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# Stability Research of an SEIR Model with Distinct General Contact Rates and Infectious Force in Latent and Recovered Period

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**Abstract:** In this work, an SEIR infectious model with distinct general contact rates and infectious force in latent and recovered period is established, and the stability of the model is studied using theoretical and numerical methods. First, we derive the basic reproduction number $R_0$, which determines whether the disease is extinct or not. Second, using the LaSalle's invariance principle, we show that the disease-free equilibrium is globally asymptotically stable and the disease always dies out when $R_0 < 1$. On the other hand, by Routh-Hurwitz criterion theory, we also prove that the disease-free equilibrium is unstable and that the unique endemic equilibrium is locally asymptotically stable when $R_0 > 1$. Third, through the method of autonomous convergence theorem, we obtain that the unique endemic equilibrium is globally asymptotically stable and the disease persists at this endemic equilibrium when $R_0 > 1$. Finally, numerical simulations are carried out to confirm the theoretical analysis.

**Key words:** basic reproductive number; equilibrium; stability; SEIR epidemic model; general contact rate

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## 0 Introduction

Epidemiology is the research hotspots in the spread of infectious diseases in order to trace the factors that cause the occurrence. Mathematical models describing the behavior of infectious population have long played an important role in better analyzing disease control and transmission patterns. To predict the development of infectious diseases between regions, the transmission behavior of infectious diseases is studied in host population through a number of epidemic models. In recent years, many mathematical models of ordinary differential equations had been proposed for the transmission dynamics of infectious diseases [1-13]. These models provided theoretical and quantitative bases for the prevention and control of infectious diseases.

However, some epidemic diseases, such as viral hepatitis, SARS, and whooping cough, become contagious only after they have been dormant in the population for a period of time.

Therefore, more general systems than the types of SIRS and SIR need to be considered to research the role of hatch in the development of infectious diseases. It can be supposed that susceptible population first experience an incubation period after infection before becoming infectious. The most general form of an epidemic model is an SEIR model consisting of four population subclasses: *S*-susceptible, *E*-exposed, *I* -infected and *R*-recovered. All other models are cases of the SEIR model under various parameter restrictions. If there is no immunity and hence no *R* class, the SEIS model is obtained, which means the

average period of immunity tends to be zero.

Many epidemic models with the infectious force in the latent period have been performed. Li [14] analyzed the global dynamics of an SEIR model with varying total population size. Sun [15] discussed the global dynamics of an SEIR model with a varying total population size and vaccination. Sun and Hsiehc[16] studied the global analysis of an SEIR model with varying population size and vaccination. Yuan *et al* [17] considered threshold dynamics in an SEIRS model with latency and temporary immunity. Zhang[18] studied global asymptotic stability of a delayed SEIRS epidemic model with saturation incidence. Zhang *et al*[19] considered an SEIR epidemic system with nonlinear transmission rate. Trawicki[20] proposed a deterministic SEIRS epidemic model for modeling vital dynamics, vaccinations, and temporary immunity. We can see that the disease models mentioned above do not consider the infectious force in the latent and recovered period. However, for infectious diseases such as hepatitis C and condyloma acuminatum, incubation and recovery periods may be contagious. Although it is difficult to measure infectivity exactly, it is generally accepted that the degree of infectivity declines with the acquisition of immunity.

The incidence of diseases plays an extremely important role in the analysis of infectious disease models. Incidence should be recorded as $\frac{\beta C(N)SI}{N}$, where $N$ is the total population size. In many models of epidemiology, the bilinear incidence rate $\beta SI$ and the standard incidence rate $\frac{\beta SI}{N}$ are often considered[20-22]. The bilinear incidence rate is duce to the law of mass action. It has been pointed out that for standard incidence rate, it may be a good approximation if the number of available partners is large enough and everybody could not make more contacts than it is practically feasible[23, 24]. It has been suggested by several authors that the disease transmission process may have a saturation incidence rate, generally written as $\frac{aSI}{1+bI}$, where $aI$ measures the infectivity of the disease and $\frac{1}{1+bI}$ measures the inhibitory effect of behavioral changes or the crowding effect of infected individuals when the number of susceptible individuals increases. Compared with bilinear and standard incidence rate, saturating incidence rate may be a better fit for our actual situation. Several authors have studied epidemic models with saturation incidence rate[25-29]. In this paper, we introduce the general contact rate $\beta(N)SI$ into the infectious disease model, which is more extensive.

In this paper, motivated by the work of Refs.[13-30], we study an SEIR epidemic model with different general contact rates, which also has the infectious force in the exposed and recovered period. In Section 1, we formulate an SEIR model with different general contact rates, which also has the infectious force in the exposed and recovered period. In Section 2, we determine the basic reproduction number, obtain the existence of equilibriums and study the global stability of disease-free equilibrium by LaSalle's invariance principle. In Section 3, we analyze the local stability of the unique endemic equilibrium by Routh-Hurwitz criterion theory. In Section 4, we prove the unique endemic equilibrium is globally asymptotically stable through the limiting equation theory and the periodic orbit stability theory. In Section 5, we carry out numerical simulations to confirm the theoretical analysis. In the last section, we give a conclusion and prospect for the research work.

# 1 Model Formulation

In this work, we are interested in an SEIR model with different general contact rates, which also has the infectious force in the exposed and recovered period.

The total population is assumed to be divided into four distinct epidemiological subgroups, namely susceptible, latent, infectious and recovered (removed), with sizes denoted by $S(t)$, $E(t)$, $I(t)$, and $R(t)$, respectively. The size of the total population is expressed in $N(t)$ at time $t$, where $N(t)=S(t)+E(t)+I(t)+R(t)$. The transfer mechanism for the class $S$ to the class $E$ is guided by the function:

$$f(t)=\beta_1(N)SE+\beta_2(N)SI+\beta_3(N)SR$$

where $\beta_i(N)(i=1,2,3)$ are general contact rate, which satisfies the following assumptions, for $N>0$, $\beta_i(N)>0$, $\beta_i'(N)\le 0$, $(N\beta_i(N))'\ge 0$, $(\beta_i(N))^2+((N\beta_i(N))')^2>0$.

