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Some Metric Properties of Sets Related to Luroth Expansion

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Abstract: This paper is mainly concerned with scrambled sets for Luroth map. It is shown that all scrambled sets have null Lebesgue measure and there exists a scrambled set with full Hausdorff dimension.

Key words: Luroth expansion; Hausdorff dimension; Li-Yorke chaotic

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0 Introduction

For any real number $x \in (0,1)$, the Luroth map $T:(0,1) \to (0,1)$ is defined by

$$T(x) := d_1(x)(d_1(x)-1)(x - \frac{1}{d_1(x)}) \tag{1}$$

where $d_1(x) = \left[\frac{1}{x}\right] + 1$ and $[x]$ denotes the greatest integer not exceeding x. We define the integer sequence $\{d_k(x): k \geq 1\}$ by

$$d_k(x) = d_1(T^{k-1}(x)) \tag{2}$$

where T^k denotes the k-th iterate of T. By (1) and (2), for any $x \in (0,1)$, this map can generate a series expansion of x, i.e.,

$$x = \frac{1}{d_1(x)} + \frac{1}{d_1(x)(d_1(x)-1)d_2(x)} + \cdots + \frac{1}{d_1(x)(d_1(x)-1)\cdots d_{n-1}(x)(d_{n-1}(x)-1)d_n(x)} + \cdots,$$

where $d_n(x) \geq 2$ are positive integer for any $n \geq 1$. We call $d_n(x)$ the digits of the Luroth expansion of x, and write the above representation as $[d_1(x), d_2(x), \cdots, d_n(x), \cdots]$ for simplicity. Such a series expansion was first studied by Luroth[1] in 1883.

Luroth series expansion plays an important role in the representation theory of real numbers and dynamical systems. It is well-known that every irrational number has a unique infinite expansion and each rational number has either a finite or a periodic expansion (see Galambos[2]). For dynamical properties, the transformation T is invariant and ergodic with respect to Lebesgue measure[3-5]. In other direction, Fan *et al* [6] obtained the

Hausdorff dimension of sets of real numbers with prescribed digit frequencies in Luroth expansion. Barreira and Iommi[7] considered the Hausdorff dimension of a class of sets defined in terms of the frequencies of digits in Luroth expansion. For more details, we can refer the reader to Refs. [8-10].

Asymptotic behavior of the orbits is one of the most important theme in dynamical systems. The first mathematical treatment of chaotic behavior of dynamical system appeared in the work of Li and Yorke in 1975[11]. A dynamical system is a pair (X, f), where X is a compact metric space with a metric d and f being a continuous map $X \to X$. A subset $S \subset X$, containing at least two points, is a scrambled set for f if every pair (x, y) of distinct points in S is a scrambled pair, i.e.,

$$\liminf_{n\to\infty} d(f^n(x), f^n(y)) = 0$$

and

$$\limsup_{n\to\infty} d(f^n(x), f^n(y)) > 0$$

If X contains an uncountable scrambled set, then the dynamical system (X, f) is called chaotic in the sense of Li-Yorke. It is well-known that the surjection continuous transformation on compact metric space with positive topological entropy is chaotic in the sense of Li-Yorke[12]. Note that the topological entropy of the Luroth map is infinite, the result in Ref. [12] can not be applied since it is not continuous.

In the following we will show that the scrambled set for Luroth map is small in the sense of Lebesgue measure and large in the sense of Haudsdorff dimension, not just uncountable.

Theorem 1 Let T be the Luroth map on (0, 1), then all scrambled sets for T have null Lebesgue measure.

Theorem 2 Let T be the Luroth map on (0, 1), then there exists a scrambled set in (0, 1) with full Hausdorff dimension.

Corollary 1 The Luroth dynamical system ((0, 1), *T*) is chaotic in the sense of Li-Yorke.

We use N to denote the set of positive integers, Λ the set of points whose Luroth expansion is finite, λ the Lebesgue measure and $\dim_H$ the Hausdorff dimension.

1 Preliminaries

In this section, we present some elementary results in the theory of Luroth expansion and some fundamental concepts in symbolic dynamics. For more details, we refer to monographs of Dajani and Kraaikamp[8] and Kurka[13].

Lemma 1[3] The digits $d_1(x), \cdots, d_n(x)$ are independent identically distributed.

