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Finite-Time Generalized Synchronization and Parameter Identification of Chaotic Systems with Different Dimensions

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Abstract: Finite-time generalized synchronization and parameter identification of chaotic systems with different dimensions are studied in this paper. Based on finite-time stability theory, adaptive controllers and parameter identification rules are proposed to realize generalized synchronization of chaotic systems and identify accurately all their unknown parameters in finite time. At the same time, the obtained synchronization time formula can estimate the required longest time for realizing the generalized synchronization of chaotic systems, which solely depends on the initial values of chaotic systems and is easily calculated in numerical simulation. At the last, some numerical examples are presented to illustrate the effectiveness and correctness of the obtained conclusions.

Key words: finite-time generalized synchronization; parameter identification; chaotic systems with different dimensions

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0 Introduction

Since the pioneering work of Pecora and Carroll in 1990[1], chaotic synchronization has been a hot topic among scientists and has been applied largely in many subjects such as chemistry, mathematics, biology, information processing, secure communication[2-4], etc. Simultaneously, many theoretical results about chaos have been obtained. One of the important results is a variety of synchronous modes, such as complete synchronization[1,5], anti-synchronization[5], phase synchronization[6], lag synchronization[7], functional projective synchronization[8,9], generalized synchronization[10], etc. By comparing we find that the generalized synchronization and functional projective synchronization are two kinds of the most complex synchronization. This complexity can make the communication information based on the functional projective synchronization or generalized synchronization of chaotic systems is more secure and reliable.

Synchronization between two similar or identical chaotic systems has been studied in recent decade years. Synchronization of chaotic systems with different dimensions can be found in nature, and synchronization between heart circulatory system and lung respiratory system is one of the classical examples[10]. At same time, applying chaotic systems with different dimensions to secure communication can make the communication information more secure than chaotic systems with same

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dimensions[11,12]. Many authors studied synchronization of chaotic systems with different dimensions by reducing order[13,14], the aim of which is to make drive system and response system have the same orders, but it is a pity that the synchronization time of chaotic systems has been not considered in these papers.

Synchronization time is a very important problem for controlling chaotic systems to realize synchronization, so the finite-time synchronization of chaotic systems has been studied in many papers. Finite-time complete synchronization of chaotic systems with different dimensions has been studied[15]. Finite-time synchronization and unknown parameter identification of the unified chaotic system have been discussed[16]. The authors have studied finite-time complete synchronization of chaotic systems with different orders[17], and achieved the robust increasing order and reduced order synchronization of the chaotic systems in finite-time.

We have found that the considered chaotic systems are either some special systems or systems without unknown parameters in these papers, and studied their complete synchronization which is the simplest synchronous pattern. Generalized synchronization of the chaotic systems with different order has been realized[10], however, it has not considered the identification of unknown parameters.

Based on the above discussion, it is meaningful to study the finite-time generalized synchronization of chaotic systems with different dimensions, and identify all unknown parameters of the drive system and response system. The obtained results could be applied to study the finite-time generalized synchronization between any two chaotic systems with unknown parameters, and also could be used in the field of secure communications.

The rest of this paper is arranged as follows: the models of the drive system and response system, the preliminary definitions and some lemmas are given in Section 1, the finite-time generalized synchronization between the drive system and response system is realized and unknown parameters of the systems are identified in Section 2; some numerical examples are employed to illustrate the correctness of the obtained results in Section 3, and conclusions are presented in Section 4.

1 The Models and Preliminary Knowledge

A class of chaotic systems with n dimensions as the drive system is given as follows,

$$\dot{\boldsymbol{x}}(t)=f(\boldsymbol{x})+F(\boldsymbol{x})\boldsymbol{\mu} \tag{1}$$

where $\boldsymbol{x}=(x_1,x_2,\cdots,x_n)^{\mathrm{T}}\in\mathbf{R}^n$ denotes the state vector, $f:\mathbf{R}^n\to\mathbf{R}^n$ is a nonlinear vector function, $F:\mathbf{R}^n\to\mathbf{R}^{n\times k}$ is a matrix function which is relevant to the uncertain parameter vector $\boldsymbol{\mu}$ with k dimensions. Similarly, the controlled response system which has the different dimensions with the drive system is described as follows:

$$\dot{\boldsymbol{y}}(t)=\boldsymbol{g}(\boldsymbol{y})+\boldsymbol{G}(\boldsymbol{y})\boldsymbol{\theta}+\boldsymbol{u}(\boldsymbol{x},\boldsymbol{y}) \tag{2}$$

where $\boldsymbol{y}=(y_1,y_2,\cdots,y_m)^{\mathrm{T}}\in\mathbf{R}^m$ is the state vector of the response system, $\boldsymbol{g}:\mathbf{R}^m\to\mathbf{R}^m$ is a nonlinear vector function, $\boldsymbol{G}:\mathbf{R}^m\to\mathbf{R}^{m\times l}$ is a matrix function, $\boldsymbol{\theta}$ is a uncertain parameter vector with l dimensions, $\boldsymbol{u}$ is a control input whose content will be given after that.