We construct the ordinary differential equation of the SEIR epidemic model as follows:

$$\begin{cases}\dfrac{\mathrm{d}S}{\mathrm{d}t}=A-\beta_1(N)SE-\beta_2(N)SI-\beta_3(N)SR-dS\\ \dfrac{\mathrm{d}E}{\mathrm{d}t}=\beta_1(N)SE+\beta_2(N)SI+\beta_3(N)SR-(\gamma+d+\alpha_1+k_1)E\\ \dfrac{\mathrm{d}I}{\mathrm{d}t}=\gamma E-(\varepsilon+d+\alpha_2+k_2)\ I\\ \dfrac{\mathrm{d}R}{\mathrm{d}t}=\varepsilon I-dR\end{cases} \tag{1}$$

where $A$ is a constant recruitment rate of the population; $\beta_1(N)SE$, $\beta_2(N)SI$, $\beta_3(N)SR$ are general contact rates in the latent, infectious and recovered period, respectively; $\alpha_1, \alpha_2$ are the rates of disease-caused death of the exposed and the infectious, respectively; $d$ is the natural death rate of the population; $k_1$, $k_2$ are the elimination rates of the exposed and the infectious, respectively; $\gamma$ is the transfer rates between the exposed and the infectious; $\varepsilon$ is the removed rate from the infective class to the recovered class. The above parameters are positive.

umming up the four equations of system (1) and denoting

$$N(t)=S(t)+E(t)+I(t)+R(t)$$

having

$$N'=A-dN-(\alpha_1+k_1)E-(\alpha_2+k_2)I$$

From the formula of $N'$, we obtain $\lim\limits_{t\to+\infty}\sup(N)=\dfrac{A}{d}$.

From a biological point of view, we study system (1) in the following feasible region

$$D=\left\{(S,E,I,R)\in\mathbf{R}_+^4\,\middle|\,S\geq 0,E\geq 0,I\geq 0,R\geq 0,S+E+I+R\leq\frac{A}{d}\right\}$$

where $\mathbf{R}_+^4$ represents the non-negative cone and its lower dimensional faces. It can be shown that $D$ is positively invariant with respect to (1).

## 2 Stability Analysis of the Disease-Free Equilibrium

In this work, we aim to show the existence of a disease-free equilibrium and an endemic equilibrium of system (1) by solving equations and study the global stability of the disease-free equilibrium $E_0(A/d,0,0,0)$ by Routh-Hurwitz criterion theory and LaSalle's invariance principle.

Let the right side of equation (1) be zero, and by calculating formula (2), we get

$$\begin{cases} S=\dfrac{A\left((\delta-d)\omega-\gamma(d+\varepsilon)\right)}{(\beta_1(N)\omega+\beta_2(N)\gamma+\beta_3(N)\dfrac{\varepsilon\gamma}{d})(A-dN)+d\left((\delta-d)\omega-\gamma(d+\varepsilon)\right)} \\ E=\dfrac{\omega(A-dN)}{(\delta-d)\omega-\gamma(d+\varepsilon)} \\ I=\dfrac{\gamma(A-dN)}{(\delta-d)\omega-\gamma(d+\varepsilon)} \\ R=\dfrac{\dfrac{\varepsilon\gamma}{d}(A-dN)}{(\delta-d)\omega-\gamma(d+\varepsilon)} \end{cases} \tag{2}$$

where $\delta=d+\alpha_1+k_1+\gamma$, $\omega=d+\alpha_2+k_2+\varepsilon$, and we have the following equation about $N$:

$$F(N)(A-dN)=0$$

where

$$\begin{aligned} F(N)=&-\frac{A}{d}\left(\beta_1(N)\omega+\beta_2(N)\gamma+\beta_3(N)\varepsilon\gamma/d\right)\\ &\cdot\left(\omega+\gamma+\varepsilon\gamma/d\right)-\delta\omega d\left((\delta-d)\omega-\gamma(d+\varepsilon)\right)\\ &+\delta\omega dN\left(\beta_1(N)\omega+\beta_2(N)\gamma+\beta_3(N)\varepsilon\gamma/d\right)\end{aligned}$$

So system (2) always has the disease-free equilibrium $E_0\left(\dfrac{A}{d},0,0,0\right)$ in the interval $\left(0,\dfrac{A}{d}\right)$. Since

$$\begin{aligned} F(0)=&-\frac{A}{d}\left(\beta_1(0)\omega+\beta_2(0)\gamma+\beta_3(0)\frac{\varepsilon\gamma}{d}\right)(\omega+\gamma)\\ &-\delta\omega d\left((\delta-d)\omega-\gamma(d+\varepsilon)\right)<0\end{aligned}$$

$$F\left(\frac{A}{d}\right)=\delta\omega d\left((\delta-d)\omega-\gamma(d+\varepsilon)\right)$$

$$\cdot\left(\frac{A}{d}\frac{\left(\beta_1\left(\dfrac{A}{d}\right)\omega+\beta_2\left(\dfrac{A}{d}\right)\gamma+\beta_3\left(\dfrac{A}{d}\right)\dfrac{\varepsilon\gamma}{d}\right)}{\delta\omega}-1\right)$$

Let

$$R_0=\frac{A}{d}\frac{\beta_1\left(\dfrac{A}{d}\right)\omega+\beta_2\left(\dfrac{A}{d}\right)\gamma+\beta_3\left(\dfrac{A}{d}\right)\dfrac{\varepsilon\gamma}{d}}{\delta\omega}$$

$R_0$ is called the basic reproduction number of system (1).

It is easy to get that if $R_0>1$, $F(0)<0$, $F\left(\dfrac{A}{d}\right)>0$.

And

$$\begin{aligned} F'(N)=&-\frac{A}{d}(\omega+\gamma)\left(\beta_1'(N)\omega+\beta_2'(N)\gamma+\beta_3'(N)\frac{\varepsilon\gamma}{d}\right)\\ &+\delta\omega d\left(\left(N\beta_1(N)\right)'\omega+\left(N\beta_2(N)\right)'\gamma+\left(N\beta_3(N)\right)'\frac{\varepsilon\gamma}{d}\right)\geq 0\end{aligned}$$

Hence, the equation $F(N)=0$ has only a positive root in the interval $\left(0, \frac{A}{d}\right)$, that is, system (1) has an unique endemic equilibrium $E^*\left(S^*, E^*, I^*, R^*\right)$, where $S^*, E^*, I^*, R^*$ are determined by (2). It is easy to obtain the following theorem.