For any $1 \le k \le n$, we call $I_n(d_1, \cdots, d_n) = \{x \in (0,1) : d_1(x) = d_1, \cdots, d_n(x) = d_n\}$ a rank-*n* basic interval. Denote by $I_n(x)$ the rank-*n* basic interval containing x. Write $|I|$ for the length of an interval I. The next proposition concerns the length of rank-*n* basic intervals.

Proposition 1[2] For any $d_1, \cdots, d_n \in N$ with $d_k \ge 2 (1 \le k \le n)$, the rank-*n* basic interval $I_n(d_1, \cdots, d_n)$ is the interval with the endpoints

$$\frac{1}{d_1} + \frac{1}{d_1(d_1-1)d_2} + \cdots + \prod_{k=1}^{n-1} \frac{1}{d_k(d_k-1)} \frac{1}{d_n}$$

and

$$\frac{1}{d_1} + \frac{1}{d_1(d_1-1)d_2} + \cdots + \prod_{k=1}^{n-1} \frac{1}{d_k(d_k-1)} \frac{1}{d_n} + \prod_{k=1}^{n} \frac{1}{d_k(d_k-1)}$$

As a result,

$$|I_n(d_1, \cdots, d_n)| = \prod_{k=1}^{n} \frac{1}{d_k(d_k-1)} \tag{3}$$

For any integer $M \ge 2$, let E_M be the set of points in (0, 1) whose digits in the Luroth expansion do not exceed M. That is, $E_M \{x \in (0,1) : 2 \le d_n(x) \le M\}$. We can see that E_M is a self-similar set, and then the Hausdorff dimension of E_M is the unique positive root of

$$\sum_{2\le d\le M} \left(\frac{1}{d(d-1)}\right)^s = 1$$

Lemma 2[14] For any integer $M \ge 2$. Let $\dim_H E_M = s_M$, then $\lim_{M\to\infty} s_M = 1$.

Now we give some notations for symbolic spaces. we denotes by A^n the set of words of A with the length of *n*. A^N is the set of infinite words and $A^* = \bigcup_{n\ge 0} A^n$, the concatenation of words $u, v \in A^*$ is written as uv. For any $M \ge 3$, let $A = \{2,3,\cdots,M\}$ or $A = \{2,3,\cdots,n,\cdots\}$, we denote the symbolic space of one-sided infinite sequence over A by $A^N = \{x = (x_1, x_2, \cdots) : x_i \in A, \forall i \in N\}$. The symbolic x_i is called the i-th coordinate of x. Let i, j be positive integers with $i < j$, write $x|_i^j = x_i, \cdots, x_j$. For any $x, y \in A^N$, we define the metric

$$d(x, y) = 2^{-n}, \text{ where } n = \inf\{i \ge 0, x_{i+1} \ne y_{i+1}\}$$

The shift map σ is defined by

$$(\sigma(x))_i = x_{i+1}, \text{ for any } x \in A^N \text{ and } i \in N.$$

2 Proofs of Theorem 1 and Theorem 2

In this section, we first prove that all scrambled sets of Luroth map have null Lebesgue measure. To deal with Theorem 2 , inspired by Xiong[15], Liu and Li[16], we will construct a scrambled set in $\{2,3,\cdots\}^N$ and then establish a continue and bijective map between $\{2,3,\cdots\}^N$ and $(0,1)$ such that the projection of the scrambled set in $\{2,3,\cdots\}^N$ has full Hausdorff dimension.

Proof of Theorem 1 Suppose that there exists a scrambled set A for Luroth map T with $\lambda(A)=c>0$, let k be the smallest positive integer such that $2^k c>1$. Let $\{I_i\}_{i\ge1}$ be all rank-k basic intervals from left to right, put $A_i = A\cap I_i (i\ge 1)$, then for any $x\in A\cap I_i$, we have

$$\sum_{i\ge1}\lambda(T^k A_i)\ge\sum_{i\ge1} d_1(x)(d_1(x)-1)\cdots d_k(x)(d_k(x)-1)\lambda(A_i)$$
$$\ge 2^k\sum_{i\ge1}\lambda(A_i)=2^k c>1$$

Thus there exist two positive integers $i\ne j$ such that $T^k(A_i)\cap T^k(A_j)\ne\varnothing$. As a result, we have $T^k(x)=T^k(y)$, where $x\in A_i$ and $y\in A_j$. Hence for all $n\ge k$, we have $T^n(x)=T^n(y)$, which contradicts the definition of scrambled pair.