Theorem 1 Assume that a continuous, differential and positive function $V(t)$ can satisfy the inequality

$$\dot{V}(t)\le -cV^{\eta}(t) \tag{3}$$

and $V(t)\equiv 0$ as $t\ge t_1$, then

$$t_1\le t_0+\frac{V^{1-\eta}(t_0)}{c(1-\eta)} \tag{4}$$

where $c>0$ and $0<\eta<1$ are constants.

Proof According to the differential inequality (3), we have

$$\frac{\mathrm{d}V(t)}{V^{\eta}(t)}\le -c\mathrm{d}t \tag{5}$$

We integrate at both sides of the inequality (5) and get

$$\int_{t_0}^{t}\frac{\mathrm{d}V(t)}{V^{\eta}(t)}\le -c\int_{t_0}^{t}\mathrm{d}t$$

$$V^{1-\eta}(t)\le V^{1-\eta}(t_0)-c(1-\eta)(t-t_0) \tag{6}$$

According to $V(t)\equiv 0$ as $t\ge t_1$, we can obtain

$$t_1\le t_0+\frac{V^{1-\eta}(t_0)}{c(1-\eta)} \tag{7}$$

Lemma 1[18] For any real numbers α_i ($i=1,2,3,\cdots,n$), and $0<r<1$, we can obtain the inequality

$$\Big(\sum_{i=1}^{n}|\alpha_i|\Big)^r\le\sum_{i=1}^{n}|\alpha_i|^r \tag{8}$$

2 Main Results

Chaotic systems have a common characteristic which is extremely sensitive to its parameter variation and initial conditions. Simultaneously, due to the complex of chaotic system, they have plenty of uncertain parameters which are important and necessary to apply

in practical situation. Therefore, it has important meanings to study the uncertain parameter identification of chaotic system. In this section, we realize finite-time generalized synchronization and parameter identification of chaotic systems with different dimensions. A definition about finite-time generalized synchronization is presented as follows.

Definition 1 If there exists a positive constant T, such that

$$\lim_{t\to T}\|\boldsymbol{y}(t)-\varphi(\boldsymbol{x}(t))\|=0 \tag{9}$$

and $\|\boldsymbol{y}(t)-\varphi(\boldsymbol{x}(t))\|\equiv 0$ as $t\geq T$, then the drive system (1) and the response system (2) are said to be finite-time generalized synchronization, where φ is an arbitrary continuous and differentiable function.

Remark 1 Generalized synchronization is that the state variables of the response system track the state variable functions of the drive system. The studies for finite-time synchronization of the chaotic systems in Refs. [16, 19] are all the special condition as $\varphi=\varepsilon$, where ε is identity mapping. Therefore, the study in the paper has promoting significance.

According to Definition 1, we give synchronous error as follows:

$$\boldsymbol{e}(t)=\boldsymbol{y}(t)-\varphi(\boldsymbol{x}(t)) \tag{10}$$

Thus finite-time generalized synchronization of the drive system (1) and the response system (2) is translated into zero solution stability of synchronous error system, which can be showed as follows:

$$\begin{aligned}\dot{\boldsymbol{e}}(t)&=\dot{\boldsymbol{y}}(t)-D\varphi(\boldsymbol{x}(t))\dot{\boldsymbol{x}}(t)\\&=\boldsymbol{g}(\boldsymbol{y})+\boldsymbol{G}(\boldsymbol{y})\boldsymbol{\theta}+\boldsymbol{u}(\boldsymbol{x},\boldsymbol{y})\\&\quad-D\varphi(\boldsymbol{x}(t))\boldsymbol{f}(\boldsymbol{x})-D\varphi(\boldsymbol{x}(t))\boldsymbol{F}(\boldsymbol{x})\boldsymbol{\mu}\end{aligned} \tag{11}$$

where the $D\varphi(\boldsymbol{x}(t))$ is the Jacobian matrix of the mapping $\varphi(\boldsymbol{x}(t))$ and is showed as follows:

$$D\varphi(\boldsymbol{x}(t))=\begin{pmatrix}\dfrac{\partial\varphi_1(\boldsymbol{x})}{\partial x_1} & \dfrac{\partial\varphi_1(\boldsymbol{x})}{\partial x_2} & \cdots & \dfrac{\partial\varphi_1(\boldsymbol{x})}{\partial x_n}\\ \dfrac{\partial\varphi_2(\boldsymbol{x})}{\partial x_1} & \dfrac{\partial\varphi_2(\boldsymbol{x})}{\partial x_2} & \cdots & \dfrac{\partial\varphi_2(\boldsymbol{x})}{\partial x_n}\\ \vdots & \vdots & \ddots & \vdots\\ \dfrac{\partial\varphi_m(\boldsymbol{x})}{\partial x_1} & \dfrac{\partial\varphi_m(\boldsymbol{x})}{\partial x_2} & \cdots & \dfrac{\partial\varphi_m(\boldsymbol{x})}{\partial x_n}\end{pmatrix}$$

In order to realize zero solution stability of synchronous error system (11) and identify unknown parameters of the two systems, we give concrete content to controller $\boldsymbol{u}(\boldsymbol{x},\boldsymbol{y})$ in the following theorem.