**Theorem 1** If $R_0 \le 1$, the disease-free equilibrium $E_0$ of system (1) is globally asymptotically stable. If $R_0 > 1$, the disease-free equilibrium $E_0$ is unstable and the solutions to system (1) starting sufficiently close to $E_0$ in $D$ move away from $E_0$ except those starting on the invariant $S$-axis approach $E_0$ along this axis.

**Proof** To prove this result, we consider the following Lyapunov function

$$V = \frac{\beta_1(N)\omega + \beta_2(N)\gamma + \beta_3(N)\frac{\varepsilon\gamma}{d}}{\delta\omega} E + \frac{\beta_2(N) + \beta_3(N)\frac{\varepsilon}{d}}{\omega} I + + \frac{\beta_3(N)}{d} R$$

Calculating the derivative of $V(t)$ along the positive solution of system (1), we get

$$\begin{aligned}\left.\frac{\mathrm{d}V}{\mathrm{d}t}\right|_{(1)} &= \left(\beta_1(N)E + \beta_2(N)I + \beta_3(N)R\right) \cdot \left(\frac{\beta_1(N)\omega + \beta_2(N)\gamma + \beta_3(N)\frac{\varepsilon\gamma}{d}}{\delta\omega} S - 1\right) \\ &\le \left(\beta_1(N)E + \beta_2(N)I + \beta_3(N)R\right) \cdot \left(\frac{\beta_1\left(\frac{A}{d}\right)\omega + \beta_2\left(\frac{A}{d}\right)\gamma + \beta_3\left(\frac{A}{d}\right)\frac{\varepsilon\gamma}{d}}{\delta\omega} \frac{A}{d} - 1\right) \\ &= \left(\beta_1(N)E + \beta_2(N)I + \beta_3(N)R\right)\left(R_0 - 1\right).\end{aligned}$$

Furthermore, $V'(t) = 0$ only if $E(t) = I(t) = R(t) = 0$ or $R_0 = 1$. The largest invariant set in $\{(S, E, I, R) \in \mathbf{R}_+^4 \mid V'(t) = 0\}$ is the singleton $\{E_0\}$. When $R_0 \le 1$, the global asymptotical stability of the disease-free equilibrium $E_0$ follows from LaSalle's invariance. When $R_0 > 1$, $S$ sufficiently closes to $A/d$ except $E(t) = I(t) = R(t) = 0$, we have the relation $V'(t) > 0$. Hence the solutions to system (1) starting sufficiently close to $E_0$ in $D$ move away from $E_0$ except those starting on the invariant $S$-axis approach $E_0$ along this axis.

## 3 Local Stability Analysis of the Endemic Equilibrium

In this section, we analyze the local stability of the endemic equilibrium $E^*$ of system (1) with the help of the Routh-Hurwitz criterion theory. We have the following result.

**Theorem 2** If $R_0 > 1$, the endemic equilibrium $E^*$ of system (1) is locally asymptotically stable.

**Proof** The Jacobian matrix of system (1) at the $E^*$ is

$$\boldsymbol{J}\left(E^*\right) = \begin{pmatrix} a_1 - d & a_2 & a_3 & a_4 \\ -a_1 & -a_2 - \delta & -a_3 & -a_4 \\ 0 & \gamma & -\omega & 0 \\ 0 & 0 & \varepsilon & -d \end{pmatrix}$$

where

$$\begin{aligned} a_1 &= -\beta_1\left(N^*\right)E^* - \beta_1'\left(N^*\right)S^*E^* - \beta_2\left(N^*\right)I^* \\ &\quad - \beta_2'\left(N^*\right)S^*I^* - \beta_3\left(N^*\right)R^* - \beta_3'\left(N^*\right)S^*R^* \\ a_2 &= -\beta_1\left(N^*\right)S^* - \beta_1'\left(N^*\right)S^*E^* - \beta_2'\left(N^*\right)S^*I^* \\ &\quad - \beta_3'\left(N^*\right)S^*R^* \\ a_3 &= -\beta_1'\left(N^*\right)S^*E^* - \beta_2\left(N^*\right)S^* \\ &\quad - \beta_2'\left(N^*\right)S^*I^* - \beta_3'\left(N^*\right)S^*R^* \\ a_4 &= -\beta_1'\left(N^*\right)S^*E^* - \beta_2'\left(N^*\right)S^*I^* \\ &\quad - \beta_3\left(N^*\right)S^* - \beta_3'\left(N^*\right)S^*R^* \end{aligned}$$

Then, the characteristic equation of $\boldsymbol{J}(E^*)$ at $E^*$ is of the form

$$\det\left(\lambda\boldsymbol{I} - \boldsymbol{J}\left(E^*\right)\right) = 0$$

where

$$S^* = \frac{\delta\omega}{\beta_1\left(N^*\right)\omega + \beta_2\left(N^*\right)\gamma + \beta_3\left(N^*\right)\varepsilon\gamma/d}, \quad E^* = \frac{\omega}{\gamma} I^*, \quad R^* = \frac{\varepsilon}{d} I^*$$

By calculation, we have

$$\left(\lambda + d\right)\left(\lambda^3 + b_1\lambda^2 + b_2\lambda + b_3 + \varepsilon\gamma a_4\right) = 0$$

where

$$b_1 = \omega + d + \beta_1\left(N^*\right)E^* + \beta_2\left(N^*\right)I^* + \beta_3\left(N^*\right)R^* + \frac{\delta\left(\beta_2\left(N^*\right)\gamma + \beta_3\left(N^*\right)\varepsilon\gamma/d\right)}{\beta_1\left(N^*\right)\omega + \beta_2\left(N^*\right)\gamma + \beta_3\left(N^*\right)\varepsilon\gamma/d} > 0$$

$$b_2 = d\omega-(\omega+d+\gamma)\left(\beta_1'\left(N^*\right)S^*E^*+\beta_2'\left(N^*\right)S^*I^*\right.$$
$$\left.+\beta_3'\left(N^*\right)S^*R^*\right)+(\delta+\omega)\left(\beta_1\left(N^*\right)E^*+\beta_1'\left(N^*\right)S^*E^*\right.$$
$$\left.+\beta_1\left(\left(N^*\right)E^*+\beta_2'\left(N^*\right)S^*I^*+\beta_3\left(N^*\right)R^*+\beta_3'\left(N^*\right)S^*R^*\right)\right.$$
$$+\frac{\mathrm{d}\delta\left(\beta_2\left(N^*\right)\gamma+\beta_3\left(N^*\right)\varepsilon\gamma/d\right)}{\beta_1\left(N^*\right)\omega+\beta_2\left(N^*\right)\gamma+\beta_3\left(N^*\right)\varepsilon\gamma/d}>0$$

