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Regularity and Energy Conservation of the Nonhomogeneous Incompressible Ideal Magnetohydrodynamics Equations

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Abstract: In this paper, we study the regularity and local energy equation of the weak solutions for nonhomogeneous incompressible ideal magnetohydrodynamics system. The conditions given on the regularity of solutions guarantee the energy to be conserved. The main method we have employed relies on the commutator estimates.

Key words: ideal magnetohydrodynamics (MHD) system; regularity; energy conservation

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0 Introduction

The goal of this paper is to study the relationship between the regularity and energy conservation of the nonhomogeneous incompressible ideal magnetohydrodynamics (MHD) equations, reading as:

$$\partial_t(\rho u) + \operatorname{div}(\rho u \otimes u) = -\nabla P + \frac{1}{\mu}\operatorname{curl}B \times B \tag{1}$$

$$\partial_t \rho + \operatorname{div}(\rho u) = 0 \tag{2}$$

$$\partial_t B = \operatorname{curl}(u \times B) \tag{3}$$

$$\operatorname{div}u = 0,\ \operatorname{div}B = 0 \tag{4}$$

where ρ, u and B represent density, speed and the magnetic field, respectively. The permeability μ is a constant, and the pressure P is unknown. We consider this system in the three-dimensional periodic domain $\mathbf{T}^3$.

When $B=0$, system (1)-(4) become the well-known nonhomogeneous incompressible Euler equations. Onsager[1] asserted that the velocity of Hölder continuity with exponential $\delta > 1/3$ guarantees energy conservation, and many researchers also studied the energy conservation of the Euler equations, which can be referred to Refs.[2- 4] etc.

For the homogeneous ideal MHD equations, Caflisch, Klapper and Steele[5] proved the conservation of energy in the periodic domain by extending the results of the Euler equations to the ideal MHD equations. Then Kang and Lee[6] proved the conservation of energy and cross-helicity of the ideal MHD equations. Subsequently, Yu[7] improved the previous results by using the special structure of the nonlinear term in the ideal MHD equations.

For the nonhomogeneous MHD equations, some results have also been obtained recently, which could be referred to Refs.[8-13] etc. Among these documents, Bie *et al*[8] studied the compressible MHD equations, and they gave two sufficient conditions on the regularity of solutions to ensure energy conservation. Wu *et al*[9] considered the incompressible MHD equations, and they proved that the regularity of the solution is sufficient to guarantee the balance of the total energy in the Besov space. However the magnetic field in Ref. [9] needs to satisfy

$$\mathrm{curl}B \in B_q^{\beta,\infty}\left((0,T)\times \mathbf{T}^3\right),$$

a natural question then is whether this condition is required or not.

Inspired by Refs. [3, 4, 8] and [9], we study the energy conservation of nonhomogeneous incompressible ideal MHD system (1)-(4) in the three-dimensional periodic domain $\mathbf{T}^3$. Our strategy relies on commutator estimates similar to those employed by Constantin *et al*[4]. In order to prove the energy conservation of system (1)-(4), we first mollify the equation (1) and then select the test function ϕ to obtain an equation that approximates the energy equation. For the treatment of the nonlinear term $\mathrm{curl}\, B\times B$ in this equation, we use the condition $\mathrm{div}B=0$ to transfer the derivative to the test function ϕ by partial integral formula, which makes the regularity assumption about $\mathrm{curl}B$ in Ref.[9] unnecessary.