Theorem 2 If the controller in the response system is

$$\begin{aligned}\boldsymbol{u}(\boldsymbol{x},\boldsymbol{y})&=-\boldsymbol{g}(\boldsymbol{y})-\boldsymbol{G}(\boldsymbol{y})\tilde{\boldsymbol{\theta}}+D\varphi(\boldsymbol{x}(t))\boldsymbol{f}(\boldsymbol{x})\\&\quad+D\varphi(\boldsymbol{x}(t))\boldsymbol{F}(\boldsymbol{x})\tilde{\boldsymbol{\mu}}-\text{sign }(\boldsymbol{e})|\boldsymbol{e}|^{\alpha}\end{aligned} \tag{12}$$

and the identification rules of the unknown parameters in the drive systems and response system are

$$\begin{cases}\dot{\tilde{\boldsymbol{\theta}}}=\boldsymbol{G}^{\mathrm{T}}(\boldsymbol{y})\boldsymbol{e}(t)+\text{sign}(\hat{\boldsymbol{\theta}})|\hat{\boldsymbol{\theta}}|^{\alpha}\\ \dot{\tilde{\boldsymbol{\mu}}}=-\boldsymbol{F}^{\mathrm{T}}(\boldsymbol{x})D\varphi^{\mathrm{T}}(\boldsymbol{x}(t))\boldsymbol{e}(t)+\text{sign}(\hat{\boldsymbol{\mu}})|\hat{\boldsymbol{\mu}}|^{\alpha}\end{cases} \tag{13}$$

then zero solution of the error system (11) is stable in finite time, where $\hat{\boldsymbol{\theta}}=\boldsymbol{\theta}-\tilde{\boldsymbol{\theta}}$, $\hat{\boldsymbol{\mu}}=\boldsymbol{\mu}-\tilde{\boldsymbol{\mu}}$, $\tilde{\boldsymbol{\theta}}$ and $\tilde{\boldsymbol{\mu}}$ are the estimate values of $\boldsymbol{\theta}$ and $\boldsymbol{\mu}$, respectively.

Proof We can easily obtain

$$\dot{\hat{\boldsymbol{\theta}}}=-\dot{\tilde{\boldsymbol{\theta}}},\quad \dot{\hat{\boldsymbol{\mu}}}=-\dot{\tilde{\boldsymbol{\mu}}} \tag{14}$$

Substitute controller (12) into the synchronous error system (11), the error system (11) is rewritten by

$$\dot{\boldsymbol{e}}(t)=\boldsymbol{G}(\boldsymbol{y})\hat{\boldsymbol{\theta}}-D\varphi(\boldsymbol{x}(t))\boldsymbol{F}(\boldsymbol{x})\hat{\boldsymbol{\mu}}-\text{sign }(\boldsymbol{e})|\boldsymbol{e}|^{\alpha} \tag{15}$$

In order to proof Theorem 2, a candidate Lyapunov function is constructed as follows:

$$V(t)=\frac{1}{2}\boldsymbol{e}^{\mathrm{T}}(t)\boldsymbol{e}(t)+\frac{1}{2}\hat{\boldsymbol{\theta}}^{\mathrm{T}}\hat{\boldsymbol{\theta}}+\frac{1}{2}\hat{\boldsymbol{\mu}}^{\mathrm{T}}\hat{\boldsymbol{\mu}} \tag{16}$$

It is obvious that $V(t)\geq 0$. The time derivative of $V(t)$ is

$$\dot{V}(t)=\boldsymbol{e}^{\mathrm{T}}(t)\dot{\boldsymbol{e}}(t)+\dot{\hat{\boldsymbol{\theta}}}^{\mathrm{T}}\hat{\boldsymbol{\theta}}+\dot{\hat{\boldsymbol{\mu}}}^{\mathrm{T}}\hat{\boldsymbol{\mu}} \tag{17}$$

Substitute formulae (13), (14) and (15) into formula (17) and use Lemma 1, we have

