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Gracefulness of Two Kinds of Unconnected Graphs with Even Vertices

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Abstract: Two kinds of unconnected double fan graphs with even vertices, $(P_1^{(1)}\vee(P_{2n}^{(1)}\cup P_{2n}^{(2)}))\cup P_{2n+1}\cup(P_1^{(2)}\vee \overline{K_{2n}})$ and $(P_1^{(1)}\vee(P_{2n}^{(1)}\cup P_{2n}^{(2)}))\cup(P_1^{(2)}\vee \overline{K_{2n}^{(1)}})\cup(P_1^{(3)}\vee \overline{K_{2n}^{(2)}})$ were presented. For natural number $n\in \mathbf{N}, n\geqslant 1$, the two graphs are all graceful graphs, where $P_{2n}^{(1)}, P_{2n}^{(2)}$ are even-vertices path, P_{2n+1} is odd-vertices path, $\overline{K_{2n}}$, $\overline{K_{2n}^{(1)}}$, $\overline{K_{2n}^{(2)}}$ are the complement of graph K_{2n}, $G_1\vee G_2$ is the join graph of G_1 and G_2.

Key words: unconnected graph; double fan graph; graceful graph; graceful label; even vertices

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0 Introduction

Research on labeling of graceful graph is on the rise following the process of computing science[1-7]. Researches on the gracefulness of fan graph, especially double fan graph, are very important for the study of graph[8]. The gracefulness of union graphs of double fan graph, linear graph and star graph has been proved[9], and the gracefulness of union graphs of double fan graph and all graceful graphs has also been proved[10]. The gracefulness of union graphs of fan graph and non-graceful graphs has rarely been studied, thus it is rather significant to have further research on the gracefulness of fan graph and non-graceful graphs.

The graphs in the following are all simple undirected graphs. Let $G(V,E)$ be a graph, $V=V(G)$ be the set of vertices of graph G, $E=E(G)$ be the set of edges of graph G, $|A|$ be the order of the set A, P_n be the path with n vertices, P_1 be the trivial graph with one vertex, $G_1\vee G_2$ be the joint graph of G_1 and G_2, $\bar{G}$ be the complementary graph of G.

Definition 1 Let $G=(V,E)$ be a graph, and let k be a positive integer. If there is a injection f: $V\to\{0,1,\cdots,|E|+k-1\}$, such that for every edge $uv\in E$, $f'(uv)=|f(u)-f(v)|$ induces a bijection $f': E\to\{k,k+1,\cdots,|E|+k-1\}$, then we call G a k-graceful graph, call f a k-graceful label of graph G. 1-graceful graph is also called graceful graph, and 1-graceful label is called graceful label.

1 The Gracefulness of Graph $P_1^{(1)}\vee(P_{2n}^{(1)}\cup P_{2n}^{(2)})\cup P_{2n+1}\cup(P_1^{(2)}\vee\overline{K_{2n}})$

Theorem 1 For natural number $n\in\mathbf{N}$, $n\geqslant 1$, $P_1^{(1)}\vee(P_{2n}^{(1)}\cup P_{2n}^{(2)})\cup P_{2n+1}\cup(P_1^{(2)}\vee\overline{K_{2n}})$ is a graceful graph.

Proof For $V(P_1^{(1)})=\{x_0\}$, $V(P_{2n}^{(1)})=\{x_1,x_2,x_3,\cdots,x_{2n}\}$, $V(P_{2n}^{(2)})=\{x_{2n+1},x_{2n+2},x_{2n+3},\cdots,x_{4n}\}$, $V(P_{2n+1})=\{y_1,y_2,y_3,\cdots,y_{2n+1}\}$, $V(P_1^{(2)})=\{z_0\}$, $V(\overline{K_{2n}})=\{z_1,z_2,z_3,\cdots,z_{2n}\}$, $E=E(P_1^{(1)}\vee(P_{2n}^{(1)}\cup P_{2n}^{(2)}))\cup P_{2n+1}\cup(P_1^{(2)}\vee\overline{K_{2n}})$, then $|E|=12n-2$, $|V|=8n+3$.