We briefly recall the definitions of Lebesgue space $L^p(\Omega)$ and Besov space $B_p^{\alpha,\infty}(\Omega)$, where $\Omega=(0,T)\times\mathbf{T}^3$ or $\Omega=\mathbf{T}^3$. $L^p(\Omega)=\{u:\Omega\to\mathbf{R}|u$ is Lebesgue measurable, $\|u\|_{L^p(\Omega)}<\infty\}$, where

$$\|u\|_{L^p(\Omega)}=\begin{cases}\left(\int_\Omega |u|^p\,\mathrm{d}x\right)^{\frac{1}{p}}, & 1\le p<\infty\\ \operatorname{ess\,sup}_{\Omega}|u|, & p=\infty\end{cases}$$

And $B_p^{\alpha,\infty}(\Omega)=\{\omega:\Omega\to\mathbf{R}\,\big|\,\|\omega\|_{B_p^{\alpha,\infty}(\Omega)}<\infty\}$, where

$$\|\omega\|_{B_p^{\alpha,\infty}(\Omega)}\overset{\mathrm{def}}{=}\|\omega\|_{L_p(\Omega)}+\sup_{\xi\in\Omega}\frac{\|\omega(\cdot+\xi)-\omega\|_{L_p(\Omega\cap(\Omega-\xi))}}{|\xi|^\alpha}$$

here $\Omega-\xi=\{x-\xi:x\in\Omega\}$.

The main result of this paper is stated as follows.

Theorem 1 Let (ρ,u,P,B) be a weak solution to system (1)-(4). Assume

$$u\in B_p^{\alpha,\infty}\cap L^\infty\left((0,T)\times\mathbf{T}^3\right), B\in B_q^{\beta,\infty}\left((0,T)\times\mathbf{T}^3\right)$$
$$\rho\in B_p^{\gamma,\infty}\cap L^\infty\left((0,T)\times\mathbf{T}^3\right), P\in L_{\mathrm{loc}}^{p^*}\left((0,T)\times\mathbf{T}^3\right)$$

for some $3\le p<\infty, 0<\gamma\le\alpha<1$ and $0<\beta<1$ such that

$$\frac{1}{p}+\frac{2}{q}=1,\frac{1}{p}+\frac{1}{p^*}=1,\min\{2\alpha+\gamma,\alpha+2\beta\}>1 \tag{5}$$

Then the energy is locally conserved in the sense of distribution, that is,

$$\partial_t\left(\frac{1}{2}\rho|u|^2+\frac{1}{2\mu}|B|^2\right)+\mathrm{div}\left[\left(\frac{1}{2}\rho|u|^2+P\right)u\right]$$
$$-\mathrm{div}\left[\frac{1}{\mu}(u\times B)\times B\right]=0 \tag{6}$$

Remark 1 If $B=0$, this system reduces to the nonhomogeneous incompressible Euler equations, and our result could recover the one for the incompressible Euler equations [3].

1 Preliminaries

In this section we introduce some properties of Besov space $B_p^{\alpha,\infty}(\Omega)$. Let $J\in C_c^\infty(\mathbf{R}^3)$ for $d=3$ or $d=4$ (according to the choice of Ω) be a standard mollifying kernel and set

$$J^\epsilon(x)=\frac{1}{\epsilon^d}J\left(\frac{x}{\epsilon}\right)$$

with the notation $\omega^\epsilon=J^\epsilon*\omega$. For any function ω, ω^ϵ is well-defined on

$$\Omega^\epsilon=\{x\in\Omega:\mathrm{dist}(x,\partial\Omega)>\epsilon\}$$

Referring to Ref. [1], we have the following facts about functions in the Besov spaces:

$$\|\omega^\epsilon-\omega\|_{L^p(\Omega)}\leqslant C\epsilon^\alpha\|\omega\|_{B_p^{\alpha,\infty}(\Omega)} \tag{7}$$

and

$$\|\nabla\omega^\epsilon\|_{L^p(\Omega)}\leqslant C\epsilon^{\alpha-1}\|\omega\|_{B_p^{\alpha,\infty}(\Omega)} \tag{8}$$

Moreover, $\left(B_p^{\alpha,\infty}\cap L^\infty\right)(\Omega)$ is an algebra, that is the product of two functions in this space is again contained in the space.

