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The Mixed Polar Orlicz-Brunn-Minkowski Inequalities

□ **LI Juan, TANG Linzeng, WANG Wei†**

School of Mathematics and Computational Science, Hunan University of Science and Technology, Xiangtan 411201, Hunan, China

Abstract: Some Orlicz-Brunn-Minkowski type inequalities for (dual) quermassintegrals of polar bodies and star dual bodies have been introduced. In this paper, we generalize the results and establish some Orlicz-Brunn-Minkowski type inequalities for mixed (dual) quermassintegrals of polar bodies and star dual bodies.

Key words: polar body; mixed quermassintegral; Orlicz-Brunn-Minkowski inequalities

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Biography: LI Juan, female, Master candidate, research direction: convex geometry. E-mail: lj2745675651@163.com

† To whom correspondence should be addressed. E-mail: wwang@hnust.edu.cn

0 Introduction

Let K^n be the set of convex bodies (compact convex sets with nonempty interior) in the n-dimensional Euclidean space $\mathbf{R}^n$. For $K, L \in K^n$, their Minkowski sum $K+L=\{x+y : x\in K, y\in L\}$, and the i-th quermassintegral of K will be denoted by $W_i(K)$ for each $i=0,1,\cdots,n-1$. The classical Brunn-Minkowski inequality for quermassintegrals states that for $K, L\in K^n$ and $0\le i\le n-1$, then

$$\left(\frac{W_i(K)}{W_i(K+L)}\right)^{\frac{1}{n-i}}+\left(\frac{W_i(L)}{W_i(K+L)}\right)^{\frac{1}{n-i}}\le 1 \qquad (1)$$

with equality if and only if K and L are homothetic. The case $i=0$ of (1) is the classical Brunn-Minkowski inequality (see Ref. [1]). It is the core of the Brunn-Minkowski theory, which is derived from questions around the isoperimetric problem. In Gardner's excellent survey[1], he summarized the history of this inequality and some applications in other related fields such as elliptic partial differential equations and algebraic geometry. In addition, this inequality helped make a great difference in studying inequalities and witnessed a rapid growth.

In the early 1960s, Firey [2] introduced the L_p-addition. Let K_o^n be the set of all convex bodies in containing the origin in their interiors. For $\mathbf{R}^n$, $K, L\in K_o^n$ and $p\ge 1$, the L_p-Minkowski addition $+_p$ is defined by (see Ref. [2])

$$h_{K+_pL}(x)^p=h_K(x)^p+h_L(x)^p,\ \ x\in\mathbf{R}^n$$

where h_K denotes the support function of the convex body K and it is defined by $h_K(x)=\sup\{x\cdot y : y\in K\}$. Here x, y denote the standard inner product of x,

$y\in \mathbf{R}^n$. Thirty years after the new L_p-addition, Lutwak[3,4] established the L_p-Brunn-Minkowski inequality for quermassintegrals: For $K,L\in K_o^n$, $p\ge 1$ and $0\le i\le n-1$, then

$$\left(\frac{W_i(K)}{W_i(K+_p L)}\right)^{\frac{p}{n-i}}+\left(\frac{W_i(L)}{W_i(K+_p L)}\right)^{\frac{p}{n-i}}\le 1$$

with equality if and only if K and L are dilates. Readers can refer to Refs. [5-9] for additional references.

The Orlicz-Brunn-Minkowski theory originated from the work of Lutwak *et al* in 2010[10,11]. As an important part of the theory, the Orlicz Brunn-Minkowski inequality has been very popular with scholars in related fields. At first, the Orlicz Busemann-Petty centroid inequality[10] was introduced as a new proof by Li and Leng[12] in 2010 and the Orlicz Petty projection inequality were established by Lutwak *et al*[11]. After that, Gardner *et al*[13] introduced the Orlicz addition and established the new Orlicz- Brunn-Minkowski inequality that implied the L_p-Brunn-Minkowski inequality in 2014. Let $\varPhi$ be the class of convex and strictly increasing functions, $\phi:[0,\infty)\to[0,\infty)$ such that $\phi(0)=0,\phi(1)=1$ and $\lim_{t\to\infty}\phi(t)=\infty$. For $K,L\in K_o^n$ and $\phi\in\varPhi$, the Orlicz addition $+_\phi$ (see Section 1 for precise definition) is defined by

$$\phi\left(\frac{h_K(x)}{h_{K+_\phi L}(x)}\right)+\phi\left(\frac{h_L(x)}{h_{K+_\phi L}(x)}\right)=\phi(1) \qquad (2)$$

for $x\in\mathbf{R}^n$. In the same year, Xiong and Zou [14] established the Orlicz-Brunn-Minkowski inequality for quermassintegrals: For $K,L\in K_o^n$, $\phi\in\varPhi$ and $0\le i\le n-1$, then

$$\phi\left(\left(\frac{W_i(K)}{W_i(K+_\phi L)}\right)^{\frac{1}{n-i}}\right)+\phi\left(\left(\frac{W_i(L)}{W_i(K+_\phi L)}\right)^{\frac{1}{n-i}}\right)\le\phi(1) \qquad (3)$$

If ϕ is strictly convex, equality holds if and only if K and L are dilates. The case $i=0$ was established by Refs. [13, 15].

In 1975, Lutwak [16] introduced dual mixed volumes and radial addition, and studied the dual Brunn-Minkowski theory for star bodies. In 2015, Gardner *et al*[17] established the dual Orlicz-Brunn-Minkowski theory and introduced the concept of radial Orlicz addtion. Let $\tilde{\varPhi}$ be the set of continuous and strictly increasing functions, $\psi:[0,\infty)\to[0,\infty)$ such that $\psi(0)=0$ and $\lim_{t\to\infty}\psi(t)=\infty$. Let $\tilde{\varPsi}$ be the set of continuous and strictly decreasing functions, $\psi:[0,\infty)\to[0,\infty)$ such that $\lim_{t\to 0^+}\psi(t)=\infty$ and $\lim_{t\to\infty}\psi(t)=0$. Let S_o^n be the set of all star bodies with the origin as an interior point. For $K,L\in S_o^n$ and $\psi\in\tilde{\varPhi}\cup\tilde{\varPsi}$, the radial Orlicz addtion $\tilde{+}_\psi$ (see Section 1 for precise definition) is defined by

$$\psi\left(\frac{\rho_K(x)}{\rho_{K\tilde{+}_\psi L}(x)}\right)+\psi\left(\frac{\rho_L(x)}{\rho_{K\tilde{+}_\psi L}(x)}\right)=\psi(1) \qquad (4)$$

for $x\in\mathbf{R}^n\setminus\{0\}$.

