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Security Steiner's Inequality on Layering Functions

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Abstract: The classical definition of Steiner symmetrizations of functions are defined according to the Steiner symmetrizations of function level sets and the layered representation of functions. In this paper, the definition is not only transformed into Steiner symmetrizations of one-dimensional parabolic functions, but also depends on the Steiner symmetrizations of the level sets of log-concave functions. To this end we prove Steiner's inequality on layering functions in the space of log-concave functions.

Key words: convex body; Steiner symmetrizations; log-concave function

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0 Introduction

Steiner symmetrizations were invented by Steiner[1] to prove the isoperimetric inequality. For well over a century Steiner symmetrizations has played a fundamental role in answering questions about isoperimetry and related geometric inequalities[2]. Steiner symmetrizations appears explicitly in the titles of numerous papers (see Refs. [3,4]) and plays a key roles in recent work such as Ref. [5].

Let Ω be an open bounded subset of $\mathbf{R}^n$. For $f:\Omega\to(0,+\infty)$, we define the layering function $\Omega_0(f)$:

$$\Omega_0(f)=\int_0^{+\infty}\int_0^{\infty}V_n([f]_h\cap rB)\mathrm{e}^{-r^2}\mathrm{d}r]\,\mathrm{d}h$$

where $r\ge 0$ denotes by rB the closed Euclidean ball of radius r centered at the origin.

Evidently the function Ω_0 is monotonic and continuous on functions defined on Ω. The layering function vanishes on functions whose support with empty interior and is strictly positive on functions whose support with non-empty interior. In this paper, we mainly prove the following theorems.

Theorem 1 Suppose that $f:\Omega\to(0,+\infty)$ is a log-concave function, and let $\boldsymbol{u}$ be a unit vector. Then its Steiner symmetrizations $S_{\boldsymbol{u}}f$ satisfies

$$\Omega_0(S_{\boldsymbol{u}}f)\ge\Omega_0(f) \tag{1}$$

The equality holds in (1) if and only if $S_{\boldsymbol{u}}f=f$.

Theorem 2 Suppose that $f:\Omega\to(0,+\infty)$ is a log-concave function. Then its symmetric decreasing rearrangement f^* satisfies

$$\Omega_0(f^*)\ge\Omega_0(f) \tag{2}$$

The equality holds in (2) if and only if $f^*=f$.

In Ref. [6], Klain proved geometric inequality which is corresponding with Theorem 1. If $S(K)$ denotes the surface area of a compact convex set K having non-empty interior, then $S(S_u K) \le S(K)$, with equality if only and if K and $S_u K$ are translations. Theorem 1 is an extension of the geometric inequality by Ref. [6]. The layering function Ω_0 is more appropriate for our purposes, because the equality case in Theorem 1 is more stringent.

1 Preliminaries

Denote n-dimensional Euclidean space by $\mathbf{R}^n$ and let K_n denote the set of all compact convex sets in $\mathbf{R}^n$. Let Ω be the domain of log-concave function f, and let sub f denote the subgraph of f. Let $[f]_h$ denote the subgraph of f. For $K \in K_n$, let $V_n(K)$ denote the n-dimensional volume of K, and let $\boldsymbol{u}$ be a unit vector. Viewing K as a family of line segments parallel to $\boldsymbol{u}$, slide these segments along $\boldsymbol{u}$ so that each is symmetrically balanced around the hyperplane $\boldsymbol{u}^{\perp}$, where $\boldsymbol{u}^{\perp}$ denotes the complementary space of $\boldsymbol{u}$. By Cavalieri's principle, the volume of K is unchanged by this rearrangement. The new set, called the Steiner symmetrizations of K in the direction of $\boldsymbol{u}$, will be denoted by $S_{\boldsymbol{u}} K$. It is not difficult to show that $S_{\boldsymbol{u}} K$ is also convex, and that $S_{\boldsymbol{u}} K \subseteq S_{\boldsymbol{u}} L$ whenever $K \subseteq L$. A little more work verifies the following intuitive assertion: if you iterate Steiner symmetrizations of K through a suitable sequence of unit directions, the successive Steiner symmetrals of K will approach a Euclidean ball in the Hausdorff topology on compact (convex) subsets of $\mathbf{R}^n$. A detailed proof of this assertion can be found in Refs. [7, 8], for example.

