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An Optimal Portfolio Problem Presented by Fractional Brownian Motion and Its Applications

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Abstract: We use the dynamic programming principle method to obtain the Hamilton-Jacobi-Bellman (HJB) equation for the value function, and solve the optimal portfolio problem explicitly in a Black-Scholes type of market driven by fractional Brownian motion with Hurst parameter $H \in (0,1)$. The results are compared with the corresponding well-known results in the standard Black-Scholes market $(H = 1/2)$. As an application of our proposed model, two optimal problems are discussed and solved, analytically.

Key words: fractional Brownian motion; Merton's optimal problem; stochastic differential equation

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0 Introduction

The idea of replacing Brownian motion with another Gaussian process in the usual financial models has been around for some time. In particular, fractional Brownian motion has been considered as it has better behaved tails and exhibits long-term dependence while remaining Gaussian.

Let $H \in (0,1)$ be a fixed constant. The fractional Brownian motion with Hurst parameter H is the Gaussian process $B_H(t,\omega); t \geqslant 0, \omega \in \Omega$ on the probability space $(\Omega, \mathcal{F}, P)$ with the property that

$$E[B_H(t)] = B_H(0) = 0$$

and

$$E[B_H(t)B_H(s)] = \frac{1}{2}\{|t|^{2H} + |s|^{2H} - |t-s|^{2H}\}; t, s \geqslant 0$$

Here E denotes the expectation with respect to the probability P. If H=1/2 then $B_H(t)$ coincides with the classical Brownian motion, denoted by $B(t)$.

Rogers[1] showed that arbitrage is possible when the risky asset has a log-normal price driven by a fractional Brownian motion if stochastic integrals are defined using pointwise products. However, using the white noise approach it is clear that stochastic integrals should be defined using Wick products. When the factors are strongly independent, a Wick product reduces to a pointwise product, and in the Brownian motion case white noise integral reduces to the usual Itô integral.

In this paper we will concentrate on the B_H-integral[2], defined by

$$\int_a^b f(t,\omega)\mathrm{d}B_H(t) = \lim_{|\Delta|\to 0}\sum_{k=0}^{n-1} f(t,\omega)\lozenge(B_H(t_{k+1}) - B_H(t_k))$$

where $f(t,\omega):\mathbf{R}\times\Omega\to\mathbf{R}$ is Skorohod integrable, $\lozenge$ denotes the Wick product. We call these fractional Itô integral because these integrals share many of the properties of the classical Itô integrals. In particular, we have

$$E\left[\int_{\mathbf{R}} f(t,\omega)\mathrm{d}B_H(t)\right] = 0 \tag{1}$$

Hu and Ksendal[3] extended this fractional Itô calculus to a white-noise calculus for fractional Brownian motion and applied it to finance, still for the case $1/2 < H < 1$ only. Then Elliott and Hoek [4] extended this theory and its application to finance to be valid for all values of $H\in(0,1)$.

There are two major mathematical techniques to find optimal controls in the field of optimal control theory. Pontryagins maximum principle and dynamic programming principle are applied to obtain the Hamilton-Jacobi-Bellman (HJB) equation [5]. In this paper, we consider the method of fractional HJB equation. This equation is a partial differential equation. The solution of this equation is the value function which gives the maximum expected utility from wealth at the time horizon.

Therefore, in this article, we shall first consider the Merton's model, in which the classical Brownian motion is replaced by a fractional Brownian motion with Hurst parameter $H\in(0,1)$. Then, as an application of this derivation, two optimal problems have discussed and solved by the method of fractional HJB equation.

1 Optimization Model

In this section, we describe the portfolio optimization problem under the fractional Brownian motion and derive the HJB equation for the value function.

We consider a continuous-time financial market consisting of two assets: a bond and a stock. Assume that the stock price $S(t)$ follows the fractional Brownian motion model and the bond price $M(t)$ satisfies the following equation:

$$\mathrm{d}S(t) = \mu S(t)\mathrm{d}t + \sigma S(t)\mathrm{d}B_H(t) \tag{2}$$

and

$$\mathrm{d}M(t) = rM(t)\mathrm{d}t \tag{3}$$

where $r > 0$ is the constant interest rate, $\mu > r > 0$ and $\sigma\neq 0$ are constants. For a portfolio comprising a stock and a risk-free bond, let π_t denote the percentages of wealth invested in the stock, $t\in[0,T]$. Then the wealth process $\{X(t)\}$ of the portfolio evolves as

$$\mathrm{d}X(t) = X(t)r\mathrm{d}t + X(t)\pi_t\sigma\left[\theta\mathrm{d}t + \mathrm{d}B_H(t)\right] \tag{4}$$

where $\theta=\dfrac{\mu-r}{\sigma}$, which is a real valued predictable process.