Let g_M,θ_M,ψ_M and Δ_M be the maps dependent on the symbol $M (M\ge 3)$, we write $\Sigma_M=\{2,3,\cdots,M\}^N$ to emphasize the dependence of M.

1) For any $x=(x_1,x_2,\cdots)\in\Sigma_M$ and $k\ge1$, we define $g_M:\Sigma_M\to\Sigma_M$ by

$$(g_M(x))_n=\begin{cases} M, n=1\\ 2,\ k(k-1)+2\le n\le 1+k^2, k\ge1\\ x_{n-1-k},\ k^2+1\le n\le 1+k+k^2, k\ge 1\end{cases}$$

One can see that

$$g_M(x)=(M,2,x_1,x_2,2,2,2,x_1,x_2,x_3,\cdots,\underbrace{2,2,\cdots,2}_{n},x_1,x_2,\cdots,x_n) \quad (4)$$

2) For any $x=(x_1,x_2,\cdots)\in\Sigma_M$, $\theta_M:\Sigma_M\to\Sigma_M$ is given by

$$\theta_M(x)=(x_1,x_1,x_2,x_1,x_2,x_3,\cdots,x_1,x_2,\cdots, x_n,\cdots,x_1,x_2,\cdots,x_n,x_{n+1},\cdots) \quad (5)$$

3) Let $L=\{l_k\}_{k\ge1}=\bigcup_{m=1}^{\infty}\bigcup_{t=1}^{m}\{m^3+t\}$ be a sequence of positive integers, for any $x=(x_1,x_2,\cdots)\in\Sigma_M$ and $y=(y_1,y_2,\cdots)\in\Sigma_M$, we define $\psi_M:\Sigma_M\times\Sigma_M\to\Sigma_M$ by

$$(\psi_M(x,y))_n=\begin{cases} y_k, n=l_k, k\ge1\\ x_{n-k+1}, l_{k-1}<n<l_k, k\ge1\end{cases} \quad (6)$$

4) For any $x\in\Sigma_M$, we define $\Delta_M:\Sigma_M\to\Sigma_M$ such that

$$\Delta_M(x)=\psi_M(x,\theta_M\circ g_M(x)) \quad (7)$$

Then we have

$$\Delta_M(x)=(x_1,(\theta_M\circ g_M(x))_1, x|_2^7,(\theta_M\circ g_M(x))|_2^3, x|_8^{24}, (\theta_M\circ g_M(x))|_4^6, x|_{25}^{48},\cdots)$$

Remark 1

1) The mappings g_M,θ_M,ψ_M are continuous and injective, the mapping Δ_M is a continuous bijective from Σ_M to $\Delta_M(\Sigma_M)$.

2) For any $x\in\Sigma_M$, $(\theta_M\circ g_M(x))_1=(\theta_M\circ g_M(x))_2=M$ and $(\theta_M\circ g_M(x))_3=2$.

For any $M\ge3$, let $S_M=\Delta_M(\Sigma_M)$ and $S=\bigcup_{M\ge3}S_M$, we have the following lemma.

Lemma 3 For any $M_1,M_2\ge3$, if $M_1\ne M_2$, then $S_{M_1}\cap S_{M_2}=\varnothing$.

Proof Without loss of generality, we assume that $M_1,M_2\ge3$. It is worth nothing that $S_{M_1}=\Delta_{M_1}(\Sigma_{M_1})=\Delta_{M_1}(\{2,3,\cdots,M_1\}^N)$ and $S_{M_2}=\Delta_{M_2}(\Sigma_{M_2})=\Delta_{M_2}(\{2,3,\cdots,M_2\}^N)$.

Hence, by the definition of Δ_{M_1} and Δ_{M_2}, for any $x\in\Sigma_{M_1}$ and $y\in\Sigma_{M_2}$, it is easy to see that M_1 appears infinitely often in $\Delta_{M_1}(x)$, but not appear in $\Delta_{M_1}(y)$, and thus $S_{M_1}\cap S_{M_2}=\varnothing$.