In this section we present some specific elementary properties of Steiner symmetrizations, together with known facts to be used in the proof. Standard references for fundamental properties of Steiner symmetrizations are the Ref. [9]. See also Ref. [10] for general background on classical convexity theory.

A non-negative function is called a log-concave when the logarithm of the function is concave. Denote log-concave function

$$f = \mathrm{e}^{-\phi(x)}$$

where $\phi(x): \mathbf{R}^n \to \mathbf{R} \cup \{+\infty\}$ is convex, and $x \in \Omega$. It is clear that

$$\log f = -\phi(x)$$

Denote the level sets of log-concave function:

$$[f]_h = \{x \in \mathbf{R}^n : f(x) \ge h\}$$

Then $[f]_h$ is convex. By the layer representation,

$$f(x) = \int_0^{+\infty} \chi_{[f]_h}(x)\mathrm{d}h \tag{3}$$

For $f: \Omega \to (0, +\infty)$, we define the symmetric decreasing rearrangement f^* of f by symmetrizing its level sets, that is

$$f^*(x) = \int_0^{+\infty} \chi_{[f]^*_h}(x)\mathrm{d}h$$

Given a compact convex set K and a unit vector $\boldsymbol{u}$, we have $S_{\boldsymbol{u}} K = K$ (or respectively, up to translation) if and only if K is symmetric under reflection across the subspace $\boldsymbol{u}^{\perp}$ (respectively, up to translation). In particular, $S_{\boldsymbol{u}} K = K$ will hold for every direction $\boldsymbol{u}$ (or even a dense set of directions) if and only if K is a Euclidean ball centered at the origin.

Given compact convex subsets $K, L \subseteq \mathbf{R}^n$ and $a, b \ge 0$, we denote

$$aK + bL = \{ax + by \mid x \in K \text{ and } y \in L\}$$

An expression of this form is called a Minkowski combination or Minkowki sum. Since K and L are convex sets, the set $aK + bL$ is also convex. Convexity also implies that $aK + bK = (a+b)K$ for all $a, b \ge 0$, although this does not hold for general sets.

2 Proof of the Main Theorem

The following crucial property of Steiner symmetrizations will be used in the main theorems.

Lemma 1 (see Ref. [11]) If $f: \Omega \to [0, +\infty)$ is a log-concave function, then there exists a sequence of successive Steiner symmetrizations $\{f_i\}_{i=1}^{\infty}$ of f which approximates its symmetric f^* in the norm $W^{1,1}(\mathbf{R}^n)$, i.e.,

$$\lim_{i \to 0} \int_{\mathbf{R}^n} (|f_i(x) - f^*(x)| + |\nabla f_i(x) - \nabla f^*(x)|)\mathrm{d}x = 0$$

Lemma 2 (see Ref. [11]) For two origin-centered ellipsoids E_1 and E_2 in $\mathbf{R}^n$, if for any $\boldsymbol{u} \in S^{n-1}$, the midpoints of the chords of E_1 and E_2 parallel to $\boldsymbol{u}$ are coplanar, then there exists $r > 0$ such that

$$E_1 = rE_2$$

Lemma 3 (see Ref. [11]) If $f: \Omega \to (0, +\infty)$ is a log-concave function, and

$$[f]_h = \left(\frac{|[f]_h|}{\omega_n}\right)^{\frac{1}{n}}(B_n)$$

Then $f(x) = f^*(x)$.

Lemma 4 If D is a ball centered at the origin, and if X is a line segment, parallel to the unit vector $\boldsymbol{u}$, having one endpoint in the interior of *D* and the other endpoint outside *D*, then Steiner symmetrizations will strictly increase the slice length; that is

$$|S_{\boldsymbol{u}}K \cap D| > |X \cap D| \tag{4}$$

Proof Let ℓ denote the line through *X*. Our conditions on the endpoints of *X* imply that $|\ell \cap D| > |X \cap D$. Meanwhile, $S_{\boldsymbol{u}}K$ fixes *D* and slides *X* parallel to *u* until it is symmetric about $\boldsymbol{u}^{\perp}$. If $|X| < |\ell \cap D|$, then $S_{\boldsymbol{u}}K$ will lie wholly inside *D*, so that $|S_{\boldsymbol{u}}K \cap D| = |X| > |X \cap D|$ and (4) follows. If $|X| \geq |\ell \cap D|$, then $S_{\boldsymbol{u}}K$ will cover the slice $\ell \cap D$ completely, so that $|S_{\boldsymbol{u}}K \cap D| = |\ell \cap D|$ and (4) follows once again.