2 Proof of Theorem 1

By smoothing (1) in space and time, we obtain

$$\partial_t(\rho u)^\epsilon+\mathrm{div}(\rho u\otimes u)^\epsilon=-\nabla P^\epsilon+\frac{1}{\mu}(\mathrm{curl}B\times B)^\epsilon \tag{9}$$

Taking $\varphi\in C_0^\infty((0,T)\times\mathbf{T}^3)$, multiplying (9) by φu^ϵ and integrating on $(0,T)\times\mathbf{T}^3$, we get

$$\int_0^T\int_{\mathbf{T}^3}\partial_t(\rho u)^\epsilon\cdot\varphi u^\epsilon\mathrm{d}x\mathrm{d}t+\int_0^T\int_{\mathbf{T}^3}\operatorname{div}(\rho u\otimes u)^\epsilon\cdot\varphi u^\epsilon\mathrm{d}x\mathrm{d}t+\int_0^T\int_{\mathbf{T}^3}\nabla P^\epsilon\cdot\varphi u^\epsilon\mathrm{d}x\mathrm{d}t-\int_0^T\int_{\mathbf{T}^3}\frac{1}{\mu}(\operatorname{curl}B\times B)^\epsilon\cdot\varphi u^\epsilon\mathrm{d}x\mathrm{d}t=0 \tag{10}$$

here we take $\epsilon>0$ small enough so that $\operatorname{supp}\varphi\in(\epsilon,T-\epsilon)\times\mathbf{T}^3$. We can rewrite (10), using appropriate commutators, as

$$\int_0^T\int_{\mathbf{T}^3}\partial_t(\rho^\epsilon u^\epsilon)\cdot\varphi u^\epsilon\mathrm{d}x\mathrm{d}t+\int_0^T\int_{\mathbf{T}^3}\operatorname{div}\left((\rho u)^\epsilon\otimes u^\epsilon\right)\cdot\varphi u^\epsilon\mathrm{d}x\mathrm{d}t+\int_0^T\int_{\mathbf{T}^3}\nabla P^\epsilon\cdot\varphi u^\epsilon\mathrm{d}x\mathrm{d}t+\int_0^T\int_{\mathbf{T}^3}\frac{1}{\mu}\varphi(u\times B)^\epsilon\cdot\operatorname{curl}B^\epsilon\mathrm{d}x\mathrm{d}t=R_1^\epsilon+R_2^\epsilon+R_3^\epsilon \tag{11}$$

where

$$R_1^\epsilon=\int_0^T\int_{\mathbf{T}^3}\partial_t\left[\rho^\epsilon u^\epsilon-(\rho u)^\epsilon\right]\cdot\varphi u^\epsilon\mathrm{d}x\mathrm{d}t$$

$$R_2^\epsilon=\int_0^T\int_{\mathbf{T}^3}\operatorname{div}\left[(\rho u)^\epsilon\otimes u^\epsilon-(\rho u\otimes u)^\epsilon\right]\cdot\varphi u^\epsilon\mathrm{d}x\mathrm{d}t$$

$$R_3^\epsilon=\int_0^T\int_{\mathbf{T}^3}\frac{1}{\mu}\varphi\left[(\operatorname{curl}B\times B)^\epsilon\cdot u^\epsilon+(u\times B)^\epsilon\cdot\operatorname{curl}B^\epsilon\right]\mathrm{d}x\mathrm{d}t$$

In order to prove our theorem, the first step is to show that the terms in the left side of (11) converge to the ones of the equation (6). For the first three terms of the left side of (11), we refer to the proof of Ref. [3](Theorem 3.1), and get

$$\int_0^T\int_{\mathbf{T}^3}\partial_t(\rho^\epsilon u^\epsilon)\cdot\varphi u^\epsilon\mathrm{d}x\mathrm{d}t=\int_0^T\int_{\mathbf{T}^3}\left(\varphi\partial_t\rho^\epsilon\left|u^\epsilon\right|^2+\frac{1}{2}\varphi\rho^\epsilon\partial_t\left|u^\epsilon\right|^2\right)\mathrm{d}x\mathrm{d}t \tag{12}$$