The inequalities for polar bodies and dual star bodies began to attract attention. For instance, Zhu[18] confirmed the conjecture[10] that the Orlicz centroid inequality for convex bodies can be extended to star bodies; Cifre and Nicol[19] proved a Brunn-Minkowski-type inequality for the polar set of the *p*-sum of convex bodies, which generalized previous results by Firey[20]; Wang and Huang [21] gave a systematic explanation of Orlicz Brunn-Minkowski inequality for polar bodies and dual star bodies and Liu [22] established some Orlicz-Brunn-Minkowski type inequalities for (dual) quermassintegrals of polar bodies and star dual bodies. Besides, the Orlicz-Brunn-Minkowski inequality for complex projection bodies[23] is also a very active field. For other generalizations on Orlicz spaces, see Refs. [17, 24, 25].

Let K^* be the polar body of a convex body K, K° the dual star body of a convex body *K*. Liu[22] established the following Orlicz-Brunn-Minkowski type inequality for dual quermassintegrals of polar bodies and star dual bodies: For $K,L\in K_o^n$, $\phi\in\varPhi$ and $0\le i\le n-1$, then

$$\phi\left(\left(\frac{\tilde{W}_i(K^*)}{\tilde{W}_i(K+_\phi L)^*}\right)^{-\frac{1}{n-i}}\right)+\phi\left(\left(\frac{\tilde{W}_i(L^*)}{\tilde{W}_i(K+_\phi L)^*}\right)^{-\frac{1}{n-i}}\right)\le\phi(1) \qquad (5)$$

If ϕ is strictly convex, equality holds if and only if K and L are dilates.

For $K,L\in S_o^n$, $\psi\in\tilde{\varPhi}\cup\tilde{\varPsi}$ and $0\le i\le n-1$, if $\psi_0(t)=\psi(t^{-\frac{1}{n-i}})$ is concave, then

$$\psi\left(\left(\frac{\tilde{W}_i(K^\circ)}{\tilde{W}_i(K\tilde{+}_\psi L)^\circ}\right)^{-\frac{1}{n-i}}\right)+\psi\left(\left(\frac{\tilde{W}_i(L^\circ)}{\tilde{W}_i(K\tilde{+}_\psi L)^\circ}\right)^{-\frac{1}{n-i}}\right)\ge\psi(1) \qquad (6)$$

while if ψ_0 is convex, the inequality is reversed. If ψ_0 is strictly concave, equality holds if and only if K and L are dilates.

The purpose of this paper is to establish the following Orlicz-Brunn-Minkowski type inequality for dual mixed quermassintegrals of polar bodies and star dual bodies.

Theorem 1 Let $K,Q,L\in K_o^n$, $\phi\in\varPhi$ and $0\le i\le n-2$. Then

$$\phi\left(\left(\frac{\tilde{W}_i(K^*,Q)}{\tilde{W}_i\left((K+_\phi L)^*,Q\right)}\right)^{-\frac{1}{n-i-1}}\right)+\phi\left(\left(\frac{\tilde{W}_i(L^*,Q)}{\tilde{W}_i\left((K+_\phi L)^*,Q\right)}\right)^{-\frac{1}{n-i-1}}\right)\le\phi(1)$$

If ϕ is strictly convex, equality holds if and only if K and L are dilates.

Theorem 2 Let $K,Q,L\in S_o^n$, $\psi\in\tilde{\varPhi}\cup\tilde{\varPsi}$, and $0\le i\le n-2$. If $\psi_0(t)=\psi\left(t^{-\frac{1}{n-i-1}}\right)$ is concave, then

$$\psi\left(\left(\frac{\tilde{W}_i(K^\circ,Q)}{\tilde{W}_i\left((K+_\psi L)^\circ,Q\right)}\right)^{-\frac{1}{n-i-1}}\right)+\psi\left(\left(\frac{\tilde{W}_i(L^\circ,Q)}{\tilde{W}_i\left((K+_\psi L)^\circ,Q\right)}\right)^{-\frac{1}{n-i-1}}\right)\ge\psi(1)$$

If ψ_0 is convex, the inequality is reversed. If ψ_0 is strictly concave (or convex, as appropriate), equality holds if and only if K and L are dilates.

Liu[22] also established the following dual Orlicz-Brunn-Minkowski type inequality dual quermassintegrals of polar bodies: Let $K,L\in K_o^n$, $\psi\in\tilde{\varPsi}$ such that $\phi(t)=\psi(t^{-1})$ is strictly convex, and $0\le i\le n-1$, then

$$\psi\left(\left(\frac{W_i(K^*)}{W_i(K\tilde{+}_\psi L)^*}\right)^{-\frac{1}{n-i}}\right)+\psi\left(\left(\frac{W_i(L^*)}{W_i(K\tilde{+}_\psi L)^*}\right)^{-\frac{1}{n-i}}\right)\le\psi(1) \qquad (7)$$

with equality if and only if K and L are dilates.

We also establish the following dual Orlicz-Brunn-Minkowski type inequality for dual mixed quermassintegrals of polar bodies which is the dual form of Theorem 1.

Theorem 3 Let $K,Q,L\in K_o^n$, $\psi\in\tilde{\varPsi}$ such that $\phi(t)=\psi(t^{-1})$ is strictly convex, and $0\le i\le n-2$, then

$$\psi\left(\left(\frac{\tilde{W}_i(K^*,Q)}{\tilde{W}_i\left((K\tilde{+}_\psi L)^*,Q\right)}\right)^{-\frac{1}{n-i-1}}\right)+\psi\left(\left(\frac{\tilde{W}_i(L^*,Q)}{\tilde{W}_i\left((K\tilde{+}_\psi L)^*,Q\right)}\right)^{-\frac{1}{n-i-1}}\right)\le\psi(1)$$

with equality if and only if K and L are dilates.

This paper is organized as follows. In Section 1, we collect some concepts and facts to be used in the proofs of our results. In Section 2, we give the integral forms of some (dual) mixed Orlicz quermassintegrals and confirm that a special case of the Aleksandrov-Fenchel inequality (with respect to three convex bodies) can be generalized to Orlicz setting. In Section 3, we introduce some Orlicz-Brunn-Minkowski inequalities for (dual) mixed quermassintegrals of polar bodies and star dual bodies.

1 Preliminaries

For $K,L\in K_o^n$, we define the Orlicz sum $K+_\phi L$ by (see Ref. [13])

$$h_{K+_\phi L}(x)=\inf\left\{\lambda>0:\phi\left(\frac{h_K(x)}{\lambda},\frac{h_L(x)}{\lambda}\right)\le 1\right\}$$

for $x\in\mathbf{R}^n$.