$$\begin{aligned}\dot{V}(t)&=\boldsymbol{e}^{\mathrm{T}}(t)\dot{\boldsymbol{e}}(t)+\dot{\hat{\boldsymbol{\theta}}}^{\mathrm{T}}\hat{\boldsymbol{\theta}}+\dot{\hat{\boldsymbol{\mu}}}^{\mathrm{T}}\hat{\boldsymbol{\mu}}\\&=\boldsymbol{e}^{\mathrm{T}}(t)[\boldsymbol{G}(\boldsymbol{y})\hat{\boldsymbol{\theta}}-D\varphi(\boldsymbol{x}(t))\boldsymbol{F}(\boldsymbol{x})\hat{\boldsymbol{\mu}}-\text{sign }(\boldsymbol{e})|\boldsymbol{e}|^{\alpha}]\\&\quad-[\boldsymbol{G}^{\mathrm{T}}(\boldsymbol{y})\boldsymbol{e}(t)+\text{sign}(\hat{\boldsymbol{\theta}})|\hat{\boldsymbol{\theta}}|^{\alpha}]^{\mathrm{T}}\hat{\boldsymbol{\theta}}+[\boldsymbol{F}^{\mathrm{T}}(\boldsymbol{x})D\varphi^{\mathrm{T}}(\boldsymbol{x}(t))\boldsymbol{e}(t)\\&\quad-\text{sign}(\hat{\boldsymbol{\mu}})|\hat{\boldsymbol{\mu}}|^{\alpha}]^{\mathrm{T}}\hat{\boldsymbol{\mu}}\\&=-\boldsymbol{e}^{\mathrm{T}}\text{sign}(\boldsymbol{e})|\boldsymbol{e}|^{\alpha}-(\text{sign}(\hat{\boldsymbol{\theta}})|\hat{\boldsymbol{\theta}}|^{\alpha})^{\mathrm{T}}\hat{\boldsymbol{\theta}}-(\text{sign}(\hat{\boldsymbol{\mu}})|\hat{\boldsymbol{\mu}}|^{\alpha})^{\mathrm{T}}\hat{\boldsymbol{\mu}}\\&=-2^{\frac{1+\alpha}{2}}[\sum_{i=1}^{m}(\frac{e_i^2}{2})^{\frac{1+\alpha}{2}}+\sum_{i=1}^{m}(\frac{\hat{\theta}_i^2}{2})^{\frac{1+\alpha}{2}}+\sum_{i=1}^{m}(\frac{\hat{\mu}_i^2}{2})^{\frac{1+\alpha}{2}}]\\&\leq-2^{\frac{1+\alpha}{2}}[(\sum_{i=1}^{m}\frac{e_i^2}{2})^{\frac{1+\alpha}{2}}+(\sum_{i=1}^{m}\frac{\hat{\theta}_i^2}{2})^{\frac{1+\alpha}{2}}+(\sum_{i=1}^{m}\frac{\hat{\mu}_i^2}{2})^{\frac{1+\alpha}{2}}]\\&\leq-2^{\frac{1+\alpha}{2}}[(\sum_{i=1}^{m}\frac{e_i^2}{2})+(\sum_{i=1}^{m}\frac{\hat{\theta}_i^2}{2})+(\sum_{i=1}^{m}\frac{\hat{\mu}_i^2}{2})]^{\frac{1+\alpha}{2}}\\&=-2^{\frac{1+\alpha}{2}}[\frac{1}{2}\boldsymbol{e}^{\mathrm{T}}\boldsymbol{e}+\frac{1}{2}\hat{\boldsymbol{\theta}}^{\mathrm{T}}\hat{\boldsymbol{\theta}}+\frac{1}{2}\hat{\boldsymbol{\mu}}^{\mathrm{T}}\hat{\boldsymbol{\mu}}]^{\frac{1+\alpha}{2}}\\&=-2^{\frac{1+\alpha}{2}}V^{\frac{1+\alpha}{2}}\end{aligned} \tag{18}$$

According to Lyapunov stability theorem, the zero solution of the synchronous error system (11) is stable, and according to Theorem 1, there exists $t_1>0$ such

that $\boldsymbol{e} \equiv 0$ as $t \geq t_1$. Comparing (18) with (3), we can find that

$$c = 2^{\frac{1+\alpha}{2}} \quad \text{and} \quad \eta = \frac{1+\alpha}{2} \tag{19}$$

Substituting (19) into (8), we can obtain the required longest time t_1 to realize synchronization of systems (1) and (2),

$$\begin{aligned} t_1 &\leq t_0 + \frac{V^{1-\eta}(t_0)}{c(1-\eta)} = t_0 + \frac{V^{\frac{1-\alpha}{2}}(t_0)}{2^{\frac{1+\alpha}{2}}(\frac{1-\alpha}{2})} \\ &= t_0 + \frac{1}{1-\alpha}(2V(t_0))^{\frac{1-\alpha}{2}} \\ &= t_0 + \frac{1}{1-\alpha}[\boldsymbol{e}^{\mathrm{T}}(t_0)\boldsymbol{e}(t_0) + \hat{\boldsymbol{\theta}}^{\mathrm{T}}(t_0)\hat{\boldsymbol{\theta}}(t_0) \\ &\quad + \hat{\boldsymbol{\mu}}^{\mathrm{T}}(t_0)\hat{\boldsymbol{\mu}}(t_0)]^{\frac{1-\alpha}{2}} \end{aligned} \tag{20}$$

Assume $X(t_0) = \boldsymbol{e}^{\mathrm{T}}(t_0)\boldsymbol{e}(t_0) + \hat{\boldsymbol{\theta}}^{\mathrm{T}}(t_0)\hat{\boldsymbol{\theta}}(t_0) + \hat{\boldsymbol{\mu}}^{\mathrm{T}}(t_0) \cdot \hat{\boldsymbol{\mu}}(t_0)$, then $t_1 \leq t_0 + \frac{1}{1-\alpha}[X(t_0)]^{\frac{1-\alpha}{2}}$.