$$\int_0^T\int_{\mathbf{T}^3}\operatorname{div}\left((\rho u)^\epsilon\otimes u^\epsilon\right)\cdot\varphi u^\epsilon\mathrm{d}x\mathrm{d}t=-\frac{1}{2}\int_0^T\int_{\mathbf{T}^3}\left(\varphi\partial_t\rho^\epsilon\left|u^\epsilon\right|^2+\left((\rho u)^\epsilon\cdot\nabla\varphi\right)\left|u^\epsilon\right|^2\right)\mathrm{d}x\mathrm{d}t \tag{13}$$

$$\int_0^T\int_{\mathbf{T}^3}\nabla P^\epsilon\cdot\varphi u^\epsilon\mathrm{d}x\mathrm{d}t=-\int_0^T\int_{\mathbf{T}^3}\nabla\varphi\cdot u^\epsilon P^\epsilon\mathrm{d}x\mathrm{d}t \tag{14}$$

And for the fourth term, we first introduce the following equality about curl,

$$\operatorname{div}(a\times b)=\operatorname{curl}a\cdot b-a\cdot\operatorname{curl}b$$

By smoothing (3) and multiplying it by B^ϵ, we obtain

$$\frac{1}{2}\partial_t\left|B^\epsilon\right|=\operatorname{curl}(u\times B)^\epsilon\cdot B^\epsilon$$

Then

$$\begin{aligned}&\int_0^T\int_{\mathbf{T}^3}\frac{1}{\mu}\varphi(u\times B)^\epsilon\cdot\operatorname{curl}B^\epsilon\mathrm{d}x\mathrm{d}t\\&=\int_0^T\int_{\mathbf{T}^3}\frac{1}{\mu}\varphi\operatorname{curl}(u\times B)^\epsilon\cdot B^\epsilon\mathrm{d}x\mathrm{d}t-\int_0^T\int_{\mathbf{T}^3}\frac{1}{\mu}\varphi\operatorname{div}\left((u\times B)^\epsilon\times B^\epsilon\right)\mathrm{d}x\mathrm{d}t\\&=\int_0^T\int_{\mathbf{T}^3}\frac{1}{\mu}\varphi\operatorname{curl}(u\times B)^\epsilon\cdot B^\epsilon\mathrm{d}x\mathrm{d}t+\int_0^T\int_{\mathbf{T}^3}\frac{1}{\mu}(u\times B)^\epsilon\times B^\epsilon\cdot\nabla\varphi\mathrm{d}x\mathrm{d}t\\&=\int_0^T\int_{\mathbf{T}^3}\frac{1}{2\mu}\varphi\partial_t\left|B^\epsilon\right|^2\mathrm{d}x\mathrm{d}t+\int_0^T\int_{\mathbf{T}^3}\frac{1}{\mu}(u\times B)^\epsilon\times B^\epsilon\cdot\nabla\varphi\mathrm{d}x\mathrm{d}t\\&=-\int_0^T\int_{\mathbf{T}^3}\frac{1}{2\mu}\partial_t\varphi\left|B^\epsilon\right|^2\mathrm{d}x\mathrm{d}t+\int_0^T\int_{\mathbf{T}^3}\frac{1}{\mu}(u\times B)^\epsilon\times B^\epsilon\cdot\nabla\varphi\mathrm{d}x\mathrm{d}t\end{aligned} \tag{15}$$

Thus, combining (11), (12), (13), (14) and (15), we find

$$\int_0^T\int_{\mathbf{T}^3}\partial_t\varphi\left(\frac{1}{2}\rho^\epsilon\left|u\right|^\epsilon+\frac{1}{2\mu}\left|B^\epsilon\right|^2\right)\mathrm{d}x\mathrm{d}t+\int_0^T\int_{\mathbf{T}^3}\nabla\varphi\cdot\left[\frac{1}{2}(\rho u)^\epsilon\left|u^\epsilon\right|^2+P^\epsilon u^\epsilon\right]\mathrm{d}x\mathrm{d}t-\int_0^T\int_{\mathbf{T}^3}\nabla\varphi\cdot\left[\frac{1}{\mu}(u\times B)^\epsilon\times B^\epsilon\right]\mathrm{d}x\mathrm{d}t=-R_1^\epsilon-R_2^\epsilon-R_3^\epsilon \tag{16}$$