Equivalently, the Orlicz sum $K+_\phi L$ can be defined implicitly by

$$\phi\left(\frac{h_K(x)}{h_{K+_\phi L}(x)},\frac{h_L(x)}{h_{K+_\phi L}(x)}\right)=1$$

If $h_K(x)+h_L(x)>0$, and by $h_{K+_\phi L}(x)=0$, if $h_K(x)=h_L(x)=0$. Here $\phi\in\varPhi_1$, the set of convex functions $\phi:[0,\infty)^2\to[0,\infty)$ that are increasing in each variable with $\phi(0,0)=0$, $\phi(1,0)=0$ and $\phi(0,0)=1$. In particular, if $\phi(x_1,x_2)=x_1^p+x_2^p$, and $p\ge 1$, then Orlicz addition reduces to L_p addition.

Gardner, Hug and Weil [13] proved that Orlicz addition is commutative if and only if $\phi(x_1,x_2)=\phi_0(x_1)+\phi_0(x_2)$. For some $\phi_0\in\varPhi$, the set of convex functions $\phi:[0,\infty)\to[0,\infty)$ satisfy $\phi(0)=0$, and $\phi(1)=1$. Therefore, (2) was defined.

For a compact star-shaped set K about the origin, the radial function $\rho_K:\mathbf{R}^n\setminus\{0\}\to\mathbf{R}$ is defined by

$$\rho_K(x)=\max\{\lambda\ge 0:\lambda x\in K\}$$

where the compact star-shaped set $K \subset \mathbf{R}^n$ is defined if the intersection of every straight line through the origin with K is a line segment. And if the ρ_K is positive and continuous, then the compact star-shaped set K about the origin is called a star body.

For $K, L \in S_o^n$, the radial addition $\tilde{+}$ is defined by

$$\rho_{K \tilde{+} L}(x) = \rho_K(x) + \rho_L(x) \tag{8}$$

and if $s > 0$, then for all $K \in S_o^n$,

$$\rho_{sK}(x) = s\rho_K(x) \tag{9}$$

For $K, L \in S_o^n$, we define the radial Orlicz sum $K \tilde{+}_\psi L$ by (see Ref.[17])

$$\rho_{K \tilde{+}_\psi L}(x) = \inf\left\{\lambda > 0 : \psi\left(\frac{\rho_K(x)}{\lambda}, \frac{\rho_L(x)}{\lambda}\right) \le 1\right\}$$

for $x \in \mathbf{R}^n \setminus \{0\}$.

Equivalently, the radial Orlicz addition $\tilde{+}_\psi$ can be defined implicitly by

$$\psi\left(\frac{\rho_K(x)}{\rho_{K \tilde{+}_\psi L}(x)}, \frac{\rho_L(x)}{\rho_{K \tilde{+}_\psi L}(x)}\right) = 1$$

If $\rho_K(x) + \rho_L(x) > 0$, and by $\rho_{K \tilde{+}_\psi L}(x) = 0$, if $\rho_K(x) = \rho_L(x) = 0$. An important special case is obtained when $\psi(x_1, x_2) = \psi_0(x_1) + \psi_0(x_2)$, for fixed $\psi_0 \in \tilde{\Phi}$.

Then by the corresponding special case

$$\psi_0\left(\frac{\rho_K(x)}{\rho_{K \tilde{+}_\psi L}(x)}\right) + \psi_0\left(\frac{\rho_L(x)}{\rho_{K \tilde{+}_\psi L}(x)}\right) = 1 \tag{10}$$

when $\rho_K(x) + \rho_L(x) > 0$, and by $\rho_{K \tilde{+}_\psi L}(x) = 0$, otherwise, and similarly by (10) when $\psi_0 \in \tilde{\Psi}$.

Therefore, (4) was defined.

We denote the unit ball in $\mathbf{R}^n$ and its surface by B, S^{n-1}, respectively. The dual mixed volume $\tilde{V}(K_1, \cdots, K_n)$ is defined by (see Ref. [16])

$$\tilde{V}(K_1, \cdots, K_n) = \frac{1}{n}\int_{S^{n-1}} \rho_{K_1}(u) \cdots \rho_{K_n}(u) \mathrm{d}S(u) \tag{11}$$

where S is the spherical Lebesgue measure ((n−1) dimensional Hausdorff measure) of S^{n-1}.

The polar body K^* of a convex body K is defined by

$$K^* = \{x \in \mathbf{R}^n : x \cdot y \le 1 \text{ for all } y \in K\}$$

and it is easy to see that K^* is a convex body and $(K^*)^* = K$. If $K \in K_o^n$ (a convex body that contains the origin in its interior), for all $u \in S^{n-1}$,

$$h_{K^*}(u) = \frac{1}{\rho_K(u)} \tag{12}$$

Suppose that μ is a probability measure on a space X and $g : X \to I \subset \mathbf{R}$ is a μ-integrable function, where I is a possible infinite interval. Jensen's inequality states that if $\phi \in I \to \mathbf{R}$ is a convex function, then

$$\int_X \phi(g(x)) \mathrm{d}\mu(x) \ge \phi\left(\int_X g(x) \mathrm{d}\mu(x)\right) \tag{13}$$

When ϕ is strictly convex, equality holds if and only if $g(x)$ is a constant for μ-almost all $x \in X$ (see Refs. [22, 26]). If ϕ is a concave function, the inequality is reversed.

For a convex body K, the i-th quermassintegral of K, $W_i(K)$, $0 \le i \le n-1$ has the following integral representation:

$$W_i(k) = \frac{1}{n}\int_{S^{n-1}} h_K(u) \mathrm{d}S_{n-i-1}(K, u)$$

where $S_{n-i-1}(K, \cdot)$ is $(n-i-1)$-th surface area measures of K. In particular, $W_0(K) = V(K)$, $nW_1(K) = S(K)$, and $W_n(K) = V(B)$, where B is the unit ball in $\mathbf{R}^n$, and V, S denote the volume and the surface area of the set involved, respectively.

For $0 \le i < n-1$, the mixed quermassintegral $W_i(K, L)$ has the following integral representation:

$$W_i(K, L) = \frac{1}{n}\int_{S^{n-1}} h_L(u) \mathrm{d}S_{n-i-1}(K, u)$$

In particular, $W_i(K, K) = W_i(K)$.