We aim to maximize the expected utility of terminal wealth:

$$\sup_{\pi\in\Pi_t} E_t[U(X(T))] \quad \text{subject to (4).}$$

To this end, we define a value function

$$V(t,x) := \sup_{\pi\in\Pi_t} E_t\left[U(X(T))\right] \tag{5}$$

where $E_t[\cdot]=E_t[\cdot\,|\,X(t)=x]$ and the utility function $U(\cdot)$ is assumed to be strictly concave and continuously differentiable on $(-\infty,+\infty)$, $\Pi_t:=\{\pi_s, s\in[t,T]\}$ is the set of all admissible strategies over $[t,T]$. Since $U(\cdot)$ is strictly concave, there exists a unique optimal trading strategy. It is obvious that V satisfies boundary condition:

$$V(T,x) = U(x), \forall x\geqslant 0 \tag{6}$$

2 The Closed-Form Solution

In this section, we apply dynamic programming principle to derive the HJB equation for the value function and investigate the optimal investment policies for problem (5) with the boundary condition (6) in the power and logarithm utility cases.

Lemma 1[6] Let $f(x,s):\mathbf{R}\times\mathbf{R}\to\mathbf{R}$ belong to $C^{1,2}(\mathbf{R}\times\mathbf{R})$, and assume that the three random variables

$$f(t,B_H(t)), \int_0^t \frac{\partial f}{\partial s}(s,B_H(s))\mathrm{d}s, \int_0^t \frac{\partial^2 f}{\partial x^2}(s,B_H(s))s^{2H-1}\mathrm{d}s$$

all belong to $L^2(P)$. Then

$$\begin{aligned} f(t,B_H(t)) &= f(0,0) + \int_0^t \frac{\partial f}{\partial s}(s,B_H(s))\mathrm{d}s \\ &\quad + H\int_0^t \frac{\partial^2 f}{\partial x^2}(s,B_H(s))s^{2H-1}\mathrm{d}s \\ &\quad + \int_0^t \frac{\partial f}{\partial x}(s,B_H(s))\mathrm{d}B_H(s) \end{aligned} \tag{7}$$

Theorem 1 The optimal value function $V(t,x)$ satisfies

$$V_t + rxV_x - \frac{1}{4Ht^{2H-1}}\frac{\theta^2 V_x^2}{V_{xx}} = 0 \tag{8}$$

on $[0,T]\times[0,\infty)$, with terminal condition (6), where V_t, V_x, V_{xx} denote partial derivative of first and second orders with respect to time and wealth.

Proof Using the dynamic programming principle[7], Eq. (5) can be read as ,

$$V(t,X(t))=\sup_{\pi} E_t\left[\sup_{\pi} E_{t+\Delta t}[U(X(T))]\right]$$
$$=\sup_{\pi} E_t\left[V(t+\Delta t, X(t+\Delta t))\right] \quad (9)$$

According to Lemma 1 and the dynamic of $\{X(t)\}$ given by (4), we have

$$V(t,X(t))=\sup_{\pi_t} E_t\left[V(t,X(t))+\int_t^{t+\Delta t}\frac{\partial V}{\partial s}(s,X(s))\mathrm{d}s\right.$$
$$+\int_t^{t+\Delta t}\frac{\partial V}{\partial x}(s,X(s))(r+\theta\sigma\pi_s)X(s)\mathrm{d}s$$
$$+H\int_t^{t+\Delta t}\frac{\partial^2 V}{\partial x^2}(s,X(s))\sigma^2\pi_s^2X(s)^2s^{2H-1}\mathrm{d}s$$
$$\left.+\int_t^{t+\Delta t}\frac{\partial V}{\partial x}(s,X(s))\sigma\pi_sX(s)\mathrm{d}B_H(s)\right] \quad (10)$$

The stochastic integral in above equation is a Quasi-Martingale, and we get

$$E_t\left[\int_t^{t+\Delta t}\frac{\partial f}{\partial x}(s,X(s))\sigma\pi_sX(s)\mathrm{d}B_H(s)\right]=0$$

Now, suppose $\Delta t\to 0$. Then $s\to t$, $X(s)\to X(t)=x$ and by intermediate value theorem for integral, the integrals in Eq.(10) are evaluated as

$$V(t,X(t))=\sup_{\pi_t}\left[V(t,X(t))+V_t\Delta t\right.$$
$$+V_x\left(r+\theta\sigma\pi_t\right)x\Delta t$$
$$\left.+HV_{xx}\sigma^2\pi_t^2x^2t^{2H-1}\Delta t\right] \quad (11)$$

Dividing both sides of Eq.(11) by Δt , we get the following partial differential equation (PDE)

$$0=\sup_{\pi_t}[V_t+V_x(r+\theta\sigma\pi_t)x+HV_{xx}\sigma^2\pi_t^2x^2t^{2H-1}] \quad (12)$$

on $[0,T]\times[0,\infty)$, with its first-order condition leading to the optimal strategy

$$\pi^*=-\frac{\theta V_x}{2Ht^{2H-1}V_{xx}\sigma x} \quad (13)$$

By substitution, we can then obtain PDE (8). The boundary condition follows immediately from (5). Furthermore, it follows from the standard verification theorem that the solution to the PDE (8) is indeed the function $V(t,x)$.

Remark 1 It follows (13) that the optimal investment strategy π^* has an analogical form of the optimal policy under a generalized Bass model (GBM) (H=1/2).