To prove our result, it suffices to show $R_1^\epsilon, R_2^\epsilon, R_3^\epsilon\to0$ as $\epsilon\to0$. For the treatment of R_1^ϵ and R_2^ϵ, we refer to the proof of Theorem 3.1 of Ref. [3], and get

$$\begin{aligned}&\left|\int_0^T\int_{\mathbf{T}^3}\partial_t\left[\rho^\epsilon u^\epsilon-(\rho u)^\epsilon\right]\cdot\varphi u^\epsilon\mathrm{d}x\mathrm{d}t\right|\\&\leqslant C\|\varphi\|_{C_1}\epsilon^\gamma\epsilon^\alpha\|\rho\|_{B_q^{\gamma,\infty}}\|u\|_{B_p^{\alpha,\infty}}^2+C\|\varphi\|_{C^0}\epsilon^\gamma\epsilon^\alpha\epsilon^{\alpha-1}\|\rho\|_{B_q^{\gamma,\infty}}\|u\|_{B_p^{\alpha,\infty}}^2\to0\end{aligned} \tag{17}$$

and

$$\begin{aligned}&\left|\int_0^T\int_{\mathbf{T}^3}\operatorname{div}\left[(\rho u)^\epsilon\otimes u^\epsilon-(\rho u\otimes u)^\epsilon\right]\cdot\varphi u^\epsilon\mathrm{d}x\mathrm{d}t\right|\\&\leqslant C\|\varphi\|_{C^0}\epsilon^\gamma\epsilon^\alpha\epsilon^{\alpha-1}\|\rho u\|_{B_q^{\gamma,\infty}}\|u\|_{B_p^{\alpha,\infty}}^2+C\|\varphi\|_{C^1}\epsilon^\gamma\epsilon^\alpha\|\rho u\|_{B_q^{\gamma,\infty}}\|u\|_{B_p^{\alpha,\infty}}^2\to0\end{aligned} \tag{18}$$

here we request $\rho u\in B_p^{\gamma,\infty}\left((0,T)\times\mathbf{T}^3\right)$. In fact when $\alpha\geqslant\gamma$, we know $B_p^{\alpha,\infty}\subset B_p^{\gamma,\infty}$ and

$$\frac{\|(\rho u)(\cdot+\xi)-\rho u\|_{L^p}}{|\xi|^\gamma}$$
$$\leqslant\frac{\|\rho(u(\cdot+\xi)-u)\|_{L^p}}{|\xi|^\gamma}+\frac{\|(\rho(\cdot+\xi)-\rho)u(\cdot+\xi)\|_{L^p}}{|\xi|^\gamma}$$
$$\leqslant\|\rho\|_{L^\infty}\|u\|_{B_p^{\gamma,\infty}}+\|u\|_{L^\infty}\|\rho\|_{B_p^{\gamma,\infty}}$$
$$\leqslant\|\rho\|_{L^\infty}\|u\|_{B_p^{\alpha,\infty}}+\|u\|_{L^\infty}\|\rho\|_{B_p^{\gamma,\infty}}$$

which yields that $\rho u\in B_p^{\gamma,\infty}\left((0,T)\times\mathbf{T}^3\right)$. As for R_3^ϵ, in view of the equality

$$\left(\text{curl}B^\epsilon\times B^\epsilon\right)\cdot u^\epsilon=-\left(u^\epsilon\times B^\epsilon\right)\cdot\text{curl}B^\epsilon$$

