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A PDE Approach to the Long-Time Asymptotic Solutions of Contact Hamilton-Jacobi Equations

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Abstract: We study the long-time asymptotic behaviour of viscosity solutions $u(x,t)$ of the Hamilton-Jacobi equation $u_t(x,t)+H(x,u(x,t),Du(x,t))=0$ in $\mathbb{T}^n\times(0,\infty)$ with a PDE approach, where $H=H(x,u,p)$ is coercive in p, non-decreasing in u and strictly convex in (u,p), and establish the uniform convergence of $u(x,t)$ to an asymptotic solution $u_\infty(x)$ as $t\to\infty$. Moreover, u_∞ is a viscosity solution of Hamilton-Jacobi equation $H(x,u(x),Du(x))=0$.

Key words: asymptotic solution; Hamilton-Jacobi equation; PDE approach

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0 Introduction

In this paper, we consider the Cauchy problem for the evolutionary Hamilton-Jacobi equation

$$\begin{cases} u_t(x,t)+H(x,u(x,t),Du(x,t))=0,(x,t)\in\mathbb{T}^n\times[0,\infty) \\ u(x,0)=f(x), x\in\mathbb{T}^n \end{cases} \tag{CP}$$

Here $u(x,t)$ is an unknown function on $\mathbb{T}^n\times[0,\infty)$, $f(x)$ is a given function, and $u_t:=\partial u/\partial t$, $Du:=(\partial u/\partial x_1,\cdots,\partial u/\partial x_n)$. We study the long-time asymptotic behavior of the viscosity solution to (CP) and furthermore, discuss the relation between the limit of the viscosity solution of (CP) and the viscosity solution of the stationary Hamilton-Jacobi equation

$$H(x,u(x),Du(x))=c, x\in\mathbb{T}^n \tag{EP}$$

There has been much study about the long-time behavior of the viscosity solutions of Hamilton-Jacobi equations either by means of dynamical techniques or by PDE methods. We occasionally suppress "viscosity" for simplicity.

The dynamical approach is based on the weak KAM theory initiated by Fathi[1,2]. It needs strong regularity assumptions on the Hamiltonian $H(x,p)$ (C^2-regularity, strict convexity and superlinearity in p) because it is based on the analysis of the associated Hamiltonian flow. Such flow is connected with the visc. solution of $\partial_t u+H(x,Du(x))=0$ through the Lax-Oleinik formula. The dynamical approach has been later modified by Roquejoffre[3], Davini and Siconolfi in Ref. [4], and others.

The PDE approach is initiated by the work of Namah and Roquejoffre[5]. It does not depend on the Lax-Oleinik formula, so it is possible to be applied to more general cases. Barles and Souganidis have obtained in Ref. [6] more general results in the case $\Omega=\mathbb{T}^n$, for possible non-convex Hamiltonians. We refer to Ref. [7] for a recent view on this approach.

In this paper, we will explore the visc. solutions' long time behavior of the Hamilton-Jacobi equation of contact type, in which the Hamiltonian $H(x,u,p)$ explicitly depends on the unknown function u. The contact Hamiltonian system is a natural extension to Hamiltonian system. Various applications of contact Hamiltonian dynamics has been found in many fields such as classic mechanics of dissipative system[8,9], mesoscopic dynamics[10], equilibrium statistical mechanics[11], and thermodynamics[12,13], etc. Su, Wang, and Yan first studied visc. solutions' long-time behavior of the contact Hamilton-Jacobi equation with implicit variational principle in Ref. [14], under Tonelli assumptions ($H(x,u,p)$ is $C^r\,(r\geqslant 2)$, strict convexity and superlinear growth in p for every (x,u), uniform Lipschitzity and monotonicity with respect to u). Their series of work are aimed at building the variational frame in the contact Hamiltonian system[15-17]. In the recent paper, the author has studied the long-time behavior of solutions of the contact Hamilton-Jacobi equations with the method combining the PDE-viscosity solutions approach and dynamical approach under more general conditions ($H(x,u,p)$ is C, strict convexity and coercive in p for every (x,u), monotonicity with respect to u)[18].

Motivated by above-mentioned results, we will continue this direction of research on the long-time behavior of visc. solutions and we want to discuss if the conditions are necessary for the convergence in this paper. We mainly use and slightly modify the PDE approach which has been introduced by Barles, Ishii and Mitake (see Ref. [7]). The main difference is that we deal with the contact Hamiltonian-Jacobi equation for the consideration of the effect of u in the proof.