Here, we notice that the stochastic control problem has been transformed into a nonlinear second-order partial differential equation; yet it is difficult to solve it. In the following subsection, we choose power utility and logarithm utility for our analysis, respectively, and try to obtain the closed-form solutions to (8).

2.1 Power Utility

Consider power utility

$$U(x)=\frac{x^p}{p},0<p<1$$

The boundary condition $V(T,x)=U(x)$ suggests that our value function has the following form

$$V(t,x)=f(t)\frac{x^p}{p} \quad (14)$$

and the function f to be determined with terminal condition $f(T)=1$. Therefore, we get

$$\frac{\partial V}{\partial t}=\frac{\mathrm{d}f}{\mathrm{d}t}\frac{x^p}{p},\frac{\partial V}{\partial x}=f(t)x^{p-1},$$
$$\frac{\partial^2 V}{\partial x^2}=f(t)(p-1)x^{p-2} \quad (15)$$

Replacing (15) into Eq. (8), yields

$$\frac{\mathrm{d}f}{\mathrm{d}t}\frac{x^p}{p}+rf(t)x^p-\frac{\theta^2}{4Ht^{2H-1}}\frac{f(t)x^p}{p-1}=0$$

Eliminating the dependence on x , we obtain

$$\frac{\mathrm{d}f}{\mathrm{d}t}=f(t)\left[\frac{\theta^2 p}{4Ht^{2H-1}(p-1)}-rp\right]$$

with $f(T)=1$, we obtain

$$f(t)=\exp\left\{rp(T-t)-\frac{\theta^2p\left(T^{2-2H}-t^{2-2H}\right)}{4H(2-2H)(p-1)}\right\} \quad (16)$$

By plugging (16) into Eq. (14), we obtain the value function

$$V(t,x)=\frac{x^p}{p}\exp\left\{rp(T-t)-\frac{\theta^2p\left(T^{2-2H}-t^{2-2H}\right)}{4H(2-2H)(p-1)}\right\} \quad (17)$$

and the optimal investment strategy

$$\pi^*=\frac{\theta}{2H\sigma(1-p)t^{2H-1}}$$

Theorem 2 If the utility function is given by

$$U(x)=\frac{x^p}{p};0<p<1$$

the value function $V(t,x)$ for problem (5) is given by

$$V(t,x)=\frac{x^p}{p}\exp\left\{rp(T-t)-\frac{\theta^2p\left(T^{2-2H}-t^{2-2H}\right)}{4H(2-2H)(p-1)}\right\}$$

And the corresponding optimal strategy can be obtained as follows:

$$\pi^*=\frac{\theta}{2H\sigma(1-p)t^{2H-1}}$$

Remark 2 It is natural to ask how the value function $V(t,x) := V_H(t,x)$ in (17) is related to the value function $V_{1/2}(t,x)$ for the corresponding problem for standard Brownian motion ($H = 1/2$). In this case it is well-known that (see Ref. [8])

$$V_{1/2}(t,x) = \frac{x^p}{p}\exp\left\{\left(rp - \frac{\theta^2 p}{2(p-1)}\right)(T-t)\right\}$$

Therefore we see that, as was to be expected

$$\lim_{H \to 1/2} V_H(t,x) = V_{1/2}(t,x)$$

2.2 Logarithm Utility

Now let us consider the following utility function

$$U(x) = \ln x$$

We can assume that our value function has the following structure:

$$V(t,x) = g(t) + \ln x$$

and the function g to be determined with terminal condition $g(T) = 0$. Hence, we have

$$\frac{\partial V}{\partial t} = \frac{\mathrm{d}g}{\mathrm{d}t}, \frac{\partial V}{\partial x} = \frac{1}{x}, \frac{\partial^2 V}{\partial x^2} = -\frac{1}{x^2} \tag{18}$$

Substituting (18) back into (8) yields

$$\frac{\mathrm{d}g}{\mathrm{d}t} = -r - \frac{\theta^2}{4Ht^{2H-1}}$$

Adding to $g(T) = 0$, we have,

$$g(t) = r(T-t) + \frac{\theta^2\left(T^{2-2H} - t^{2-2H}\right)}{4H(2-2H)}$$

Hence, we obtain the value function

$$V(t,x) = \ln x + r(T-t) + \frac{\theta^2\left(T^{2-2H} - t^{2-2H}\right)}{4H(2-2H)}$$

with the optimal investment strategy

$$\pi^* = \frac{\theta}{2H\sigma t^{2H-1}}$$

When $H = 1/2$, we obtain

$$V_{1/2}(t,x) = \ln x + \left(r + \frac{\theta^2}{2}\right)(T-t)$$

At the end, we can conclude the optimal portfolio problem for logarithm utility in the following theorem.

Theorem 3 If the utility function is given by

$$U(x) = \ln x$$

the value function $V(t,x)$ for problem (5) is given by

$$V(t,x) = \ln x + r(T-t) + \frac{\theta^2 (T^{2-2H} - t^{2-2H})}{4H(2-2H)}$$

and the corresponding optimal controls can be obtained as follows

$$\pi^* = \frac{\theta}{2H\sigma t^{2H-1}}.$$

□

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