$$b_3 = d\omega\delta+\omega\delta\left(\beta_1\left(N^*\right)E^*+\beta_1'\left(N^*\right)S^*E^*+\beta_2\left(N^*\right)I^*\right.$$
$$\left.+\beta_2'\left(N^*\right)S^*I^*+\beta_3\left(N^*\right)R^*+\beta_3'\left(N^*\right)S^*R^*\right)-(\omega+\gamma)d$$
$$\cdot\left(\beta_1'\left(N^*\right)S^*E^*+\beta_2'\left(N^*\right)S^*I^*+\beta_3'\left(N^*\right)S^*R^*\right)>0$$

By calculation, we have $b_1b_2-\left(b_3+\varepsilon\gamma a_4\right)>0$. By Routh-Hurwitz stability, all the four eigenvalues have negative real parts. Thus, the endemic equilibrium $E^*$ of system (1) is locally asymptotically stable in $D$, when $R_0>1$.

## 4 Global Stability Analysis of the Endemic Equilibrium

In this work, we aim to study the global dynamics of the endemic equilibrium $E^*$ of system (1) by means of the limiting equation theory and the method of autonomous convergence theorem.

In order to study the global stability of the endemic equilibrium $E^*$ of system (1), the following results can be found in Ref. [30]. We consider the following two systems

$$\dot{x}=f(t,x) \tag{3}$$

$$\dot{y}=g(y) \tag{4}$$

where $x,y\in\mathbf{R}^n$, $f\in C(\mathbf{R}^n\times\mathbf{R}^n)$, $g\in C(\mathbf{R}^n)$, $x$ satisfies conditions of local Lipschitz. If $f(t,x)\to g(x)$ as $t\to\infty$, system (4) is the limit system of system (3).

**Lemma 1** Let $P$ be a locally asymptotically stable equilibrium of system (4) and $\omega$ be the $\omega$ limit set of a forward bounded solution $x(t)$ of system (3). If $y_0\in\omega$ and the solution $y(t)\to P$ as $t\to\infty$ satisfying the initial condition $y(0)=y_0$, then $\omega=\{P\}$, i.e. $x(t)\to P$ as $t\to\infty$.

**Corollary 1** If solutions to system (3) are bounded and the equilibrium $P$ of the limit system (4) is globally asymptotically stable, then any solution $x(t)$ to system (3) satisfies $x(t)\to P$ as $t\to\infty$.

Let $\mathrm{d}t=\tau$, system (1) becomes

$$\begin{cases}\dfrac{\mathrm{d}S}{\mathrm{d}\tau}=\dfrac{A}{d}-\beta_{10}(N)SE-\beta_{20}(N)SI-\beta_{30}(N)SR-S\\ \dfrac{\mathrm{d}E}{\mathrm{d}\tau}=\beta_{10}(N)SE+\beta_{20}(N)SI+\beta_{30}(N)SR-\delta E\\ \dfrac{\mathrm{d}I}{\mathrm{d}\tau}=\gamma_0E-\omega I\\ \dfrac{\mathrm{d}R}{\mathrm{d}\tau}=\varepsilon_0I-R\end{cases} \tag{5}$$

where $\beta_{10}=\dfrac{\beta_1(N)}{d}$, $\beta_{20}=\dfrac{\beta_2(N)}{d}$, $\beta_{30}=\dfrac{\beta_3(N)}{d}$, $\delta=1+\gamma_0+\alpha_{10}+k_{10}$, $\omega=1+\varepsilon_0+\alpha_{20}+k_{20}$, $\gamma_0=\gamma/d$, $\varepsilon_0=\varepsilon/d$, $\alpha_{10}=\alpha_1/d$, $\alpha_{20}=\alpha_2/d$, $k_{10}=k_1/d$, $k_{20}=k_2/d$.

By calculation, we have the following differential equation

$$\frac{\mathrm{d}N}{\mathrm{d}\tau}=\frac{A}{d}-N-(\alpha_{10}+k_{10})E-(\alpha_{20}+k_{20})I$$

We use $N$ as a variable in place of the variable $S$ to give the following system

$$\begin{cases}\dfrac{\mathrm{d}E}{\mathrm{d}\tau}=\left(\beta_{10}(N)E+\beta_{20}(N)I+\beta_{30}(N)R\right)(N-E-I-R)\\ \qquad\qquad -\delta E\\ \dfrac{\mathrm{d}I}{\mathrm{d}\tau}=\gamma_0E-\omega I\\ \dfrac{\mathrm{d}R}{\mathrm{d}\tau}=\varepsilon_0I-R\\ \dfrac{\mathrm{d}N}{\mathrm{d}\tau}=\dfrac{A}{d}-N-(\alpha_{10}+k_{10})E-(\alpha_{20}+k_{20})I\end{cases} \tag{6}$$

System (6) is equivalent to system (5). From biological considerations, we study system (6) in the closed set $T=\left\{(E,I,R,N)\in\mathbf{R}_+^4\middle|0\le E+I+R\le N\le\dfrac{A}{d}\right\}$. We denote by $\partial T$ and $\dot{T}$ the boundary and the interior of $T$ in $\mathbf{R}_+^4$, respectively. For system (6), the global stability of the endemic equilibrium $E^*$ is considered when $\alpha_{10}=\alpha_{20}=k_{10}=k_{20}=0$. Since $N\to\dfrac{A}{d}$ as $\tau\to\infty$, we can obtain the following limit system

$$\begin{cases}\dfrac{\mathrm{d}E}{\mathrm{d}\tau}=\left(\beta_{10}\left(\dfrac{A}{d}\right)E+\beta_{20}\left(\dfrac{A}{d}\right)I+\beta_{30}\left(\dfrac{A}{d}\right)R\right)\\ \qquad\cdot\left(\dfrac{A}{d}-E-I-R\right)-\delta E\\ \dfrac{\mathrm{d}I}{\mathrm{d}\tau}=\gamma_0E-\omega I\\ \dfrac{\mathrm{d}R}{\mathrm{d}\tau}=\varepsilon_0I-R\end{cases} \tag{7}$$