Lemma 4 The set S is a scrambled set of shift σ on $\{2,3,\cdots\}^N$.

Proof Let $u\in S_{M_1}$ and $v\in S_{M_2}$ such that $u\ne v$. We shall prove that (u,v) is a scrambled pair for shift σ.

Case (i) $M_1=M_2$. Let $u,v\in S_{M_1}=\Delta_{M_1}(\Sigma_M)$, then there exist two different points $x,y\in\Sigma_{M_1}$ such that $u=\Delta_{M_1}(x)$ and $v=\Delta_{M_1}(y)$. Since $x\ne y$, there exists $k\ge1$ such that $x_k\ne y_k$. Notice that the symbols x_k and y_k appear infinitely often in the same location of $\theta_{M_1}\circ g_{M_1}(x)$ and $\theta_{M_1}\circ g_{M_1}(y)$ respectively. Using the same method, we obtain that x_k and

y_k appear infinitely often in the same location of $\Delta_{M_1}(x)$ and $\Delta_{M_1}(y)$, respectively. As a result, there exists an increasing sequence $\{n_j\}_{j\geq 1}$ such that $(\sigma^{n_j}(u))_1 = x_k \neq y_k = (\sigma^{n_j}(v))_1$.

On the other hand, by (4) and (5), there exits an increasing sequence $\{m_j\}_{j\geq 1}$ such that

$$(\theta_{M_1} \circ g_{M_1}(x))\Big|_{m_j+1}^{m_j+j} = (\underbrace{2,2,\cdots,2}_{j})$$

$$= (\theta_{M_1} \circ g_{M_1}(y))\Big|_{m_j+1}^{m_j+j}$$

By the definition of Δ_{M_1}, there exists an increasing sequence $\{t_j\}_{j\geq 1}$ such that

$$u\Big|_{t_j+1}^{t_j+j} = (\Delta_{M_1}(x))\Big|_{t_j+1}^{t_j+j} = (\theta_{M_1} \circ g_{M_1}(x))\Big|_{m_j+1}^{m_j+j}$$

$$= (\underbrace{2,2,\cdots,2}_{j}) = (\theta_{M_1} \circ g_{M_1}(y))\Big|_{m_j+1}^{m_j+j}$$

$$= (\Delta_{M_1}(y))\Big|_{t_j+1}^{t_j+j} = v\Big|_{t_j+1}^{t_j+j}$$

Then we have $d(\sigma^{t_j}(u),\sigma^{t_j}(uv)) \leq 2^{-j}$ and thus $\liminf\limits_{n\to\infty} d(\sigma^n(u),\sigma^n(v)) = 0$.

Case (ii) $M_1 \neq M_2$. Let $u \in S_{M_1}$ and $v \in S_{M_2}$. Then there exist $x \in \Sigma_{M_1}$ and $y \in \Sigma_{M_2}$ such that $u = \Sigma_{M_1}(x)$ and $v = \Sigma_{M_2}(y)$. From 1) of Remark 1 we have

$$(\theta_{M_1} \circ g_{M_1}(x))_1 = M_1 \text{ and } (\theta_{M_2} \circ g_{M_2}(y))_1 = M_2$$

By the definition of Δ_M, there exists an increasing sequence $\{n_j\}_{j\geq 1}$ such that $u_{n_j+1}(\theta_{M_1} \circ g_{M_1}(x))_1 = M_1$ and $v_{n_j+1} = (\theta_{M_2} \circ g_{M_2}(y))_1 = M_2$.

It follows that $\lim\limits_{j\to\infty} d(\sigma^{n_j}(u),\sigma^{n_j}(v)) = 1$, thus $\limsup\limits_{n\to\infty} d(\sigma^n(u),\sigma^n(v)) > 0$.

The proof of lower limits is similar to the case $M_1 = M_2$. In fact, there exists an increasing sequence $\{t_i\}_{i\geq 1}$ such that $u\Big|_{t_j+1}^{t_j+j} = (\underbrace{2,2,\cdots,2}_{j}) = v\Big|_{t_j+1}^{t_j+j}$, with the same method, we get

$$\lim_{j\to\infty} d(\sigma^{t_j}(u) - \sigma^{t_j}(v)) = 0$$

and thus $\liminf\limits_{n\to\infty} d(\sigma^n(u) - \sigma^n(v)) = 0$.