Define the vertex label f of graph $P_1^{(1)}\vee(P_{2n}^{(1)}\cup P_{2n}^{(2)})\cup P_{2n+1}\cup(P_1^{(2)}\vee\overline{K_{2n}})$ as follows:

$$f(x_0)=0,\ f(x_{2n-1})=12n-2\ ,\ f(x_{2n})=2,$$

$$f(x_{2n-i})=\begin{cases} i-1\ , & i=2,4,6,\cdots,2n-2\ ,\\ 12n-2-i\ , & i=3,5,7,\cdots,2n-1\ .\end{cases}$$

$$f(x_{2n+i})=\begin{cases} 2n-3+(i+1)\ , & i=1,3,5,\cdots,2n-1\ ,\\ 10n-1-i\ , & i=2,4,6,\cdots,2n\ .\end{cases}$$

$$f(y_{2n+1})=12n-6\ ,\ f(y_{2n})=4n-6\ ,\ f(y_{2n-1})=4n-2\ ,$$

$$f(y_{2n-i})=\begin{cases} 4n+6-i\ , & i=2,4,6,\cdots,2n-2\ ,\\ 4n+9+i\ , & i=3,5,7,\cdots,2n-1\ .\end{cases}$$

$$f(z_0)=12n-4,\ f(z_i)=4n-3+2i,\quad i=1,2,3,\cdots,2n$$

We shall prove that label f is a graceful label of graph $P_1^{(1)}\vee(P_{2n}^{(1)}\cup P_{2n}^{(2)})\cup P_{2n+1}\cup(P_1^{(2)}\vee\overline{K_{2n}})$.

(i) By the definition of f:

$$f(x_0)<f(x_{2n-2})<f(x_{2n})<f(x_{2n-4})<\cdots<f(x_2)<f(x_{2n+1})<f(x_{2n+3})<\cdots<f(x_{2n+9})$$
$$<f(y_2)<f(x_{2n+11})<f(y_4)<\cdots<f(x_{4n-5})<f(y_{2n})<f(x_{4n-3})<f(x_{4n-1})<f(y_{2n-1})$$
$$<f(z_1)<f(y_{2n-6})<f(z_2)<f(y_{2n-4})<f(z_3)<f(y_{2n-2})<\cdots<f(z_{11})<f(y_{2n-3})<f(z_{13})<f(y_{2n-5})$$
$$<f(y_1)<f(z_{2n})<f(x_{4n})<f(x_{4n-2})<f(x_{4n-4})<\cdots<f(x_{2n+2})<f(x_1)<f(x_3)$$
$$<f(x_5)<\cdots<f(x_{2n-5})<f(y_{2n+1})<f(x_{2n-3})<f(z_0)<f(x_{2n-1})$$

Hence mapping f: $V((P_1^{(1)}\vee(P_{2n}^{(1)}\cup P_{2n}^{(2)}))\cup P_{2n+1}\cup(P_1^{(2)}\vee\overline{K_{2n}}))\to\{0,1,\cdots,12n-2\}$ is an injection.

(ii) For every edge $uv\in E(P_1^{(1)}\vee(P_{2n}^{(1)}\cup P_{2n}^{(2)}))\cup P_{2n+1}\cup(P_1^{(2)}\vee\overline{K_{2n}})$, let $f'(uv)=|f(u)-f(v)|$.