We make the change of variable $x=\frac{A}{d}-E-I-R$, $y=E$, $z=I$, then the following (5) is equivalent to the above system

$$\begin{cases}\dfrac{\mathrm{d}x}{\mathrm{d}\tau}=\dfrac{A}{d}-x-\left(\beta_{30}\left(\dfrac{A}{d}\right)A/d+\left(\beta_{10}\left(\dfrac{A}{d}\right)-\beta_{30}\left(\dfrac{A}{d}\right)\right)y\right.\\ \quad\left.+\left(\beta_{20}\left(\dfrac{A}{d}\right)-\beta_{30}\left(\dfrac{A}{d}\right)\right)z-\beta_{30}\left(\dfrac{A}{d}\right)x\right)x\\ \dfrac{\mathrm{d}y}{\mathrm{d}\tau}=\left(\beta_{30}\left(\dfrac{A}{d}\right)\dfrac{A}{d}+\left(\beta_{10}\left(\dfrac{A}{d}\right)-\beta_{30}\left(\dfrac{A}{d}\right)\right)y\right.\\ \quad\left.+\left(\beta_{20}\left(\dfrac{A}{d}\right)-\beta_{30}\left(\dfrac{A}{d}\right)\right)z-\beta_{30}\left(\dfrac{A}{d}\right)x\right)x-\delta y\\ \dfrac{\mathrm{d}z}{\mathrm{d}\tau}=\gamma_0 y-\omega z\end{cases} \tag{8}$$

**Theorem 3** Consider the following system[31]:

$$x'=f(x),\ f\in C'(\mathbf{R}^n),\ x\in T\subset\mathbf{R}^n \tag{9}$$

where $T$ is an open set. If system (9) satisfies the following conditions: ① System (9) exists a compact absorbing set $K\subset T$ and has a unique equilibrium $P$ in $\dot{T}$; ② $P$ is locally asymptotically stable; ③ System (9) satisfies the Poincare-Bendixson criterion; ④ The periodic orbit of system (9) is asymptotically orbitally stable. Then the only equilibrium $P$ is the globally asymptotically stable in $T$.

**Theorem 4** A sufficient condition for a periodic orbit $P=\{P(t):\ 0\le t\le\tau\}$ of system (9) to be asymptotically orbitally stable with asymptotic phase is that the linear system $z'(t)=\dfrac{\partial f^{[2]}}{\partial t}(P(t))z(t)$ is asymptotically stable, where $\dfrac{\partial f^{[2]}}{\partial t}$ is the second additive compound matrix of the Jacobian matrix $\dfrac{\partial f}{\partial t}$ of $f$. System (9) is called the second compound system of the orbit $P(t)$.

**Lemma 2** Any periodic solution to system (8), if it exists, is asymptotically orbitally stable.

**Proof** Suppose that the solution $(x(t), y(t), z(t))$ is periodic of least period $\tau>0$ such that $(x(0), y(0), z(0))\in\dot{T}$. The periodic orbit is $P=\{P(t):0\le t\le\tau\}$. We have the second compound system $x'=\boldsymbol{J}^{[2]}(P)x$ of the differential system $y'=\boldsymbol{J}(P)y$ in the periodic solution is the following periodic linear system

$$\begin{cases}X'=a_{11}X+\left(\beta_{20}\left(\dfrac{A}{d}\right)-\beta_{30}\left(\dfrac{A}{d}\right)\right)x(Y+Z)\\ Y'=\gamma_0 X+a_{22}Y-\left(\beta_{10}\left(\dfrac{A}{d}\right)-\beta_{30}\left(\dfrac{A}{d}\right)\right)xZ\\ Z'=a_{32}Y-\left(\delta+\omega-\left(\beta_{10}\left(\dfrac{A}{d}\right)-\beta_{30}\left(\dfrac{A}{d}\right)\right)x\right)Z\end{cases} \tag{10}$$

where

$$a_{11}=-1+\beta_{30}\left(\frac{A}{d}\right)x-\left(\beta_{30}\left(\frac{A}{d}\right)\frac{A}{d}+\left(\beta_{10}\left(\frac{A}{d}\right)-\beta_{30}\left(\frac{A}{d}\right)\right)y+\left(\beta_{20}\left(\frac{A}{d}\right)-\beta_{30}\left(\frac{A}{d}\right)\right)z-\beta_{30}\left(\frac{A}{d}\right)x\right)-\delta$$

$$a_{22}=-\left(1+\omega+\left(\beta_{10}\left(\frac{A}{d}\right)-\beta_{30}\left(\frac{A}{d}\right)\right)x-\beta_{30}\left(\frac{A}{d}\right)x+\beta_{30}\left(\frac{A}{d}\right)\frac{A}{d}+\left(\beta_{10}\left(\frac{A}{d}\right)-\beta_{30}\left(\frac{A}{d}\right)\right)y\right)$$

$$+\left(\left(\beta_{20}\left(\frac{A}{d}\right)-\beta_{30}\left(\frac{A}{d}\right)\right)z-\beta_{30}\left(\frac{A}{d}\right)x\right)$$

$$a_{32}=-\beta_{30}\left(\frac{A}{d}\right)x+\beta_{30}\left(\frac{A}{d}\right)\frac{A}{d}+\left(\beta_{10}\left(\frac{A}{d}\right)-\beta_{30}\left(\frac{A}{d}\right)\right)y+\left(\beta_{20}\left(\frac{A}{d}\right)-\beta_{30}\left(\frac{A}{d}\right)\right)z-\beta_{30}\left(\frac{A}{d}\right)x$$

$$\boldsymbol{J}(P)=\begin{pmatrix}a_{11}+\delta & -\left(\beta_{10}\left(\dfrac{A}{d}\right)-\beta_{30}\left(\dfrac{A}{d}\right)\right)x & -\left(\beta_{20}\left(\dfrac{A}{d}\right)-\beta_{30}\left(\dfrac{A}{d}\right)\right)x\\ a_{32} & \left(\beta_{10}\left(\dfrac{A}{d}\right)-\beta_{30}\left(\dfrac{A}{d}\right)\right)x-\delta & \left(\beta_{20}\left(\dfrac{A}{d}\right)-\beta_{30}\left(\dfrac{A}{d}\right)\right)x\\ 0 & \delta & -\omega\end{pmatrix}$$

$$\boldsymbol{J}^{[2]}(P)=\begin{pmatrix} a_{11} & \left(\beta_{20}\left(\frac{A}{d}\right)-\beta_{30}\left(\frac{A}{d}\right)\right)x & \left(\beta_{20}\left(\frac{A}{d}\right)-\beta_{30}\left(\frac{A}{d}\right)\right)x \\ \gamma_0 & a_{22} & -\left(\beta_{10}\left(\frac{A}{d}\right)-\beta_{30}\left(\frac{A}{d}\right)\right)x \\ 0 & a_{32} & -\delta-\omega+\left(\beta_{10}\left(\frac{A}{d}\right)-\beta_{30}\left(\frac{A}{d}\right)\right)x \end{pmatrix}$$