According to (20), when the initial values of the errors and estimated parameters are given, the synchronous convergent time t_1 can be computed via α. Therefore the convergent time t_1 can be controlled by changing the initial values of state variables and uncertain parameters.

Remark 2 Because the parameter $1-\alpha \in (0,1)$ is known, t_1 is an increasing function of $X(t_0)$. If the $X(t_0)$ is known and $0 < X(t_0) < 1$, then t_1 is also an increasing function of α. If the $X(t_0)$ is known and $X(t_0) > 1$, then t_1 is also a decreasing function of α.

3 Numerical Simulation

The dimensions n of system (1) and system (2) are different, so there are two cases $n > m$ and $m < n$ needed to be discussed, but we only discuss the case $n > m$ because the processes of the two cases are very similar .

We select a new hyper-chaotic system proposed in Ref. [20] as the drive system, and its dynamical equations are showed as follows:

$$\begin{cases} \dot{x}_1 = a_1(y_1 - x_1) \\ \dot{y}_1 = b_1 x_1 - x_1 z_1 - y_1 + c_1 w_1 \\ \dot{z}_1 = x_1 y_1 - d_1 z_1 \\ \dot{w}_1 = -k_1 x_1 \end{cases} \tag{21}$$

where x_1, y_1, z_1, w_1 are state variables and a_1, b_1, c_1, d_1, k_1 are system parameters. When $a_1 = 10$, $b_1 = 28$, $c_1 = 2$, $d_1 = 4$ and $k_1 = 8$, the system is hyper-chaotic and has beautiful attractors shown in Fig.1. Because shapes of the attractors like signs “3” and “8”, the author named the hyper-chaotic system (21) as a three eight hyper-chaotic system.

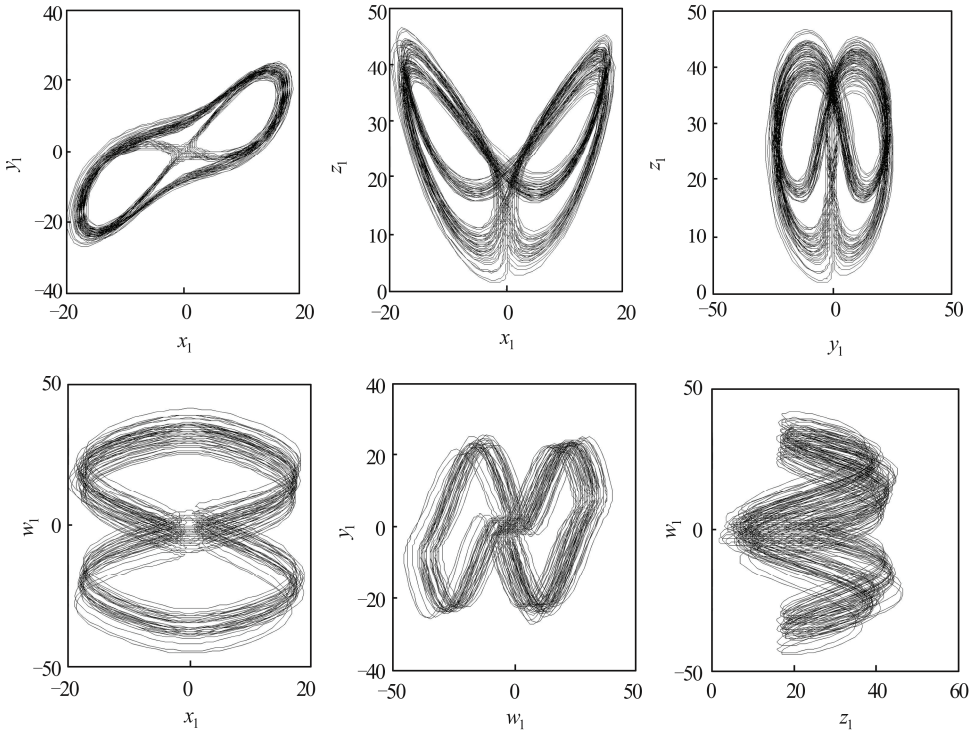


Fig.1 Attractors of the three eight hyper-chaotic system (21)

The drive system (21) is a four dimensional system, and the Chen system[21] with three dimensions is selected as the response system, and its dynamical equations are

$$\begin{cases}\dot{x}_2 = a_2(y_2 - x_2)\\ \dot{y}_2 = (c_2 - a_2)x_2 - x_2 z_2 + c_2 y_2\\ \dot{z}_2 = x_2 y_2 - b_2 z_2\end{cases} \tag{22}$$

where x_2, y_2, z_2 are state variables and a_2, b_2, c_2 are system parameters, and the system is chaotic as $a_2 = 35$, $b_2 = 3$, and $c_2 = 28$.