Then

$$f'(x_0x_{2n-i})=\begin{cases} i-1, & i=2,4,\cdots,2n-2\ ,\\ 12n-2-i\ , & i=3,5,\cdots,2n-1\ .\end{cases}$$

$$f'(x_0x_{2n+i})=\begin{cases} 2n-1+(i-1), & i=1,3,\cdots,2n-1\ ,\\ 10n-1-i\ , & i=2,4,\cdots,2n\ .\end{cases}$$

$$f'(x_ix_{i+1})=8n+2i\ ,\quad i=1,2,3,\cdots,2n-3,$$

$$f'(x_{2n+i}x_{2n+i+1})=8n-2i,\ \ i=1,2,3,\cdots,2n-1,$$

$$f'(y_{2n-i}y_{2n-i+1})=2(i+1)\ ,\ \ i=1,2,3,\cdots,2n-1,$$

$$f'(z_0z_i)=8n-1-2i\ ,\ \ i=1,2,3,\cdots,2n\ ,$$

$$f'(x_0x_{2n})=2,\quad f'(x_0x_{2n-1})=12n-2,\quad f'(x_{2n-1}x_{2n})=12n-4,$$

$$f'(x_{2n-2}x_{2n-1})=12n-3,\quad f'(y_{2n}y_{2n+1})=8n.$$

Hence

$$f'(x_0x_{2n-2})<f'(x_0x_{2n})<f'(x_0x_{2n-4})<f'(y_{2n-1}y_{2n})<f'(x_0x_{2n-6})<f'(y_{2n-2}y_{2n-1})$$
$$<\cdots<f'(x_0x_2)<f'(y_{n+2}y_{n+3})<f'(x_0x_{2n+1})<f'(y_{n+1}y_{n+2})<f'(x_0x_{2n+3})<f'(y_ny_{n+1})$$
$$<\cdots<f'(x_0x_{4n-1})<f'(y_2y_3)<f'(z_0z_{2n})<f'(y_1y_2)<f'(z_0z_{2n-1})<f'(x_{4n-1}x_{4n})$$
$$<f'(z_0z_{2n-2})<f'(x_{4n-2}x_{4n-1})<\cdots<f'(z_0z_1)<f'(x_{2n+1}x_{2n+2})<f'(x_0x_{4n})$$
$$<f'(y_{2n}y_{2n+1})<f'(x_0x_{4n-2})<f'(x_1x_2)<f'(x_0x_{4n-4})<f'(x_2x_3)<\cdots<f'(x_0x_1)<f'(x_nx_{n+1})$$
$$<f'(x_0x_3)<f'(x_{n+1}x_{n+2})<\cdots<f'(x_0x_{2n-3})<f'(x_{2n-1}x_{2n})<f'(x_{2n-2}x_{2n-1})<f'(x_0x_{2n-1})$$

Hence

f' : $E(P_1^{(1)}\vee(P_{2n}^{(1)}\cup P_{2n}^{(2)}))\cup P_{2n+1}\cup(P_1^{(2)}\vee\overline{K_{2n}})\rightarrow\{1,2,\cdots,12n-2\}$ is a bijection, graph $(P_1^{(1)}\vee(P_{2n}^{(1)}\cup P_{2n}^{(2)}))\cup P_{2n+1}\cup(P_1^{(2)}\vee\overline{K_{2n}})$ is a graceful graph.

2 The Gracefulness of Graph $(P_1^{(1)}\vee(P_{2n}^{(1)}\cup P_{2n}^{(2)}))\cup(P_1^{(2)}\vee\overline{K_{2n}^{(1)}})\cup(P_1^{(3)}\vee\overline{K_{2n}^{(2)}})$

Theorem 2 For natural number $n\in\mathbf{N}$, $n\geqslant 1$, $(P_1^{(1)}\vee(P_{2n}^{(1)}\cup P_{2n}^{(2)}))\cup(P_1^{(2)}\vee\overline{K_{2n}^{(1)}})\cup(P_1^{(3)}\vee\overline{K_{2n}^{(2)}})$ is a graceful graph.