Suppose that $(x(t),y(t),z(t))$ is a solution to system (10). Let

$$V(X,Y,Z,x,y,z)=\sup\left\{|X|,\ \frac{y}{z}(|Y|+|Z|)\right\}$$

From the condition (1) of Theorem 3, we can know there exists constant $\eta>0$ such that

$$V(X,Y,Z,x,y,z)\geq\eta|(X,Y,Z)|$$

for all $(X,Y,Z)\in\mathbf{R}^3$ and $(x,y,z)\in P$. By direct calculations, we can obtain the following differential inequalities:

$$D_+|X(t)|\leq a_{11}|X(t)|+\left(\beta_{20}\left(\frac{A}{d}\right)-\beta_{30}\left(\frac{A}{d}\right)\right)x(|Y(t)|+|Z(t)|) \tag{11}$$

$$D_+|Y(t)|\leq\gamma_0|X(t)|+a_{22}|Y(t)|-\left(\beta_{10}\left(\frac{A}{d}\right)-\beta_{30}\left(\frac{A}{d}\right)\right)x|Z(t)| \tag{12}$$

$$D_+|Z(t)|\leq a_{32}|Y(t)|-\left(\delta+\omega-\left(\left(\beta_{10}\left(\frac{A}{d}\right)-\beta_{30}\left(\frac{A}{d}\right)\right)x\right)\right)|Z(t)| \tag{13}$$

Using (12) and (13), we have

$$D_+\frac{y}{z}(|Y(t)|+|Z(t)|)=\frac{y\gamma_0}{z}|X(t)|+\left(\frac{y'}{y}-\frac{z'}{z}-\omega-1\right)\frac{y}{z}(|Y(t)|+|Z(t)|) \tag{14}$$

From (11) and (14), we get

$$D_+V(t)\leq\sup\{g_1,g_2\}V(t) \tag{15}$$

where

$$g_1=-1-\delta+\beta_{10}(A/d)x-\beta_{30}(A/d)A/d-(\beta_{10}(A/d)-\beta_{30}(A/d))y-(\beta_{20}(A/d)-\beta_{30}(A/d))z+\beta_{30}(A/d)x+\frac{(\beta_{20}(A/d)-\beta_{30}(A/d))xz}{y} \tag{16}$$

$$g_2=\frac{y\gamma_0}{z}+\left(\frac{y'}{y}-\frac{z'}{z}-\omega-1\right) \tag{17}$$

Rewrite the last two equations of system (8), then we obtain

$$\frac{\left(\beta_{20}\left(\frac{A}{d}\right)-\beta_{30}\left(\frac{A}{d}\right)\right)xz}{y}=\frac{y'}{y}+\delta-\beta_{30}\left(\frac{A}{d}\right)\frac{A}{d}\frac{x}{y}-\left(\beta_{10}\left(\frac{A}{d}\right)-\beta_{30}\left(\frac{A}{d}\right)\right)x+\beta_{30}\left(\frac{A}{d}\right)\frac{x^2}{y} \tag{18}$$

$$\frac{y\gamma_0}{z}=\omega+\frac{z'}{z} \tag{19}$$

Substituting (18) into (16) and (19) into (17), we have

$$g_1=\frac{y'}{y}-1-\beta_{30}\left(\frac{A}{d}\right)\left(\frac{A}{d}-x-y-z\right)-\beta_{30}(A/d)x\cdot\left(\frac{A}{d}-x-y\right)-\beta_{10}\left(\frac{A}{d}\right)y-\beta_{20}\left(\frac{A}{d}\right)z \tag{20}$$

$$g_2=\frac{y'}{y}-1 \tag{21}$$

Thus

$$\sup\{g_1(t),g_2(t)\}\leq\frac{y'}{y}-1$$

and

$$\int_0^\tau\sup\{g_1(t),g_2(t)\}\leq\ln y(t)\Big|_0^\tau-\tau=-\tau$$

From (15), we have $V(t)\to 0$ as $t\to\infty$, and in turn that $(X(t),Y(t),Z(t))\to 0$ as $t\to\infty$. As a result, the second compound system (10) is asymptotically stable and the periodic solution $(x(t),y(t),z(t))$ is asymptotically orbit ally stable by Theorem 3.

**Lemma 3** System (8) is uniformly persistent when $R_0>1$.

**Proof** Let $G=\{P_0\}$, when $R_0>1$, the stable set $G^s$ is just contained in the *S*-axis and thus in the boundary of $\Gamma$. It also implies that the stable set $G^s$ is isolated in $\Gamma$. Then, when $R_0>1$, system (8) satisfies the conditions of Theorem 2 of Ref. [1], namely, the maximal compact invariant set *G* in the boundary of $\Gamma$ is isolated and the stable set $G^s$ of *G* is contained in the boundary of $\Gamma$. Therefore, system (8) is uniformly persistent in $\Gamma$ when $R_0>1$.