It is the fact that the Luroth expansion of $x \in \Lambda^c \cap (0,1)$ is infinite and unique. Then for any $(d_1, d_2, \cdots) \in \{2,3,\cdots\}^N$, we define a continuous bijective map Φ from $\{2,3,\cdots\}^N$ to $\Lambda^c \cap (0,1)$ by $\Phi(d_1, d_2, \cdots) = [d_1, d_2, \cdots]$, then we have $\Phi \circ \sigma = T \circ \Phi$.

Lemma 5 The set $\Phi(S)$ is a scrambled set of T on $(0,1)$.

Proof For any $u, v \in S$, $u \neq v$. Let $u \in S_{M_1}$ and $v \in S_{M_2}$, by the definition of scrambled set, we shall prove that $(\Phi(u), \Phi(v))$ is a scrambled pair of T.

Lower limits By Lemma 3, we have $\liminf\limits_{n\to\infty} d(\sigma^n(u) - \sigma^n(v)) = 0$, thus there exists an increasing sequence $\{n_i\}_{i\geq 1}$ such that $\lim\limits_{i\to\infty} d(\sigma^{n_i}(u) - \sigma^{n_i}(v)) = 0$. Recall that the map Φ is continuous and $\Phi \circ \sigma = T \circ \Phi$. Then we have

$$|T^{n_i} \circ \Phi(u) - T^{n_i} \circ \Phi(v)| = |\Phi(\sigma^{n_i}(u)) - \Phi(\sigma^{n_i}(v))|$$

It is easy to see that

$$\lim_{i\to\infty} |T^{n_i} \circ \Phi(u) - T^{n_i} \circ \Phi(v)| = 0$$

and thus $\liminf\limits_{n\to\infty} |T^n \circ \Phi(u) - T^n \circ \Phi(v)| = 0$.

Upper limits We divide the proof into two cases.

Case (i) $M_1 = M_2$. Let $u, v \in S_{M_1}$, then for any $i \geq 1$, $u_i, v_i \in \{2,3,\cdots,M_1\}$. By Lemma 4, we have $\limsup\limits_{n\to\infty} d(\sigma^n(u) - \sigma^n(v)) = 1$, thus there exists an increasing sequence $\{m_i\}_{i\geq 1}$ such that $(\sigma^{m_i}(u))_1 \neq (\sigma^{m_i}(v))_1$. Without loss of generality, let $(\sigma^{m_i}(u))_1 < (\sigma^{m_i}(v))_1$, we obtain

$$\begin{aligned}
&|T^{m_i} \circ \Phi(u) - T^{m_i} \circ \Phi(v)| \\
&= |\Phi(\sigma^{m_i}(u)) - \Phi(\sigma^{m_i}(v))| \\
&\geq |I_2(\sigma^{m_i}(u))_1, M_1 + 1| \\
&= |I_1(\sigma^{m_i}(u))_1| \cdot |I_1(M_1 + 1)| \\
&\geq |I_1(M_1 + 1)|^2 > 0
\end{aligned}$$

Hence, we get $\limsup\limits_{n\to\infty} |T^n \circ \Phi(u) - T^n \circ \Phi(v)| > 0$.

Case (ii) $M_1 \neq M_2$. Let $u \in S_{M_1}$ and $v \in S_{M_2}$. From 2) of Remark 1, there exists an increasing sequence $\{t_i\}_{i\geq 1}$ such that $(\sigma^{t_i}(u))_1(\sigma^{t_i}(u))_2 = M_1$ and $(\sigma^{t_i}(v))_1 \ (\sigma^{t_i}(v))_2 = M_2$. Suppose that $M_1 < M_2$, with the same method, we have

Therefore, $\limsup\limits_{n\to\infty} |T^n \circ \Phi(u) - T^n \circ \Phi(v)| > 0$.