Proof For $V(P_1^{(1)})=\{x_0\}, V(P_{2n}^{(1)})=\{x_1,x_2,x_3,\cdots,x_{2n}\}$,

$$V(P_{2n}^{(2)})=\{x_{2n+1},x_{2n+2},x_{2n+3},\cdots,x_{4n}\},$$
$$V(P_1^{(2)})=\{y_0\}, V(\overline{K_{2n}^{(1)}})=\{y_1,y_2,y_3,\cdots,y_{2n}\},$$
$$V(P_1^{(3)})=\{z_0\}, V(\overline{K_{2n}^{(2)}})=\{z_1,z_2,z_3,\cdots,z_{2n}\},$$

$E=E((P_1^{(1)}\vee(P_{2n}^{(1)}\cup P_{2n}^{(2)}))\cup(P_1^{(2)}\vee\overline{K_{2n}^{(1)}})\cup(P_1^{(3)}\vee\overline{K_{2n}^{(2)}}))$, then $|E|=12n-2$.

Define the vertex label f of graph $(P_1^{(1)}\vee(P_{2n}^{(1)}\cup P_{2n}^{(2)}))\cup(P_1^{(2)}\vee\overline{K_{2n}^{(1)}})\cup(P_1^{(3)}\vee\overline{K_{2n}^{(2)}})$ as follows:

$$f(x_0)=0,\ f(x_{2n-1})=12n-2,\ f(x_{2n})=2,$$
$$f(x_{2n-i})=\begin{cases}i-1, & i=2,4,6,\cdots,2n-2,\\ 12n-2-i, & i=3,5,7,\cdots,2n-1.\end{cases}$$
$$f(x_{2n+i})=\begin{cases}2n-1+(i-1), & i=1,3,5,\cdots,2n-1,\\ 10n-1-i, & i=2,4,6,\cdots,2n.\end{cases}$$
$$f(y_0)=12n-4,\ f(z_0)=12n-6,\ f(z_{2n})=4n-6,$$
$$f(y_i)=8n-1-2i,\quad i=1,2,3,\cdots,2n,$$
$$f(z_i)=12n-6-2(i+1),\quad i=1,2,3,\cdots,2n-1.$$

We shall prove that label f is a graceful label of graph $(P_1^{(1)}\vee(P_{2n}^{(1)}\cup P_{2n}^{(2)}))\cup(P_1^{(2)}\vee\overline{K_{2n}^{(1)}})\cup(P_1^{(3)}\vee\overline{K_{2n}^{(2)}})$.

(i) By the definition of f:

$$\begin{aligned}&f(x_0)<f(x_{2n-2})<f(x_{2n})<f(x_{2n-4})<\cdots<f(x_2)<f(x_{2n+1})<f(x_{2n+3})<f(z_{2n})\\&<f(x_{2n+5})<\cdots<f(x_{4n-5})<f(z_{2n})<f(x_{4n-3})<f(y_{2n})<f(y_{2n-1})<\cdots<f(y_3)\\&<f(z_{2n-1})<f(y_2)<f(z_{2n-2})<f(y_1)<f(z_{2n-3})<f(x_{4n})<f(z_{2n-4})<f(x_{4n-2})\\&<f(z_{2n-6})<\cdots<f(z_n)<f(x_{2n+6})<f(z_{n-1})<f(x_{2n+4})<f(x_1)<f(z_{n-4})<f(x_3)\\&<f(z_{n-5})<\cdots<f(z_2)<f(z_1)<f(x_{2n-7})<f(x_{2n-5})<f(z_0)<f(x_{2n-3})<f(y_0)<f(x_{2n-1})\end{aligned}$$

Hence mapping f: $V((P_1^{(1)}\vee(P_{2n}^{(1)}\cup P_{2n}^{(2)}))\cup(P_1^{(2)}\vee\overline{K_{2n}^{(1)}})\cup(P_1^{(3)}\vee\overline{K_{2n}^{(2)}}))\rightarrow\{0,1,\cdots,12n-2\}\}$ is an injection.