**Lemma 4** System (5) is competitive when $R_0>1$.

**Proof** Set $x_1=S,x_2=E,x_3=I,x_4=R$, system (5) is replaced by

$$\begin{cases} x_1' = \dfrac{A}{d} - \beta_{10}(N)S\left(\dfrac{A}{d} - x_1 - x_3 - x_4\right) - \beta_{20}(N)S\left(\dfrac{A}{d} - x_1 - x_2 - x_4\right) - \beta_{30}(N)S\left(\dfrac{A}{d} - x_1 - x_2 - x_3\right) - x_1 \\ x_2' = \beta_{10}(N)Sx_2 + \beta_{20}(N)Sx_3 + \beta_{30}(N)Sx_4 - \delta x_2 \\ x_3' = \gamma_0 x_2 - \omega x_3 \\ x_4' = \varepsilon_0 x_3 - x_4 \end{cases}$$

Furthermore, the system has the following form

$$\boldsymbol{X}' = (\boldsymbol{A}(t) - \boldsymbol{I})\boldsymbol{X} + \boldsymbol{C}(t)$$

where $\boldsymbol{X} = (x_1, x_2, x_3, x_4) \in \mathbf{R}^4$, $\boldsymbol{I}$ denotes the $4\times 4$ unit matrix, $\boldsymbol{C}(t)$ is a vector function,

$$\boldsymbol{A}(t) = \begin{pmatrix} (\beta_{10}(N) + \beta_{20}(N) + \beta_{30}(N))x_1 & (\beta_{20}(N) + \beta_{30}(N))x_1 & (\beta_{10}(N) + \beta_{30}(N))x_1 & (\beta_{10}(N) + \beta_{20}(N))x_1 \\ 0 & \beta_{10}(N)x_1 - \gamma_0 & \beta_{20}(N)x_1 & \beta_{30}(N)x_1 \\ 0 & \gamma_0 & -\varepsilon_0 & 0 \\ 0 & 0 & \varepsilon_0 & 0 \end{pmatrix}$$

The off-diagonal entries in this matrix are non-negative, the system as a whole is quasimonotone. Thus we can verify that system (5) is competitive with respect to the partial ordering defined by the orthant $K = \{(S,E,I,R) \in \mathbf{R}^4 \,|\, S \geq 0, E \geq 0, I \geq 0, R \geq 0\}$, see Refs. [29, 32].

**Theorem 5** If $R_0 > 1$, the endemic equilibrium $E^*$ of system (1) is globally asymptotically stable.

## 5 Example and Numerical Simulation

In this section, we provide the numerical simulation of the model to illustrate the main theoretical results above to better explain the occurrence and development of the infectious disease. The number simulations for system (1) are shown in Figs. 1-3. Choose $\beta_1(N) = \dfrac{\beta_1}{1+N}, \beta_2(N) = \dfrac{\beta_2}{3+N}, \beta_3(N) = \dfrac{\beta_3}{2+N}$. Let $A = 0.6$, $d = 0.05$, $\varepsilon = 0.07$, $\gamma = 0.15$, $\beta_1 = 0.08$, $\beta_2 = 0.15$, $\beta_3 = 0.03$. We randomly choose six initial conditions $(5,1.9,3.2,2.6)$, $(3.8,0.6,2,5.4)$, (2,1.5,2.6, 5), $(3.5,2.5,0.8,4.6)$, (0.5,4,2.2,3.6) and (4.5,1.6, 2.4,3.1) in $D = \{(S,E,I,R) \in \mathbf{R}_+^4 \,|\, 0 < S+E+I+R < 12\}$. By computing, we derive $R_0 = 0.475 < 1$ and system (1) has a disease-free equilibrium $E_0(12,0,0,0)$. We give the trajectory plot and its tridimensional figure for system (1) in Fig.1 and Fig. 3(a). From Theorem 1, it can be known that $E_0$ is globally asymptotically stable. The dynamic behavior of system (1) is shown in Fig.1.

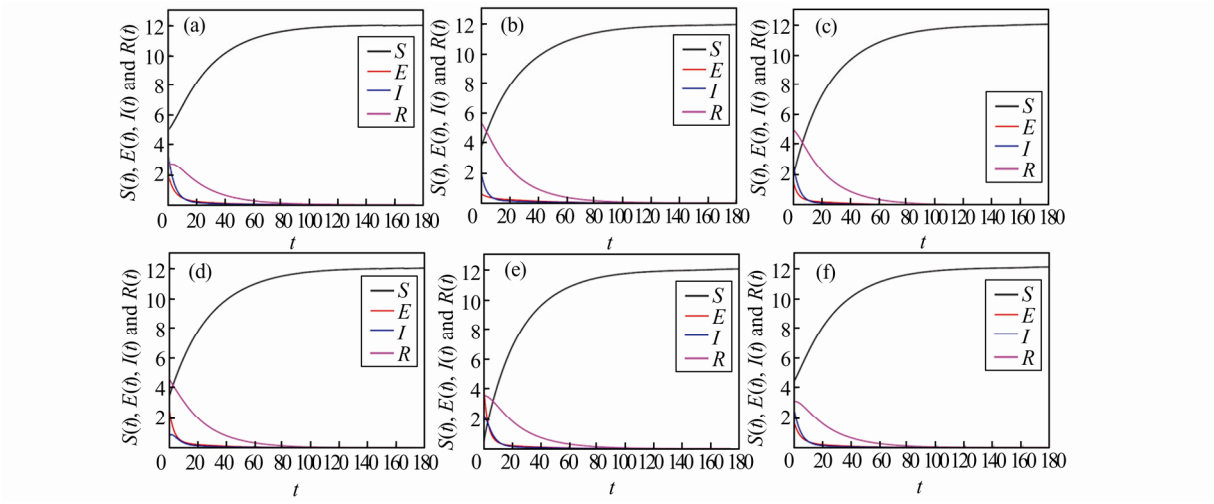


**Fig.1 Variational curves of *S*, *E*, *I* and *R* with time *t* when $R_0 = 0.475 < 1$**

Six initial conditions are (a) (5, 1.9, 3.2, 2.6), (b) (3.8,0.6,2,5.4), (c) (2.1, 5, 2.6, 5), (d) (3.5, 2.5, 0.8, 4.6), (e) (0.5, 4, 2.2, 3.6) and (f) (4.5, 1.6, 2.4, 3.1), respectively

Choose $\beta_1(N)=\frac{\beta_1}{2+N}$, $\beta_2(N)=\frac{\beta_2}{4+N}$, $\beta_3(N)=\frac{\beta_3}{3+N}$. Let $A=1.8$, $d=0.05$, $\varepsilon=0.07$, $\gamma=0.15$, $\beta_1=0.1$, $\beta_2=0.2$, $\beta_3=0.05$. We randomly choose six initial conditions $(11.5,9.2,7.3,8)$, $(9.8,6.6,8,11.4)$, $(8,7.5,8.6,11)$, $(6.5,10,8.2,9.1)$, $(10.5,7.6,8.4,9.1)$ and $(5.8,9.6,7.4,12)$ in $D=\{(S,E,I,R)\in\mathbf{R}_+^4 \mid 0<S+E+I+R<36\}$. By computing, we derive $R_0=5.508>1$ and system (1) has a disease-free equilibrium $E^*=(17.54,4.621,5.766,8.074)$. We give the trajectory plot and its tridimensional figure for system (1) in Fig.2 and Fig. 3(b). According to Theorem 1, $E^*$ is globally asymptotically stable. The dynamic behavior of system (1) is shown in Fig.2.