For $0 \le i < n-2$, $K, L, Q \in K_o^n$, the mixed volume $V\left(\underbrace{K, \cdots, K}_{n-i-2}, \underbrace{B, \cdots, B}_{i}, L, Q\right)$ is written as $W_i(K, L, Q)$. In particular, $W_i(K, K, Q) = W_i(K, Q)$, $W_i(K, L, B) = W_{i+1}(K, L)$. The mixed quermassintegral $W_i(K, L, Q)$ has the following integral representation (see Ref. [9]):

$$W_i(K, L, Q) = \frac{1}{n}\int_{S^{n-1}} h_L(u) \mathrm{d}S_{n-i-1}(K, Q, u) \tag{14}$$

where the measure

$$S_{n-i-1}(K, Q, u) = S\left(\underbrace{K, \cdots, K}_{n-i-2}, \underbrace{B, \cdots, B}_{i}, Q, u\right)$$

An important special case of the Aleksandrov-Fenchel inequality [9] is stated as follows:

Suppose $K, L, Q \in K_o^n$, then for $0 \le i < n-2$,

$$W_i(K, L, Q) \ge W_i(K, Q)^{\frac{n-i-2}{n-i-1}} W_i(L, Q)^{\frac{1}{n-i-1}} \tag{15}$$

and the inequality can be rewritten as

$$\frac{W_i(K,L,Q)}{W_i(K,Q)} \geq \left(\frac{W_i(L,Q)}{W_i(K,Q)}\right)^{\frac{1}{n-i-1}} \qquad (16)$$

We will extend the inequality (16) to the Orlicz setting in Theorem 4. Clearly, the equality holds in (15) and (16) if K and L are homothetic. In particular, for $Q=B$, we have

$$W_i(K)^{\frac{n-i-1}{n-i}} W_i(L)^{\frac{1}{n-i}} \leq W_i(K,L),\ 0\leq i\leq n-2$$

which is the fundamental inequality for mixed quermassintegrals. For $0\leq i<n-1$, and $\phi\in\Phi$, the mixed Orlicz-quermassintegral $W_{\phi,i}(K,L)$ has the following integral representation (see Ref.[14]):

$$W_{\phi,i}(K,L) = \frac{1}{n}\int_{S^{n-1}}\phi\left(\frac{h_L(u)}{h_K(u)}\right)h_K(u)\mathrm{d}S_{n-i-1}(K,u)$$

For $\phi\in\Phi$ and $0\leq i<n-2$, the mixed Orlicz-quermassintegral about three convex bodies $W_{\phi,i}(K,Q,L)$ is defined by

$$\frac{n-i-1}{n\phi_l'(1)}W_{\phi,i}(K,Q,L) = \lim_{\varepsilon\to 0^+}\frac{W_{\phi,i}(K+_\phi L,Q)-W_{\phi,i}(K,Q)}{\varepsilon} \qquad (17)$$

Here $\phi_l'(1)$ denotes the left derivative of $\phi(t)$ at $t=1$.

We will give the integral representation of $W_{\phi,i}(K,Q,L)$ in Section 2.

From (11), we see that if $K_1=\cdots=K_{n-i}=K$ and $K_{n-i-1}=\cdots=K_n=B$, then the dual mixed volume $\tilde{V}(\underbrace{K,\cdots,K}_{n-i},\underbrace{B,\cdots,B}_{i})$ is written as $\tilde{W}_i(K)$ (the dual quermassintegral of K). In particular, $\tilde{W}_0(K)=V(K)$ and $\tilde{W}_n=V(B)$. The dual mixed quermassintegral $\tilde{W}_i(K,L)$ has the following integral representation:

$$\tilde{W}_i(K,L)=\frac{1}{n}\int_{S^{n-1}}\rho_K^{n-i-1}(u)\rho_L(u)\mathrm{d}S(u) \qquad (18)$$

Then, let us introduce the dual mixed quermassintegrals $\tilde{W}_i(K,L,Q)$.

For $K,Q,L\in S_o^n$ and $0\leq i\leq n-2$, we define the dual quermassintegrals $\tilde{W}_i(K,L,Q)$ by

$$\tilde{W}_i(K,L,Q) = \frac{1}{n-i-1}\lim_{\varepsilon\to 0^+}\frac{\tilde{W}_i(K\tilde{+}\varepsilon\cdot L,Q)-\tilde{W}_i(K,Q)}{\varepsilon} \qquad (19)$$

For $K,L\in S_o^n$, $\psi\in\tilde{\Phi}\cup\tilde{\Psi}$ and $0\leq i\leq n-1$, the dual mixed Orlicz-quermassintegrals $\tilde{W}_{\psi,i}(K,L)$ has the following integral representation:

$$\tilde{W}_{\psi,i}(K,L)=\frac{1}{n}\int_{S^{n-1}}\psi\left(\frac{\rho_L(u)}{\rho_K(u)}\right)\rho_K^{n-i}(u)\mathrm{d}S(u)$$

Let $\psi\in\tilde{\Phi}\cup\tilde{\Psi}$, $K,Q,L\in S_o^n$ and $0\leq i\leq n-2$, we define the dual mixed Orlicz-quermassintegrals $\tilde{W}_{\psi,i}(K,L,Q)$ by

$$\tilde{W}_{\psi,i}(K,L,Q) = \frac{\psi_r'(1)}{n-i-1}\lim_{\varepsilon\to 0^+}\frac{\tilde{W}_i(K\tilde{+}_\psi\varepsilon\cdot L,Q)-\tilde{W}_i(K,Q)}{\varepsilon} \qquad (20)$$

Here $\psi_r'(1)$ denotes the right derivative of $\psi(t)$ at $t=1$.

2 The (Dual) Mixed Orlicz Quermassintegrals

Lemma 1 If $K,Q,L\in K_o^n$ and $\phi\in\Phi$ then for $0\leq i\leq n-2$,

$$W_{\phi,i}(K,L,Q) = \frac{1}{n}\int_{S^{n-1}}\phi\left(\frac{h_L(u)}{h_K(u)}\right)h_K(u)\mathrm{d}S_{n-i-1}(K,L,u)$$

Proof We write $K+_\phi\varepsilon\cdot L$ as K_ε, and define $g:[0,\infty)\to(0,\infty)$ by $g(\varepsilon)=W_i(K_\varepsilon,Q)^{\frac{1}{n-i-1}}$. Let

$$l_{\inf}=\liminf_{\varepsilon\to 0^+}\frac{W_i(K_\varepsilon,Q)-W_i(K_\varepsilon,K,Q)}{\varepsilon}$$

and

$$l_{\sup}=\limsup_{\varepsilon\to 0^+}\frac{W_i(K,K_\varepsilon,Q)-W_i(K,Q)}{\varepsilon}$$

