) = 2$, $d_3 = \gcd(6,8) = 2$, $D = \text{lcm}(d_1, d_2, d_3) = 2$, $M = \text{lcm}(5, 2, 6) = 30$, $d_0 = \gcd(d, k) = 2$ and $d = \gcd(3M/m, -kM/(q-1)/D) = 2$. One can immediately conclude that (9) has 6 817 solutions in $F_2$, by Corollary 1.

**References**