In order to estimate the Hausdorff dimension of $\Phi(S_M)$, we shall make use of a kind of symbolic space described as follow: for any $n \geq 1$, set

$$A_n=\{(d_1,d_2,\cdots,d_n)\in\{2,3,\cdots,M\}^n:$$
$$(d_1,d_2,\cdots,d_n)=(x_1,x_2,\cdots,x_n),\forall(x_1,x_2,\cdots,x_n)\in S_M\}$$

For any $n\ge 1$ and $(d_1,\cdots,d_n)\in A_n$, let

$$J_n(d_1,\cdots,d_n)=\bigcup_{d_{n+1}} I_{n+1}(d_1,\cdots,d_n,d_{n+1})$$

where the union is taken over all d_{n+1} such that $(d_1,\cdots,d_n,d_{n+1})\in A_{n+1}$. It is obvious that

$$\Phi(S_M)=\bigcap_{n\ge1}\bigcup_{(d_1,\cdots,d_n)\in A_n} I_n(d_1,d_2,\cdots,d_n)$$
$$=\bigcap_{n\ge1}\bigcup_{(d_1,\cdots,d_n)\in A_n} J_n(d_1,d_2,\cdots,d_n)$$

Recall that $L=\{l_k\}_{k\ge1}=\bigcup_{m=1}^{\infty}\bigcup_{t=1}^{m}\{m^3+t\}$. For any $n\ge1$ and $(d_1,\cdots,d_n)\in A_n$, let $t(n)$ be the number of k such that $l_k\le n$ and $l_k\in L$. Let $\overline{(d_1,\cdots,d_n)}$ be the block obtained by eliminating the terms $\{d_{l_k}:l_k\le n,l_k\in L\}$ in $(d_1,\cdots,d_n)$, then the length of $\overline{(d_1,\cdots,d_n)}$ is $n-t(n)$. For simplicity, set

$$\overline{I_n}(d_1,\cdots,d_n)=I_{n-t(n)}(d_1,\cdots,d_n) \tag{8}$$

Then we have $\overline{(d_1,\cdots,d_n)}\in D^{n-t(n)}$, where $D=\{2,3,\cdots,M\}$.

By the definition of $t(n)$, it is easy to check that for large enough n there exist two positive constants c_1,c_2 such that $c_1n^{\frac{2}{3}}\le t(n)\le c_2n^{\frac{2}{3}}$.

Lemma 5 For any $\varepsilon>0$, there exists N_1 such that for any $n\ge N_1$, $(d_1,\cdots,d_n)\in A_n$. We have $|I_n(d_1,\cdots,d_n)|\ge|\overline{I_n}(d_1,\cdots,d_n)|^{1+\varepsilon}$.

Proof Let $\varepsilon>0$, by (1) and (8), we have

$$|\overline{I_n}(d_1,\cdots,d_n)|^{\varepsilon}\le\frac{1}{2^{(n-t(n))\varepsilon}}\le\frac{1}{(M+1)^{2t(n)}} \tag{9}$$

By (9) and Lemma 1, it is easy to see that

$$|I_n(d_1,\cdots,d_n)|=|\overline{I_n}(d_1,\cdots,d_n)|\cdot\frac{1}{d_1(d_1-1)\cdots d_{l_k}(d_{l_k}-1)}$$
$$\ge|\overline{I_n}(d_1,\cdots,d_n)|\cdot\frac{1}{(M+1)^{2t(n)}}\ge|\overline{I_n}(d_1,\cdots,d_n)|^{1+\varepsilon}$$

For any $x\in[\eta_1,\eta_2,\cdots]\in\Phi(S_M)$, $y\in[\xi_1,\xi_2,\cdots]\in\Phi(S_M)$, without loss of generality, we assume that $x<y$. Notice that the points x and y can not be contained in the same $I_k(d)$ for any $d\in A_k$ and large enough k. Thus there exists a greatest integer n such that x,y are contained in the same basic interval of rank n, that is to say, there exists $l_{n+1}>r_{n+1}$ such that $(d_1,\cdots,d_n,l_{n+1})\in A_{n+1}$, $(d_1,\cdots,d_n,r_{n+1})\in A_{n+1}$ and $x\in I_{n+1}(d_1,\cdots,d_n,l_{n+1})$, $y\in I_{n+1}(d_1,\cdots,d_n,r_{n+1})$. Since

$$I_{n+1}(d_1,\cdots,d_n,l_{n+1})\cap\Phi(S_M)=J_{n+1}(d_1,\cdots,d_n,l_{n+1})\cap\Phi(S_M)$$

and

$$I_{n+1}(d_1,\cdots,d_n,r_{n+1})\cap\Phi(S_M)=J_{n+1}(d_1,\cdots,d_n,r_{n+1})\cap\Phi(S_M)$$

We have $x\in J_{n+1}(d_1,\cdots,d_n,l_{n+1})$, $y\in J_{n+1}(d_1,\cdots,d_n,r_{n+1})$. As a consequence, the distance $y-x$ is not less than the gap between $J_{n+1}(d_1,\cdots,d_n,l_{n+1})$ and $J_{n+1}(d_1,\cdots,d_n,r_{n+1})$.