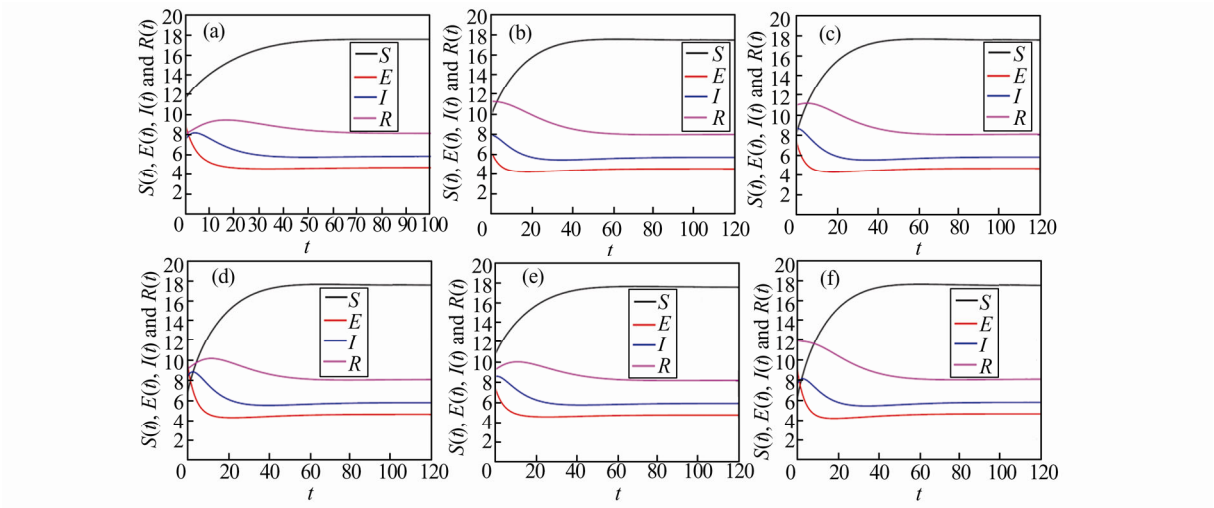


**Fig. 2 Variational curves of *S*, *E*, *I* and *R* with time *t* when $R_0=5.508>1$**

Six initial conditions are (a) (11.5, 9.2, 2.7, 3.8), (b) (9.8, 6.6, 8, 11.4), (c) (8, 7.5, 8.6, 11), (d) (6.5, 10, 8.2, 9.1), (e) (10.5, 7.6, 8.4, 9.1) and (f) (5.8, 9.6, 7.4, 12), respectively

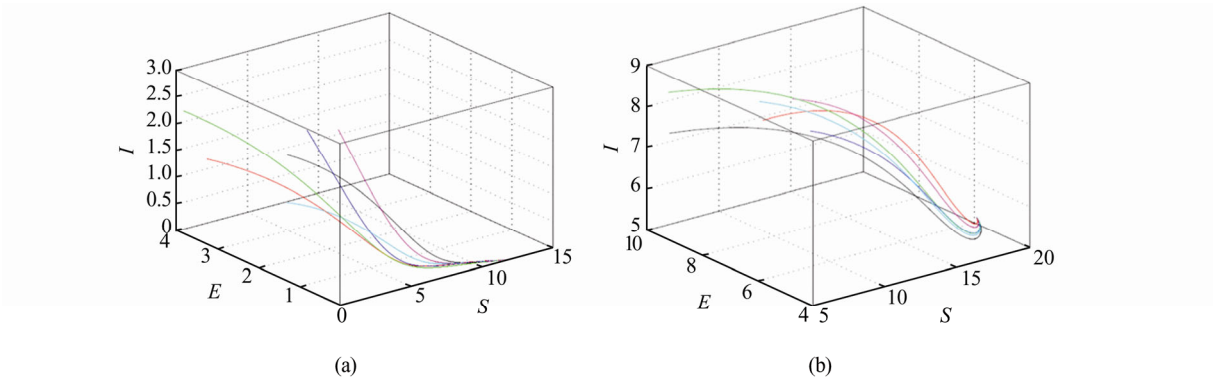


**Fig.3 The graph of the trajectory in (*S*, *E*, *I*)-space**

(a) and (b) correspond with Fig.1 and Fig.2, respectively, and the six curves correspond to six initial conditions

## 6 Conclusion

In this work, we have performed a complete mathematical analysis of the global stability problem at equilibrium with SEIR models with general contact rates and latent period, both theoretically and numerically. For the model, we give the expression for the basic reproduction number $R_0$. When $R_0<1$, as shown in Theorem 1, the disease-free equilibrium is globally asymptotically stable by LaSalle's invariance principle (see Fig. 1 and Fig. 3(a)), and the disease always disappears eventually. When $R_0>1$, Theorem 5 tells us that by limit equation

theory and periodic orbit stability theory, the only endemic equilibrium is globally asymptotically stable (see Fig. 2 and Fig. 3(b)), and the disease persists at the endemic equilibrium if it exists initially. If the infectivity of the disease in the exposed and recovered period is smaller, the smaller the basic regeneration number, the more conducive to the elimination of the disease. Therefore, for the infectious diseases of the exposed and recovered period, not only the patients in the infected period, but also the patients in the exposed and recovery period should be controlled, so as to control and eliminate the spread of the disease more effectively. Finally, numerical simulations are carried out to confirm the correctness of the theoretical analysis.

Interestingly, the stability of the model equilibrium is affected by the general contact rate. We believe that our findings offer guidance in face of the disease. In addition, we would like to point out here that the model (1) leaves us a problem that we only take the general form of contact rate, a more abstract situation in the model into consideration, but the incidence of other specific forms such as standard or nonlinear remains undiscussed. We leave this for our future work.

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