we have

$$R_3^\epsilon=\int_0^T\int_{\mathbf{T}^3}\frac{1}{\mu}\varphi\left[\left(\text{curl}B\times B\right)^\epsilon\cdot u^\epsilon+\left(u\times B\right)^\epsilon\cdot\text{curl}B^\epsilon\right]\text{d}x\text{d}t$$
$$=\int_0^T\int_{\mathbf{T}^3}\frac{1}{\mu}\varphi\left[\left(\text{curl}B\times B\right)^\epsilon\cdot u^\epsilon-\left(\text{curl}B^\epsilon\times B^\epsilon\right)\cdot u^\epsilon\right]\text{d}x\text{d}t$$
$$+\int_0^T\int_{\mathbf{T}^3}\frac{1}{\mu}\varphi\left[\left(u\times B\right)^\epsilon\cdot\text{curl}B^\epsilon-\left(u^\epsilon\times B^\epsilon\right)\cdot\text{curl}B^\epsilon\right]\text{d}x\text{d}t$$
$$\overset{\text{def}}{=}R_{31}^\epsilon+R_{32}^\epsilon \tag{19}$$

Here R_{31}^ϵ can be estimated as

$$\int_0^T\int_{\mathbf{T}^3}\frac{1}{\mu}\varphi\left[\left(\text{curl}B\times B\right)^\epsilon\cdot u^\epsilon-\left(\text{curl}B^\epsilon\times B^\epsilon\right)\cdot u^\epsilon\right]\text{d}x\text{d}t$$
$$=\int_0^T\int_{\mathbf{T}^3}\frac{1}{\mu}\left[\left(B\cdot\nabla B\right)^\epsilon-\frac{1}{2}\nabla\left(|B|^2\right)^\epsilon\right]\cdot\varphi u^\epsilon\text{d}x\text{d}t$$
$$-\int_0^T\int_{\mathbf{T}^3}\frac{1}{\mu}\left[\left(B^\epsilon\cdot\nabla B^\epsilon\right)-\frac{1}{2}\nabla\left(|B|^\epsilon\right)^2\right]\cdot\varphi u^\epsilon\text{d}x\text{d}t$$
$$=\int_0^T\int_{\mathbf{T}^3}\frac{1}{\mu}\left[\left(B\cdot\nabla B\right)^\epsilon-\left(B^\epsilon\cdot\nabla B^\epsilon\right)\right]\cdot\varphi u^\epsilon\text{d}x\text{d}t$$
$$-\int_0^T\int_{\mathbf{T}^3}\frac{1}{\mu}\left[\frac{1}{2}\nabla\left(|B|^2\right)^\epsilon-\frac{1}{2}\nabla\left(|B|^\epsilon\right)^2\right]\cdot\varphi u^\epsilon\text{d}x\text{d}t$$
$$=\int_0^T\int_{\mathbf{T}^3}\frac{1}{\mu}\left[\partial_i\left(B_iB_j\right)^\epsilon-\partial_i\left(B_i^\epsilon B_j^\epsilon\right)\right]\cdot\varphi u^\epsilon\text{d}x\text{d}t$$
$$-\int_0^T\int_{\mathbf{T}^3}\frac{1}{\mu}\left[\frac{1}{2}\partial_i\left(|B_i|^2\right)^\epsilon-\frac{1}{2}\partial_i\left(|B_i|^\epsilon\right)^2\right]\cdot\varphi u^\epsilon\text{d}x\text{d}t$$
$$\overset{\text{def}}{=}I_1+I_2$$

Because the estimates of I_1 and I_2 are the same, we only calculate I_1. Firstly we observe that

$$B_i^\epsilon B_j^\epsilon-\left(B_iB_j\right)^\epsilon=\left(B_i^\epsilon-B_i\right)\left(B_j^\epsilon-B_j\right)$$
$$-\int_{-\epsilon}^{\epsilon}\int_{\mathbf{T}^3}J^\epsilon(x,\xi)\left(B_i(t-\tau,x-\xi)-B_i(t,x)\right)$$
$$\times\left(B_j(t-\tau,x-\xi)-B_j(t,x)\right)\text{d}\xi\text{d}\tau$$