According to the forms of (1) and (2), the drive system (21) and the controlled response system (22) can be written respectively as follows:

$$\begin{pmatrix}\dot{x}_1\\ \dot{y}_1\\ \dot{z}_1\\ \dot{w}_1\end{pmatrix} = \begin{pmatrix}0\\ -x_1 z_1 - y_1\\ x_1 y_1\\ 0\end{pmatrix} + \begin{pmatrix}y_1 - x_1 & 0 & 0 & 0 & 0\\ 0 & x_1 & w_1 & 0 & 0\\ 0 & 0 & 0 & -z_1 & 0\\ 0 & 0 & 0 & 0 & -x_1\end{pmatrix}\begin{pmatrix}a_1\\ b_1\\ c_1\\ d_1\\ k_1\end{pmatrix} \tag{23}$$

$$\begin{pmatrix}\dot{x}_2\\ \dot{y}_2\\ \dot{z}_2\end{pmatrix} = \begin{pmatrix}0\\ -x_2 z_2\\ x_2 y_2\end{pmatrix} + \begin{pmatrix}y_2 - x_2 & 0 & 0\\ -x_2 & 0 & x_2 + y_2\\ 0 & -z_2 & 0\end{pmatrix}\begin{pmatrix}a_2\\ b_2\\ c_2\end{pmatrix} + \begin{pmatrix}u_1\\ u_2\\ u_3\end{pmatrix} \tag{24}$$

Without the loss of general, we suppose $\varphi(\boldsymbol{x}(t)) = (x_1 + w_1, y_1, z_1)^{\mathrm{T}}$ in synchronous error (10), then the Jacobian matrix of $\varphi(\boldsymbol{x}(t))$ is described as follows:

$$D\varphi(\boldsymbol{x}(t)) = \begin{pmatrix}1 & 0 & 0 & 1\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\end{pmatrix} \tag{25}$$

According to $\varphi(\boldsymbol{x}(t))$, the synchronous error (10) and the synchronous error system (11) are written respectively as follows:

$$\begin{cases}e_1 = x_2 - x_1 - w_1\\ e_2 = y_2 - y_1\\ e_3 = z_2 - z_1\end{cases} \tag{26}$$

$$\begin{cases}\dot{e}_1 = a_2(y_2 - x_2) - a_1(y_1 - x_1) + k_1 x_1 + u_1\\ \dot{e}_2 = (c_2 - a_2)x_2 - x_2 z_2 + c_2 y_2 + x_1 z_1 + y_1\\ \qquad - b_1 x_1 - c_1 w_1 + u_2\\ \dot{e}_3 = x_2 y_2 - b_2 z_2 - x_1 y_1 + d_1 z_1 + u_3\end{cases} \tag{27}$$

According to Theorem 2, the synchronization controller (12) can be written as follows:

$$\begin{pmatrix}u_1\\ u_2\\ u_3\end{pmatrix} = -\begin{pmatrix}0\\ -x_2 z_2\\ x_2 y_2\end{pmatrix} - \begin{pmatrix}y_2 - x_2 & 0 & 0\\ -x_2 & 0 & x_2 + y_2\\ 0 & -z_2 & 0\end{pmatrix}\begin{pmatrix}\tilde{a}_2\\ \tilde{b}_2\\ \tilde{c}_2\end{pmatrix} + \begin{pmatrix}1 & 0 & 0 & 1\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\end{pmatrix}$$

$$\cdot\begin{pmatrix}0\\ -x_1 z_1 - y_1\\ x_1 y_1\\ 0\end{pmatrix} + \begin{pmatrix}1 & 0 & 0 & 1\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\end{pmatrix}\begin{pmatrix}y_1 - x_1 & 0 & 0 & 0 & 0\\ 0 & x_1 & w_1 & 0 & 0\\ 0 & 0 & 0 & -z_1 & 0\\ 0 & 0 & 0 & 0 & -x_1\end{pmatrix}$$

$$\cdot\begin{pmatrix}\tilde{a}_1\\ \tilde{b}_1\\ \tilde{c}_1\\ \tilde{d}_1\\ \tilde{k}_1\end{pmatrix} - \operatorname{sign}\left(\begin{pmatrix}x_2 - x_1 - w_1\\ y_2 - y_1\\ z_2 - z_1\end{pmatrix}\right)\begin{pmatrix}|x_2 - x_1 - w_1|\\ |y_2 - y_1|\\ |z_2 - z_1|\end{pmatrix}^{\alpha} \tag{28}$$

At the same time, the identified rules of unknown parameters (13) of the drive system (23) and the response system (24) can be respectively shown by following equations in numerical simulation:

$$\begin{pmatrix}\dot{\tilde{a}}_1\\ \dot{\tilde{b}}_1\\ \dot{\tilde{c}}_1\\ \dot{\tilde{d}}_1\\ \dot{\tilde{k}}_1\end{pmatrix} = -\begin{pmatrix}y_1 - x_1 & 0 & 0 & 0 & 0\\ 0 & x_1 & w_1 & 0 & 0\\ 0 & 0 & 0 & -z_1 & 0\\ 0 & 0 & 0 & 0 & -x_1\end{pmatrix}^{\mathrm{T}}\begin{pmatrix}1 & 0 & 0 & 1\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\end{pmatrix}^{\mathrm{T}}$$