Since $W_i(K_{\varepsilon_1})<W_i(K_{\varepsilon_2})$, for $0\leq\varepsilon_1<\varepsilon_2\leq\infty$, the existence of $l_{\inf}$ and $l_{\sup}$ is obtained. By (15), then

$$\liminf_{\varepsilon\to 0^+}W_i(K_\varepsilon,Q)^{\frac{n-i-2}{n-i-1}}\frac{W_i(K_\varepsilon,Q)^{\frac{1}{n-i-1}}-W_i(K,Q)^{\frac{1}{n-i-1}}}{\varepsilon} \geq l_{\inf}$$

and

$$\limsup_{\varepsilon\to 0^+}W_i(K_\varepsilon,Q)^{\frac{n-i-2}{n-i-1}}\frac{W_i(K_\varepsilon,Q)^{\frac{1}{n-i-1}}-W_i(K,Q)^{\frac{1}{n-i-1}}}{\varepsilon} \leq l_{\sup}$$

The continuity of the mixed quermassintegral W_i implies that g is continuous at origin o. Thus

$$W_i(K,Q)^{\frac{n-i-2}{n-i-1}} \cdot\liminf_{\varepsilon\to 0^+}\frac{W_i(K_\varepsilon,Q)^{\frac{1}{n-i-1}}-W_i(K,Q)^{\frac{1}{n-i-1}}}{\varepsilon}\geq l_{\inf} \qquad (21)$$

and

$$W_i(K,Q)^{\frac{n-i-2}{n-i-1}}$$

$$\cdot \limsup_{\varepsilon\to 0^+} \frac{W_i(K_\varepsilon,Q)^{\frac{1}{n-i-1}}-W_i(K,Q)^{\frac{1}{n-i-1}}}{\varepsilon}\le l_{\sup} \quad (22)$$

The weak continuity of surface area measures as well as $\lim\limits_{\varepsilon\to 0^+}K_\varepsilon=K$ (see Ref. [13]) implies that $\lim\limits_{\varepsilon\to 0^+}S_{n-i-1}(K_\varepsilon,Q,u)=S_{n-i-1}(K,Q,u)$, weakly on S^{n-1}.

From $\left.\frac{\partial}{\partial_\varepsilon}\right|_{\varepsilon=0^+} h_{K+_\phi\varepsilon\cdot L}(u)=\frac{1}{\phi_l'}\phi\left(\frac{h_L(u)}{h_K(u)}\right)h_K(u)$ (see Ref. [13]), we can obtain that

$$\begin{aligned}&\lim_{\varepsilon\to 0^+}\frac{W_i(K_\varepsilon,Q)-W_i(K_\varepsilon,K,Q)}{\varepsilon}\\&=\lim_{\varepsilon\to 0^+}\frac{W_i(K_\varepsilon,K_\varepsilon,Q)-W_i(K_\varepsilon,K,Q)}{\varepsilon}\\&=\frac{1}{n}\lim_{\varepsilon\to 0^+}\int_{S^{n-1}}\frac{h_{K_\varepsilon}(u)-h_K(u)}{\varepsilon}\mathrm{d}S_{n-i-1}(K_\varepsilon,Q,u)\\&=\frac{1}{n\phi_1'}\int_{S^{n-1}}\phi\left(\frac{h_L(u)}{h_K(u)}\right)h_K(u)\mathrm{d}S_{n-i-1}(K,Q,u)\end{aligned} \quad (23)$$

Similarly, we have

$$\begin{aligned}&\lim_{\varepsilon\to 0^+}\frac{W_i(K,K_\varepsilon,Q)-W_i(K_\varepsilon,K,Q)}{\varepsilon}\\&=\frac{1}{n\phi_1'}\int_{S^{n-1}}\phi\left(\frac{h_L(u)}{h_K(u)}\right)h_K(u)\mathrm{d}S_{n-i-1}(K,Q,u)\end{aligned} \quad (24)$$

Combining (21), (22), (23), and (24), we know that $g(\varepsilon)$ is differential at o. In fact, a bit more than $l_{\inf}\ge l_{\sup}$ will be proved, then $l_{\inf}=l_{\sup}$. Therefore, $g(\varepsilon)^{n-i-1}$ is differential at o, furthermore,

$$\begin{aligned}&\lim_{\varepsilon\to 0^+}\frac{W_i(K+_\phi\varepsilon\cdot L,Q)-W_i(K,Q)}{\varepsilon}\\&=\lim_{\varepsilon\to 0^+}\frac{g(\varepsilon)^{n-i-1}-g(0)^{n-i-1}}{\varepsilon}\\&=(n-i-1)g(0)^{n-i-2}\lim_{\varepsilon\to 0^+}\frac{g(\varepsilon)-g(0)}{\varepsilon}\\&=(n-i-1)l_{\inf}=(n-i-1)l_{\sup}\\&=\frac{n-i-1}{n\phi_1'}\int_{S^{n-1}}\phi\left(\frac{h_L(u)}{h_K(u)}\right)h_K(u)\mathrm{d}S_{n-i-1}(K,Q,u)\end{aligned}$$

By (17), we can complete the proof of Lemma 1.

Remark 1 For Lemma 1, if $\phi(t)=t^p(p\ge 1)$, then $+_\phi=+_p$ by (2), and the case $+_p$ was introduced by Wang [9] in 2013. If $Q=B$, $0\le i\le n-1$, then we have the integral representation of $W_{\phi,i}(K,L)$ (see also Ref. [14]).

Theorem 4 If $K,L,Q\in K_{\mathrm{o}}^n$, and $\phi\in\Phi$, then for $0\le i\le n-2$,

$$\frac{W_{\phi,i}(K,L,Q)}{W_i(K,L)}\ge\phi\left(\left(\frac{W_i(L,Q)}{W_i(K,Q)}\right)^{\frac{1}{n-i-1}}\right)$$

If ϕ is strictly convex, the equality holds if and only if K and L are dilates.

Proof If $\phi\in\Phi$, then by (13) and (16), we have

$$\begin{aligned}&\frac{W_{\phi,i}(K,L,Q)}{W_i(K,L)}\\&=\frac{1}{nW_i(K,Q)}\int_{S^{n-1}}\phi\left(\frac{h_L(u)}{h_K(u)}\right)h_K(u)\mathrm{d}S_{n-i-1}(K,Q,u)\\&\ge\phi\left(\frac{1}{nW_i(K,Q)}\int_{S^{n-1}}h_L(u)\mathrm{d}S_{n-i-1}(K,Q,u)\right)\\&=\phi\left(\frac{W_i(K,L,Q)}{W_i(K,L)}\right)\ge\phi\left(\left(\frac{W_i(L,Q)}{W_i(K,Q)}\right)^{\frac{1}{n-i-1}}\right)\end{aligned}$$

Now, we verify the equality conditions. First, from the equality condition of Jensen's inequality (13), the sufficiency is easy to prove, then we prove the necessity.