By using Hölder inequality and Fubini theorem, we estimate I_1 as

$$\left|\int_0^T\int_{\mathbf{T}^3}\frac{1}{\mu}\partial_i\left[\left(B_i^\epsilon-B_i\right)\left(B_j^\epsilon-B_j\right)\right]\cdot\varphi u^\epsilon\text{d}x\text{d}t\right|$$
$$\leqslant C\|\varphi\|_{C^0}\|B_i^\epsilon-B_i\|_{L^p}\|B_j^\epsilon-B_j\|_{L^p}\|\text{div}u^\epsilon\|_{L^p}$$
$$+C\|\varphi\|_{C^1}\|B_i^\epsilon-B_i\|_{L^p}\|B_j^\epsilon-B_j\|_{L^p}\|u^\epsilon\|_{L^p}$$
$$\leqslant C\|\varphi\|_{C^0}\epsilon^\beta\epsilon^\beta\epsilon^{\alpha-1}\|B\|_{B_p^{\beta,\infty}}^2\|u\|_{B_p^{\alpha,\infty}}$$
$$+C\|\varphi\|_{C^1}\epsilon^\beta\epsilon^\beta\epsilon^\alpha\|B\|_{B_p^{\beta,\infty}}^2\|u\|_{B_p^{\alpha,\infty}} \tag{20}$$

and

$$\left|\int_0^T\int_{\mathbf{T}^3}\partial_i\left(\int_{-\epsilon}^{\epsilon}\int_{\mathbf{T}^3}J^\epsilon(x,\xi)\left(B_i(t-\tau,x-\xi)-B_i(t,x)\right)\right.\right.$$
$$\left.\left.\times\left(B_j(t-\tau,x-\xi)-B_j(t,x)\right)\text{d}\xi\text{d}\tau\right)\cdot\varphi u^\epsilon\text{d}x\text{d}t\right|$$
$$\leqslant C\|\varphi\|_{C^0}\epsilon^\beta\epsilon^\beta\epsilon^{\alpha-1}\|B\|_{B_p^{\beta,\infty}}^2\|u\|_{B_p^{\alpha,\infty}}$$
$$+C\|\varphi\|_{C^1}\epsilon^\beta\epsilon^\beta\epsilon^\alpha\|B\|_{B_p^{\beta,\infty}}^2\|u\|_{B_p^{\alpha,\infty}} \tag{21}$$

The estimate for R_{32}^ϵ is similar to that of I_1. Then

$$\left|\int_0^T\int_{\mathbf{T}^3}\frac{1}{\mu}\varphi\left[\left(u\times B\right)^\epsilon\cdot\text{curl}B^\epsilon-\left(u^\epsilon\times B^\epsilon\right)\cdot\text{curl}B^\epsilon\right]\text{d}x\text{d}t\right|$$
$$\leqslant C\|\varphi\|_{C^0}\epsilon^\alpha\epsilon^\beta\epsilon^{\beta-1}\|u\|_{B_p^{\alpha,\infty}}\|B\|_{B_p^{\beta,\infty}}^2 \tag{22}$$

Collecting all the above estimates and putting them into (16), we obtain

$$\int_0^T\int_{\mathbf{T}^3}\partial_t\varphi\left(\frac{1}{2}\rho|u|^2+\frac{1}{2\mu}|B|^2\right)\text{d}x\text{d}t$$
$$+\int_0^T\int_{\mathbf{T}^3}\nabla\varphi\cdot\left[\frac{1}{2}(\rho u)|u|^2+Pu\right]\text{d}x\text{d}t$$
$$-\int_0^T\int_{\mathbf{T}^3}\nabla\varphi\cdot\left[\frac{1}{\mu}(u\times B)\times B\right]\text{d}x\text{d}t=0$$

which completes the proof of Theorem 1.

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