$$\cdot\begin{pmatrix}x_2 - x_1 - w_1\\ y_2 - y_1\\ z_2 - z_1\end{pmatrix} + \operatorname{sign}\left(\begin{pmatrix}a_1 - \tilde{a}_1\\ b_1 - \tilde{b}_1\\ c_1 - \tilde{c}_1\\ d_1 - \tilde{d}_1\\ k_1 - \tilde{k}_1\end{pmatrix}\right)\begin{pmatrix}|a_1 - \tilde{a}_1|\\ |b_1 - \tilde{b}_1|\\ |c_1 - \tilde{c}_1|\\ |d_1 - \tilde{d}_1|\\ |k_1 - \tilde{k}_1|\end{pmatrix}^{\alpha} \tag{30}$$

and

$$\begin{pmatrix}\dot{\tilde{a}}_2\\ \dot{\tilde{b}}_2\\ \dot{\tilde{c}}_2\end{pmatrix} = \begin{pmatrix}y_2 - x_2 & 0 & 0\\ -x_2 & 0 & x_2 + y_2\\ 0 & -z_2 & 0\end{pmatrix}^{\mathrm{T}}\begin{pmatrix}x_2 - x_1 - w_1\\ y_2 - y_1\\ z_2 - z_1\end{pmatrix}$$

$$+\operatorname{sign}\left(\begin{pmatrix}a_2 - \tilde{a}_2\\ b_2 - \tilde{b}_2\\ c_2 - \tilde{c}_2\end{pmatrix}\right)\begin{pmatrix}|a_2 - \tilde{a}_2|\\ |b_2 - \tilde{b}_2|\\ |c_2 - \tilde{c}_2|\end{pmatrix}^{\alpha} \tag{31}$$

And that is to say,

$$\begin{cases}\dot{\tilde{a}}_1 = (x_1 - y_1)(x_2 - x_1 - w_1) + \operatorname{sign}(a_1 - \tilde{a}_1)|a_1 - \tilde{a}_1|^{\alpha}\\ \dot{\tilde{b}}_1 = -x_1(y_2 - y_1) + \operatorname{sign}(b_1 - \tilde{b}_1)|b_1 - \tilde{b}_1|^{\alpha}\\ \dot{\tilde{c}}_1 = -w_1(y_2 - y_1) + \operatorname{sign}(c_1 - \tilde{c}_1)|c_1 - \tilde{c}_1|^{\alpha}\\ \dot{\tilde{d}}_1 = z_1(z_2 - z_1) + \operatorname{sign}(d_1 - \tilde{d}_1)|d_1 - \tilde{d}_1|^{\alpha}\\ \dot{\tilde{k}}_1 = x_1(x_2 - x_1 - w_1) + \operatorname{sign}(k_1 - \tilde{k}_1)|k_1 - \tilde{k}_1|^{\alpha}\end{cases} \tag{32}$$

and

$$\begin{cases}\dot{\tilde{a}}_2=(y_2-x_2)(x_2-x_1-w_1)-x_2(y_2-y_1)\\ \quad +\text{sign}(a_2-\tilde{a}_2)\left|a_2-\tilde{a}_2\right|^{\alpha}\\ \dot{\tilde{b}}_2=-z_2(z_2-z_1)+\text{sign}(b_2-\tilde{b}_2)\left|b_2-\tilde{b}_2\right|^{\alpha}\\ \dot{\tilde{c}}_2=(x_2+y_2)(y_2-y_1)+\text{sign}(c_2-\tilde{c}_2)\left|c_2-\tilde{c}_2\right|^{\alpha}\end{cases} \tag{33}$$

Assume α=1 / 5 and the initial value of (21) and (22) are $x_1(0)=3$, $y_1(0)=-1$, $z_1(0)=-2$, $w_1(0)$ =1.5, $x_2(0)=1$, $y_2(0)=-0.5$, $z_2(0)=2$, $\tilde{a}_1(0)=-1$, $\tilde{b}_1(0)$ =0, $\tilde{c}_1(0)=-0.1$, $\tilde{d}_1(0)=0.2$, $\tilde{k}_1(0)=-1$, $\tilde{a}_2(0)=2$, $\tilde{b}_2(0)=1$, $\tilde{c}_2(0)=4$, then the required longest time t_1 to realize the generalized synchronization of the chaotic systems (21) and (22) is estimated as follows:

$$\begin{aligned}t_1&\le t_0+\frac{1}{1-\alpha}[\boldsymbol{e}^{\mathrm{T}}(t_0)\boldsymbol{e}(t_0)+\hat{\boldsymbol{\theta}}^{\mathrm{T}}(t_0)\hat{\boldsymbol{\theta}}(t_0)+\hat{\boldsymbol{\mu}}^{\mathrm{T}}(t_0)\hat{\boldsymbol{\mu}}(t_0)]^{\frac{1-\alpha}{2}}\\ &=0+\frac{5}{4}[(-3.5)^2+(0.5)^2+4^2+(10+1)^2+28^2+(2+0.1)^2\\ &\quad+(4-0.2)^2+(8+1)^2+(35-2)^2+(3-1)^2+(28-4)^2]^{\frac{2}{5}}\\ &=29.485\,4\text{ s}\end{aligned} \tag{34}$$

The finite synchronization time t_1 relies solely on the parameter α and the initial values of the systems, and the generalized synchronization and parameters identification can be realized in 29.485 4 s from the starting simulation.