Suppose the equality holds. From the injectivity of ϕ, we have the equality in (16). Then, K and L are homothetic, so there exist $x\in\mathbf{R}^n$ and $r>0$ such that $L=rK+x$. Hence, by the definition of the support function, we have

$$h_L(u)=rh_K(u)+x\cdot u,\ \text{for all } u\in S^{n-1}$$

And then we just have to prove that $x=o$. Since ϕ is strictly convex, by the equality condition of Jensen's inequality, we have

$$\frac{1}{nW_i(K,Q)}\int_{S^{n-1}}\frac{h_L(u)}{h_K(u)}h_K(u)\mathrm{d}S_{n-i-1}(K,Q,u)=\frac{h_L(v)}{h_K(v)}$$

for $S_{n-i-1}(K,Q,\cdot)$-almost all $v\in S^{n-1}$. Thus,

$$\begin{aligned}&\frac{1}{nW_i(K,Q)}\int_{S^{n-1}}\left(r+\frac{x\cdot u}{h_K(u)}\right)h_K(u)\mathrm{d}S_{n-i-1}(K,Q,u)\\&=r+\frac{x\cdot v}{h_K(v)}\end{aligned}$$

Note that the centroid of $S_{n-i-1}(K,Q,\cdot)$ is at the origin, so it follows that

$$\begin{aligned}0&=x\cdot\left(\frac{1}{nW_i(K,Q)}\int_{S^{n-1}}u\mathrm{d}S_{n-i-1}(K,Q,u)\right)\\&=\frac{1}{nW_i(K,Q)}\int_{S^{n-1}}x\cdot u\mathrm{d}S_{n-i-1}(K,Q,u)=\frac{x\cdot v}{h_K(v)}\end{aligned}$$

Thus x is the origin, and therefore K and L are dilates.

Remark 2 The case $Q=B$ of Theorem 4 was established by Xiong and Zou [14], and when $i=0$, it is the Orlicz-Minkowski inequality (see Ref. [13]).

Lemma 2 Suppose $K,Q,L\in S_o^n$, then for $0\le i\le n-2$,

$$\tilde{W}_i(K,L,Q)=\frac{1}{n}\int_{S^{n-1}}\rho_K^{n-i-2}(u)\rho_L(u)\rho_Q(u)\mathrm{d}S(u) \quad (25)$$

In particular, $\tilde{W}_i(K,K,K)=\tilde{W}_i(K)$.

Proof By (7) and (8), we have

$$\lim_{\varepsilon\to0^+}\frac{\rho_{K\tilde{+}\varepsilon L}^{n-i-1}(u)-\rho_K^{n-i-1}(u)}{\varepsilon}$$
$$=(n-i-1)\rho_{K\tilde{+}\varepsilon L}^{n-i-2}(u)\Big|_{\varepsilon=0}\cdot\lim_{\varepsilon\to0^+}\frac{\rho_{K\tilde{+}\varepsilon L}(u)-\rho_K(u)}{\varepsilon}$$
$$=(n-i-1)\rho_K^{n-i-2}(u)\rho_L(u)$$

then using (18),

$$\frac{\tilde{W}_i(K\tilde{+}\varepsilon\cdot L,Q)-\tilde{W}_i(K,Q)}{\varepsilon}$$
$$=\frac{n-i-1}{n}\int_{S^{n-1}}\rho_K^{n-i-2}(u)\rho_L(u)\rho_Q(u)S(u)$$

Hence, by (19)

$$\tilde{W}_i(K,L,Q)=\frac{1}{n}\int_{S^{n-1}}\rho_K^{n-i-2}(u)\rho_L(u)\rho_Q(u)S(u)$$

Lemma 3 [27] Let $K,L\in S_o^n$ and $\psi\in\tilde{\Phi}\cup\tilde{\Psi}$. Then

$$\lim_{\varepsilon\to0^+}\frac{\rho_{K\tilde{+}_\psi\varepsilon\cdot L}(u)-\rho_K(u)}{\varepsilon}=\frac{\rho_K(u)}{\psi_r'(1)}\psi\left(\frac{\rho_L(u)}{\rho_K(u)}\right)$$

uniformly for all $u\in S^{n-1}$.

Lemma 4 Let $K,Q,L\in S_o^n$, $\psi\in\tilde{\Phi}\cup\tilde{\Psi}$ and $0\le i\le n-2$. Then

$$\tilde{W}_{\psi,i}(K,L,Q)$$
$$=\frac{1}{n}\int_{S^{n-1}}\psi\left(\frac{\rho_L(u)}{\rho_K(u)}\right)\rho_K^{n-i-1}(u)\rho_Q(u)\mathrm{d}S(u) \quad (26)$$

Proof Let $\varepsilon>0$, $K,Q,L\in S_o^n$, and $u\in S^{n-1}$. From Lemma 3, we have

$$\lim_{\varepsilon\to0^+}\frac{\rho_{K\tilde{+}_\psi\varepsilon\cdot L}^{n-i-1}(u)-\rho_K^{n-i-1}(u)}{\varepsilon}$$
$$=(n-i-1)\rho_K^{n-i-2}(u)\cdot\lim_{\varepsilon\to0^+}\frac{\rho_{K\tilde{+}_\psi\varepsilon\cdot L}(u)-\rho_K(u)}{\varepsilon}$$
$$=\frac{(n-i-1)\rho_K^{n-i-1}(u)}{\psi_r'(1)}\psi\left(\frac{\rho_L(u)}{\rho_K(u)}\right)$$

Then, by (18),

$$\lim_{\varepsilon\to0^+}\frac{\tilde{W}_i(K\tilde{+}_\psi\varepsilon\cdot L,Q)-\tilde{W}_i(K,Q)}{\varepsilon}$$
$$=\lim_{\varepsilon\to0^+}\frac{\frac{1}{n}\int_{S^{n-1}}\left(\rho_{K\tilde{+}_\psi\varepsilon\cdot L}^{n-i-1}(u)-\rho_K^{n-i-1}(u)\right)\rho_Q(u)\mathrm{d}S(u)}{\varepsilon}$$
$$=\frac{n-i-1}{n\psi_r''(1)}\int_{S^{n-1}}\psi\left(\frac{\rho_L(u)}{\rho_K(u)}\right)\rho_K^{n-i-1}(u)\rho_Q(u)\mathrm{d}S(u)$$

Hence, by (20), we have

$$\tilde{W}_{\psi,i}(K,L,Q)$$
$$=\frac{1}{n}\int_{S^{n-1}}\psi\left(\frac{\rho_L(u)}{\rho_K(u)}\right)\rho_K^{n-i-1}(u)\rho_Q(u)\mathrm{d}S(u)$$

Theorem 5 Let $K,Q,L\in S_o^n$, $\psi\in\tilde{\Phi}\cup\tilde{\Psi}$ and $0\le i\le n-2$. If $\psi_0(t)=\psi\left(t^{\frac{1}{n-i-1}}\right)$ is concave, then

$$\frac{\tilde{W}_{\psi,i}(K,L,Q)}{\tilde{W}_i(K,Q)}\le\psi\left(\left(\frac{\tilde{W}_i(L,Q)}{\tilde{W}_i(K,Q)}\right)^{\frac{1}{n-i-1}}\right)$$

while if $\psi_0(t)$ is convex, the inequality is reversed. When ψ_0 is strictly concave (or convex, as appropriate), the equality holds if and only if K and L are dilates.