(ii) Let

$$f'(uv)=|f(u)-f(v)|,$$
$$f'(x_0x_{2n-i})=\begin{cases}i-1, & i=2,4,6,\cdots,2n-2,\\ 12n-2-i, & i=3,5,7,\cdots,2n-1.\end{cases}$$
$$f'(x_0x_{2n+i})=\begin{cases}2n-1+(i-1), & i=1,3,\cdots,2n-1,\\ 10n-1-i, & i=2,4,\cdots,2n.\end{cases}$$
$$f'(x_ix_{i+1})=8n+2i,\ \ i=1,2,3,\cdots,2n-3,$$
$$f'(x_{2n+i}x_{2n+i+1})=8n-2i,\ \ i=1,2,3,\cdots,2n-1,$$
$$f'(y_0y_i)=4n-3+2i,\ \ i=1,2,3,\cdots,2n,$$
$$f'(z_0z_i)=2(i+1),\ \ i=1,2,3,\cdots,2n-1,$$
$$f'(x_0x_{2n})=2,\quad f'(x_0x_{2n-1})=12n-2,\quad f'(x_{2n}x_{2n-1})=12n-4,$$
$$f'(x_{2n-1}x_{2n-2})=12n-3,\qquad f'(z_0z_{2n})=8n.$$

Hence

$$\begin{aligned}&f'(x_0x_{2n-2})<f'(x_0x_{2n})<f'(x_0x_{2n-4})<f'(z_0z_1)<f'(x_0x_{2n-6})<f'(z_0z_2)<\cdots\\&<f'(x_0x_2)<f'(z_0z_{n-2})<f'(x_0x_{2n+1})<f'(z_0z_{n-1})<f'(x_0x_{2n+3})<f'(z_0z_n)<\cdots\end{aligned}$$

$$<f'(x_0x_{4n-1})<f'(z_0z_{2n-2})<f'(y_0y_1)<f'(z_0z_{2n-1})<f'(y_0y_2)<f'(x_{4n-1}x_{4n})$$
$$<f'(x_{4n-1}x_{4n})<f'(y_0y_3)<f'(x_{4n-2}x_{4n-1})<f'(y_0y_4)<\cdots<f'(y_0y_{2n})<f'(x_{2n+1}x_{2n+2})$$
$$<f'(x_0x_{4n})<f'(z_0z_{2n})<f'(x_0x_{4n-2})<f'(x_1x_2)<f'(x_0x_{4n-4})<f'(x_2x_3)<\cdots$$
$$<f'(x_0x_{2n+2})<f'(x_{n-1}x_n)<f'(x_0x_1)<f'(x_nx_{n+1})<\cdots<f'(x_0x_{2n-5})$$
$$<f'(x_{2n-3}x_{2n-2})<f'(x_0x_{2n-3})<f'(x_{2n}x_{2n-1})<f'(x_{2n-1}x_{2n-2})<f'(x_0x_{2n-1})$$

Hence

$f':E((P_1^{(1)}\vee(P_{2n}^{(1)}\cup P_{2n}^{(2)}))\cup(P_1^{(2)}\vee\overline{K_{2n}^{(1)}})\cup(P_1^{(3)}\vee\overline{K_{2n}^{(2)}}))\to\{1,2,\cdots,12n-2\}$ is a bijection, and graph $(P_1^{(1)}\vee(P_{2n}^{(1)}\cup P_{2n}^{(2)}))\cup(P_1^{(2)}\vee\overline{K_{2n}^{(1)}})\cup(P_1^{(3)}\vee\overline{K_{2n}^{(2)}})$ is a graceful graph.

3 Conclusion

The gracefulness on two kinds of unconnected double fan graphs with even vertices, $(P_1^{(1)}\vee(P_{2n}^{(1)}\cup P_{2n}^{(2)}))\cup P_{2n+1}\cup(P_1^{(2)}\vee\overline{K_{2n}})$ and $(P_1^{(1)}\vee(P_{2n}^{(1)}\cup P_{2n}^{(2)}))\cup(P_1^{(2)}\vee\overline{K_{2n}^{(1)}})\cup(P_1^{(3)}\vee\overline{K_{2n}^{(2)}})$, were proved. The vertex label f of graph was defined by the definition of graceful graph, and f was proved as injection/bijection. The graceful theorem of the union of unconnected triple graphs was proved. The research will be beneficial to the study of gracefulness on union graph of multiple graphs.

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