Lemma 6 $y-x\ge\dfrac{|I_n(d_1,\cdots,d_n)|}{M^3}$

Proof Let δ_1,δ_2 denote the right endpoint of $J_{n+1}(d_1,\cdots,d_n,l_{n+1})$ and the left endpoint of $J_{n+1}(d_1,\cdots,d_n,r_{n+1})$, respectively, then

$$\delta_1=\frac{1}{d}+\sum_{j=2}^{n}\frac{1}{d_1(d_1-1)\cdots d_{j-1}(d_{j-1}-1)d_j}$$
$$+\frac{1}{d_1(d_1-1)\cdots d_n(d_n-1)l_{n+1}}$$
$$+\frac{1}{d_1(d_1-1)\cdots d_n(d_n-1)l_{n+1}(l_{n+1}-1)(2-1)},$$
$$\delta_2=\frac{1}{d}+\sum_{j=2}^{n}\frac{1}{d_1(d_1-1)\cdots d_{j-1}(d_{j-1}-1)d_j}$$
$$+\frac{1}{d_1(d_1-1)\cdots d_n(d_n-1)r_{n+1}}$$
$$+\frac{1}{d_1(d_1-1)\cdots d_n(d_n-1)r_{n+1}(r_{n+1}-1)M}$$

So $y-x$ is greater than the distance between δ_1 and δ_2, then

$$y-x\ge\delta_2-\delta_1=|I_n(d_1,\cdots,d_n)|$$
$$\cdot\left(\frac{1}{r_{n+1}}-\frac{1}{l_{n+1}}+\frac{1}{r_{n+1}(r_{n+1}-1)M}-\frac{1}{l_{n+1}(l_{n+1}-1)(2-1)}\right)$$
$$\ge\frac{|I_n(d_1,\cdots,d_n)|}{M^3}$$

Proof of Theorem 2 Consider a map $f:\Phi(S_M)\to E_M\cap\Lambda^c$ defined as follows: For any $x=[d_1,\cdots,d_n]\in\Phi(S_M)$, let $f(x)=\tilde{x}=\lim_{n\to\infty}\overline{[d_1,\cdots,d_n]}$.

For any $\varepsilon>0$, by Lemma 4, when $x,y\in\Phi(S_M)$ satisfy

$$|x-y|<\frac{1}{M^3}\min_{(d_1,\cdots,d_n)\in A_{N_1}}\{|I_{N_1}(d_1,\cdots,d_{N_1})|\}$$

where N_1 is the same as in Lemma 5, we have

$$|f(x)-f(y)|\le(M^3)^{\frac{1}{1+\varepsilon}}\cdot|x-y|^{\frac{1}{1+\varepsilon}}$$

As a result, by Ref.[17] Proposition 2.3 and $\dim_H E(S_M\cap\Lambda^c)=\dim_H E_M$, we obtain

$$\dim_H \Phi(S) \geq \frac{1}{1+\varepsilon}\dim_H E(S_M \cap \Lambda^c) = \frac{1}{1+\varepsilon}\dim_H E_M$$
$$= \frac{1}{1+\varepsilon} S_M \tag{10}$$

Recall that $S = \bigcup_{M\geq 3} S_M$, then $\dim_H \Phi(S) \geq \dim_H \Phi(S_M)$. Combining with (10), we have

$$\dim_H \Phi(S) \geq \dim_H \Phi(S_M) \geq \frac{1}{1+\varepsilon}\dim_H E_M = \frac{1}{1+\varepsilon} s_M$$

Let $\varepsilon \to 0$ and then $M \to 0$, we conclude that $\dim_H \Phi(S) = 1$.

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