Proof If $\psi_0(t)=\psi\left(t^{\frac{1}{n-i-1}}\right)$ is concave, by (26) and (13), it follows that

$$\frac{\tilde{W}_{\psi,i}(K,L,Q)}{\tilde{W}_i(K,Q)}$$
$$=\frac{1}{n\tilde{W}_i(K,Q)}\int_{S^{n-1}}\psi\left(\frac{\rho_L(u)}{\rho_K(u)}\right)\rho_K^{n-i-1}(u)\rho_Q(u)\mathrm{d}S(u)$$
$$=\frac{1}{n\tilde{W}_i(K,Q)}\int_{S^{n-1}}\psi_0\left(\left(\frac{\rho_L(u)}{\rho_K(u)}\right)^{n-i-1}\right)\rho_K^{n-i-1}(u)\rho_Q(u)\mathrm{d}S(u)$$
$$\le\psi_0\left(\frac{1}{n\tilde{W}_i(K,Q)}\int_{S^{n-1}}\rho_L^{n-i-1}(u)\rho_Q(u)\mathrm{d}S(u)\right)$$
$$=\psi_0\left(\frac{\tilde{W}_i(L,Q)}{\tilde{W}_i(K,Q)}\right)=\psi\left(\left(\frac{\tilde{W}_i(L,Q)}{\tilde{W}_i(K,Q)}\right)^{\frac{1}{n-i-1}}\right)$$

When ψ_0 is strictly concave, from the process of proving the equality of Theorem 4, we have that K and L are dilates.

Remark 3 For Theorem 5, taking $Q=B$ and $0\le i\le n-1$, $\psi_0(t)=\psi\left(t^{\frac{1}{n-i}}\right)$, we can obtain an inequality which was established by Liu [22]. Furthermore, the case $i=0$ is the dual Orlicz-Minkowski inequality (see Refs. [17, 27]).

3 Proof of the Main Results

Theorem 6 Let $K,L,Q\in K_{\text{o}}^{n}$, $\phi\in\varPhi$ and $0\le i\le n-2$. Then

$$\phi\left(\left(\frac{W_i\left(K,Q\right)}{W_i\left(K+_{\phi}L,Q\right)}\right)^{\frac{1}{n-i-1}}\right)+\phi\left(\left(\frac{W_i\left(L,Q\right)}{W_i\left(K+_{\phi}L,Q\right)}\right)^{\frac{1}{n-i-1}}\right)\le\phi(1)$$

If ϕ is strictly convex, the equality holds if and only if K and L are dilates.

Proof We write $K+_{\phi}L$ as $\tilde{K}$. From (2), (14) and Lemma 1, it follows that

$$\begin{aligned}&\phi(1)W_i\left(\tilde{K},Q\right)\\&=\phi(1)W_i\left(\tilde{K},\tilde{K},Q\right)\\&=\frac{1}{n}\int_{S^{n-1}}\left(\phi\left(\frac{h_K(u)}{h_{\tilde{K}}(u)}\right)+\phi\left(\frac{h_L(u)}{h_{\tilde{K}}(u)}\right)\right)h_{\tilde{K}}(u)\mathrm{d}S_{n-i-1}\left(\tilde{K},Q;u\right)\\&=W_{\phi,i}\left(\tilde{K},K,Q\right)+W_{\phi,i}\left(\tilde{K},L,Q\right)\end{aligned}$$

By Theorem 4, we have

$$\begin{aligned}&W_{\phi,i}\left(\tilde{K},Q,K\right)+W_{\phi,i}\left(\tilde{K},Q,L\right)\\&\ge W_i\left(\tilde{K},Q\right)\phi\left(\left(\frac{W_i\left(K,Q\right)}{W_i\left(K+_{\phi}L,Q\right)}\right)^{\frac{1}{n-i-1}}\right)\\&+W_i\left(\tilde{K},Q\right)\phi\left(\left(\frac{W_i\left(L,Q\right)}{W_i\left(K+_{\phi}L,Q\right)}\right)^{\frac{1}{n-i-1}}\right)\end{aligned}$$

Thus, the proof of the inequality of this theorem is completed. From Theorem 4, the equality conditions can be obtained immediately.

Remark 4 For Theorem 6, the case $Q=B$ and $0\le i\le n-1$ is (3).

Theorem 7 Let $K,Q,L\in S_{\text{o}}^{n}$, $\psi\in\tilde{\varPhi}\cup\tilde{\varPsi}$ and $0\le i\le n-2$. If $\psi_0(t)=\psi\left(t^{\frac{1}{n-i-1}}\right)$ is concave, then

$$\psi\left(\left(\frac{\tilde{W}_i\left(K,Q\right)}{\tilde{W}_i\left(K\tilde{+}_{\psi}L,Q\right)}\right)^{\frac{1}{n-i-1}}\right)+\psi\left(\left(\frac{\tilde{W}_i\left(L,Q\right)}{\tilde{W}_i\left(K\tilde{+}_{\psi}L,Q\right)}\right)^{\frac{1}{n-i-1}}\right)\ge\psi(1)$$

while if $\psi_0(t)$ is convex, the inequality is reversed. When ψ_0 is strictly concave (or convex, as appropriate), the equality holds if and only if K and L are dilates.