Proof of Theorem 1 Let $\boldsymbol{u}$ be a unit vector. The monotonicity of Steiner symmetrizations implies that

$$S_{\boldsymbol{u}}\left([f]_h \cap rB\right) \subset S_{\boldsymbol{u}}[f]_h \cap S_{\boldsymbol{u}}rB = S_{\boldsymbol{u}}[f]_h \cap rB$$

Thus

$$V_n\left(S_{\boldsymbol{u}}[f]_h \cap rB\right) \geq V_n\left(S_{\boldsymbol{u}}([f]_h \cap rB)\right) = V_n([f]_h \cap rB)$$

whence $\Omega_0\left(S_{\boldsymbol{u}}[f]_h\right) \geq \Omega_0\left([f]_h\right)$. It follows that

$$\Omega_0(S_{\boldsymbol{u}}f) \geq \Omega_0(f)$$

Evidently equality holds if $S_{\boldsymbol{u}}[f]_h = [f]_h$. For the converse, suppose that $[f]_h$ has non-empty interior, and that $S_{\boldsymbol{u}}[f]_h \neq [f]_h$. Let ψ denote the reflection of $\mathbf{R}^n$ across the subspace $\boldsymbol{u}^{\perp}$. Since $\psi[f]_h \neq [f]_h$ and $[f]_h$ has non-empty interior, there is a point $x \in \mathrm{int}([f]_h)$ such that $\psi\, x \notin [f]_h$. Let *D* denote the ball around the origin of radius $|x|$, and let ℓ denote the line through *x* and parallel to $\boldsymbol{u}$. The slice $[f]_h \cap \ell$ meets the boundary of *D* at *x* on one side of $\boldsymbol{u}^{\perp}$, and has an endpoint $x \pm \varepsilon\boldsymbol{u}$ outside *D* and another endpoint $x - \delta\boldsymbol{u}$ in the interior of *D*, where $\varepsilon, \delta > 0$. It follows from (4) of Lemma 4 that

$$|S_{\boldsymbol{u}}[f]_h \cap \ell \cap D| > |[f]_h \cap \ell \cap D|$$

Moreover, this holds for parallel slices through point x' in an open neighborhood of *x*. After integration of parallel slice lengths to compute volumes, we obtain

$$V_n\left(S_{\boldsymbol{u}}[f]_h \cap rB\right) > V_n([f]_h \cap rB)$$

for values of *r* in an open neighborhood of $|x|$. It follows that $\Omega_0\left(S_{\boldsymbol{u}}[f]_h\right) > \Omega_0\left([f]_h\right)$.

Hence,

$$\Omega_0(S_{\boldsymbol{u}}f) > \Omega_0(f)$$

So, if $\Omega_0(S_{\boldsymbol{u}}f) = \Omega_0(f)$, then $S_{\boldsymbol{u}}[f]_h = [f]_h$ for any $h > 0$. Therefore, by (3) that $S_{\boldsymbol{u}}f = f$.

Therefore, we complete the proof of Theorem 1.

Proof of Theorem 2 Suppose $f(x) \neq f^*(x)$. By Lemma 2 and Lemma 3, there exist some direction $\boldsymbol{u}$ such that the midpoints of the chords of sub *f* parallel to $\boldsymbol{u}$ do not lie in any linear subspace of $\mathbf{R}^{n+1}$ parallel to e_{n+1}. Let $f_1 = S_{\boldsymbol{u}}f$. From Theorem 1 it follows that

$$\Omega_0(f) < \Omega_0(f_1)$$

By Lemma 1, there exists a sequence of directions $\{u_i\}, i = 1, 2, \cdots, \infty$, such that the sequence defined by $f_{i+1} = S_{u_i}f_i$ converges to f^* in the $W^{1,1}(\mathbf{R}^n)$ norm. Thus, by the continuity of $\Omega_0(f)$ in the space $W^{1,1}(\mathbf{R}^n)$, we have

$$\Omega_0(f) < \Omega_0(f_1) \leq \cdots \leq \Omega_0(f_i) \to \Omega_0(f^*)$$

This implies that if $\Omega_0(f) = \Omega_0(f^*)$, then $f = f^*$. Using the same argument, we can get that for any log-concave function $f: \Omega \to (0, +\infty)$, $\Omega_0(f) \leq \Omega_0(f^*)$. This proves the result of Theorem 2.

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