The synchronization errors of the drive system (21) and the response system (22) are shown in Fig. 2.

As we can see in Fig.2, all synchronization error trajectories converge to zero in 29.485 4 s, which means that the finite-time generalized synchronization of the drive system (21) and the response system (22) can be realized under the adaptive controllers (28). All uncertain parameters of the drive system (21) and the response system (22) are identified in Fig.3 and Fig.4 , respectively.

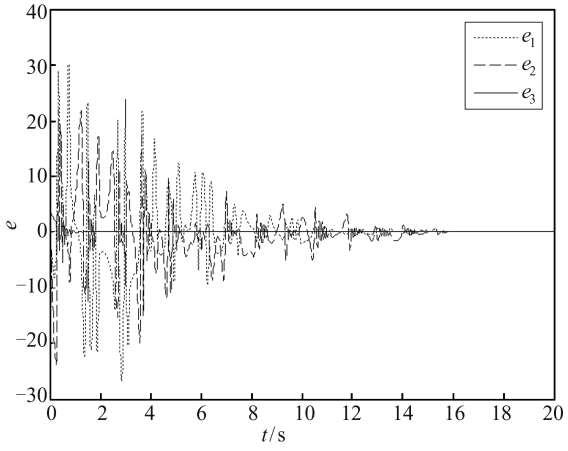


Fig.2 Error trajectories of the drive system (21) and the response system (22)

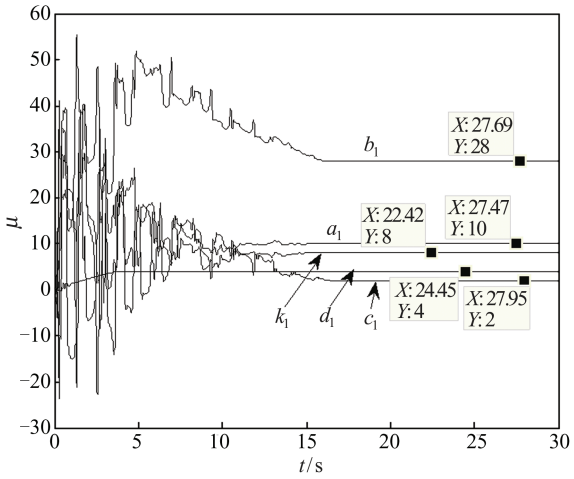


Fig.3 The uncertain parameters identification of the drive system (21)

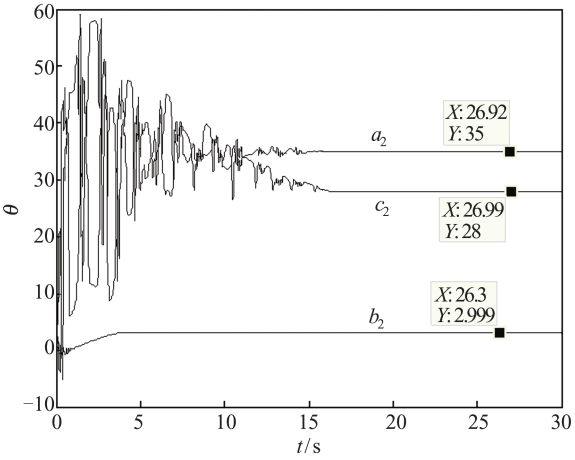


Fig.4 The uncertain parameters identification of the response system (22)

We find that identified curves of all uncertain parameters exactly tend to inherent values ($a_1=10$, $b_1=28$, $c_1=2$, $d_1=4$, $k_1=8$, $a_2=35$, $b_2=3$, $c_2=28$) of the systems in Fig.3 and Fig.4 in 29.485 4 s, which indicates that the identification rules of unknown parameters (30) and (31) are correct and effective.

4 Conclusion

Finite-time generalized synchronization of chaotic systems with uncertain parameters is realized under the adaptive controllers. Meanwhile, parameter identification rules are proposed to exactly identify all uncertain pa-

rameters of drive and response chaotic systems. Synchronization time which solely depends on the initial conditions of chaotic systems is easily estimated in numerical simulation.

The main innovation is a content innovation based on already existing finite-time generalized synchronization of chaotic systems whose parameters are all certain and given. However, we discuss the problem of finite-time generalized synchronization of chaotic systems with uncertain parameters, construct adaptive controllers and identification rules of all uncertain parameters, and give the required longest time to implement chaotic synchronization and identify uncertain parameters. Studied results can apply in any chaotic synchronization with uncertain parameters.

In this paper, in order to realize generalized synchronization of chaotic systems, adaptive controllers are constructed to control all state variables of the response system. However, it is still an important direction to realize finite-time generalized synchronization by the controlling number of variables as little as possible.

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