Proof We only prove the case in which $\psi_0(t)$ is concave, and the case in which $\psi_0(t)$ is convex is analogous. Let $K_\psi=K\tilde{+}_{\psi}L$. If $\psi_0(t)=\psi\left(t^{\frac{1}{n-i-1}}\right)$ is concave, by (4), (18), (26) and Theorem 5, it follows that

$$\begin{aligned}&\psi(1)\\&=\frac{1}{n\tilde{W}_i\left(K_\psi,Q\right)}\int_{S^{n-1}}\psi\left(\frac{\rho_K(u)}{\rho_{K_\psi}(u)}\right)\rho_{K_\psi}^{n-i-1}(u)\rho_Q(u)\mathrm{d}S(u)\\&+\frac{1}{n\tilde{W}_i\left(K_\psi,Q\right)}\int_{S^{n-1}}\psi\left(\frac{\rho_L(u)}{\rho_{K_\psi}(u)}\right)\rho_{K_\psi}^{n-i-1}(u)\rho_Q(u)\mathrm{d}S(u)\\&=\frac{\tilde{W}_{\psi,i}\left(K_\psi,K,Q\right)}{\tilde{W}_i\left(K_\psi,Q\right)}+\frac{\tilde{W}_{\psi,i}\left(K_\psi,L,Q\right)}{\tilde{W}_i\left(K_\psi,Q\right)}\\&\le\psi\left(\left(\frac{\tilde{W}_i\left(K,Q\right)}{\tilde{W}_i\left(K_\psi,Q\right)}\right)^{\frac{1}{n-i-1}}\right)+\psi\left(\left(\frac{\tilde{W}_i\left(L,Q\right)}{\tilde{W}_i\left(K_\psi,Q\right)}\right)^{\frac{1}{n-i-1}}\right)\end{aligned}$$

When ψ_0 is strictly concave, from the process of proving the equality of Theorem 4, we know the equality holds if and only if that K and L are dilates.

Remark 5 For Theorem 7, taking $Q=B$ and $0\le i\le n-1$, $\psi_0(t)=\psi\left(t^{\frac{1}{n-i-1}}\right)$, we can obtain an inequality which was established by Liu [22]. Furthermore, the case $i=0$ is the dual Orlicz-Minkowski inequality (see Refs. [17, 27]).

Lemma 5 [21] Let $K,L\in K_{\text{o}}^{n}$ and $\phi\in\varPhi$. If $\psi(t)=\phi(t^{-1})$, then

$$K^*\tilde{+}_{\psi}L^*=\left(K+_{\phi}L\right)^*$$

Proof of Theorem 1 Suppose $\psi(t)=\phi(t^{-1})$. We clearly have that $\psi\in\tilde{\varPsi}$ and, moreover, that $\psi_0(t)=\psi\left(t^{\frac{1}{n-i-1}}\right)$ is convex. From Theorem 7 (for K^* and L^*) together with Lemma 5, we get

$$\begin{aligned}\phi(1) &= \psi(1)\\ &\geq \psi\left(\left(\frac{\tilde{W}_i(K^*,Q)}{\tilde{W}_i(K^* \tilde{+}_\psi L^*,Q)}\right)^{\frac{1}{n-i-1}}\right) + \psi\left(\left(\frac{\tilde{W}_i(L^*,Q)}{\tilde{W}_i(K^* \tilde{+}_\psi L^*,Q)}\right)^{\frac{1}{n-i-1}}\right)\\ &= \phi\left(\left(\frac{\tilde{W}_i(K^*,Q)}{\tilde{W}_i(K^* \tilde{+}_\phi L^*,Q)}\right)^{-\frac{1}{n-i-1}}\right) + \phi\left(\left(\frac{\tilde{W}_i(L^*,Q)}{\tilde{W}_i(K^* \tilde{+}_\phi L^*,Q)}\right)^{-\frac{1}{n-i-1}}\right)\end{aligned}$$

By the equality condition of Theorem 7, equality holds if and only if K and L are dilates.

Remark 6 For Theorem 1, the case $Q=B$ and $0\le i\le n-1$ is (5). Furthermore, when $\psi(t)=t^p$, $p\ge 1$, the case $i=0$ is stated by Firey [28].

Lemma 6[22] Let $K,L\in S_o^n$ and $\psi\in\tilde{\Phi}\cup\tilde{\Psi}$. If $\phi(t)=\psi(t^{-1})$, then

$$K^\circ \tilde{+}_\phi L^\circ = (K \tilde{+}_\psi L)^\circ$$

Proof of Theorem 2 Without the loss of generality, we may consider that ψ_0 is concave. Suppose $\psi\in\tilde{\Phi}$, $\phi(t)=\psi(t^{-1})$, so $\phi\in\tilde{\Psi}$. Thus, from Theorem 6 (for ϕ, K° and L°) together with Lemma 7, we get

$$\begin{aligned}\psi(1) &= \phi(1)\\ &\le \phi\left(\left(\frac{\tilde{W}_i(K^\circ,Q)}{\tilde{W}_i(K^\circ \tilde{+}_\phi L^\circ,Q)}\right)^{\frac{1}{n-i-1}}\right) + \phi\left(\left(\frac{\tilde{W}_i(L^\circ,Q)}{\tilde{W}_i(K^\circ \tilde{+}_\phi L^\circ,Q)}\right)^{\frac{1}{n-i-1}}\right)\\ &= \psi\left(\left(\frac{\tilde{W}_i(K^\circ,Q)}{\tilde{W}_i((K \tilde{+}_\psi L)^\circ,Q)}\right)^{-\frac{1}{n-i-1}}\right) + \psi\left(\left(\frac{\tilde{W}_i(L^\circ,Q)}{\tilde{W}_i((K \tilde{+}_\psi L)^\circ,Q)}\right)^{-\frac{1}{n-i-1}}\right)\end{aligned}$$

with equality if and only if K and L are dilates.

Remark 7 For Theorem 3, the case $Q=B$ and $0\le i\le n-1$ is (6).

Lemma 7[22] Let $K,L\in K_o^n$ and $\phi\in\tilde{\Phi}$ such that $\phi(t)=\psi(t^{-1})$ is convex, then

$$K +_\phi L = (K^* \tilde{+}_\psi L^*)^*$$

Proof of Theorem 3 Suppose $\phi(t)=\psi(t^{-1})$. We clearly have $\phi\in\Phi$. From Theorem 6 (for K^* and L^*) together with Lemma 6, we get

$$\begin{aligned}\psi(1) &= \phi(1)\\ &\geq \phi\left(\left(\frac{W_i(K^*,Q)}{W_i(K^* \tilde{+}_\phi L^*,Q)}\right)^{\frac{1}{n-i-1}}\right) + \phi\left(\left(\frac{W_i(L^*,Q)}{W_i(K^* \tilde{+}_\phi L^*,Q)}\right)^{\frac{1}{n-i-1}}\right)\\ &= \psi\left(\left(\frac{W_i(K^*,Q)}{W_i((K \tilde{+}_\psi L)^*,Q)}\right)^{-\frac{1}{n-i-1}}\right) + \psi\left(\left(\frac{W_i(L^*,Q)}{W_i((K \tilde{+}_\psi L)^*,Q)}\right)^{-\frac{1}{n-i-1}}\right)\end{aligned}$$

From the equality condition of Theorem 6, equality holds if and only if K and L are dilates.

Remark 8 For Theorem 2, the case $Q=B$ and $0\le i\le n-1$ is (7).

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