We assume that

$$H\in C(\mathbb{T}^n\times\mathbb{R}\times\mathbb{R}^n) \tag{C}$$

$$\lim_{R\to+\infty}\inf\left\{H(x,u,p): x\in\mathbb{T}^n, |p|>R\right\}=\infty, \text{ for all } u\in\mathbb{R} \tag{CER}$$

The function $u\to H(x,u,p)$ is non-decreasing on $\mathbb{R}$, for all $(x,p)\in\mathbb{T}^n\times\mathbb{R}^n$ (MON)

(EP) with $c=0$ has a visc. solution $\omega_0\in C(\mathbb{T}^n)$ (Z)

There exist positive constants $\eta_0>0,\ \theta_0>1$ and a positive constant $\psi=\psi(\eta,\theta)$ with $(\eta,\theta)\in(0,\eta_0)\times(1,\theta_0)$, such that for all $x,p,q\in\mathbb{R}^n, u,v\in\mathbb{R}$, if $H(x,u,q)\geqslant\eta$ and $H(x,u,q)\leqslant 0$, then $H(x,v+\theta(u-v),p+\theta(q-p))\geqslant\eta\theta+\psi$. (DSTC+)

There exist positive constants $\eta_0>0,\ \theta_0>1$ and a positive constant $\psi=\psi(\eta,\theta)$ with $(\eta,\theta)\in(0,\eta_0)\times(1,\theta_0)$, such that for all $x,p,q\in\mathbb{R}^n$, $u,v\in\mathbb{R}$, if $H(x,u,q)\geqslant-\eta$ and $H(x,u,q)\leqslant 0$, then $H(x,v+\theta(u-v),p+\theta(q-p))\geqslant-\eta\theta+\psi$. (DSTC−)

Condition (DSTC+) ((DSTC−)) means some kind of strict convexity of $H(x,u,p)$ in (u,p). Indeed, if H is strictly convex in (u,p), then

$$\begin{aligned}
&\eta\leqslant H(x,u,q)\\
&=H\{x,\theta^{-1}[v+\theta(u-v)]+(1-\theta^{-1})v,\\
&\theta^{-1}[p+\theta(q-p)]+(1-\theta^{-1})p\}\\
&<\theta^{-1}H(x,v+\theta(u-v),p+\theta(q-p))\\
&+(1-\theta^{-1})H(x,v,p)\\
&<\theta^{-1}H(x,v+\theta(u-v),p+\theta(q-p)).
\end{aligned}$$

A condition similar to DSTC+(DSTC−) has appeared in Ref. [1]. The difference is the strict convexity about (u,p) in this paper and the strict convexity about p both in the u-independent case in Ref. [7] and in the u-dependent case in Ref. [18]. The convexity in (u,p) is the necessary condition for using this PDE approach, bcause we cannot fix $H(x,u(x),\ Du(x))=c$ to $H(x,u_\infty,Du(x))=c$ as we have done with the PDE approach in Ref. [18].

In this case, we can deal with the convergence problem of the Hamiltonian $H(x,u,p)$ which is not strictly convex in p in contrast to what happens in Ref.[14, 18]. Our main result is:

Theorem 1 Assume (C), (CER), (MON), (Z) and (DSTC+) ((DSTC−)). Let $f\in C(\mathbb{T}^n)$, and let $u\in C(\mathbb{T}^n\times[0,\infty))$ be the visc. solution of (CP). Then there exists $u_\infty\in C(\mathbb{T}^n)$ such that

$\lim_{t\to\infty}u(x,t)=u_\infty(x)$ uniformly on $\mathbb{T}^n$.

Moreover, u_∞ is a visc. solution of (EP), with $c=0$.

The paper is organized as follows. In Section 1, we will give some classical results about visc. solution theory which are needed for the next proof. Based on the comparison theorem and the Perron method, we get the existence theorem of (CP) (Theorem 3). Assuming moreover (Z), $u(x,t)$ is bounded and uniformly continuous on $\mathbb{T}^n\times[0,\infty)$. In Section 2, we will give the proof of Theorem 1 with the condition (DSTC+) ((DSTC−)) instead of the condition (CON) in the Ref.[18].

1 The Preliminary Results

As the basis of the existence theorem and the uniqueness theorem, we first introduce the comparison theorem.

Theorem 2[18] Assume (C), (CER) and (MON). Let $u \in \mathrm{USC}(\mathbb{T}^n \times [0,T))$ and $v \in \mathrm{LSC}(\mathbb{T}^n \times [0,T))$ be a visc. subsolution and a visc. supersolution of (CP), respectively, where $0 < T \leqslant \infty$. Then

$$u(x,t) - v(x,t) \leqslant \max\{\max_{\mathbb{T}^n}(u(\cdot,0) - v(\cdot,0)), 0\}$$

for all $(x,t) \in \mathbb{T}^n \times (0,T)$.

Corollary 1[18] If, in addition, u, v are both visc. solutions of (CP), then we have

$$\sup_{\mathbb{T}^n \times [0,T)} |u - v| \leqslant \max_{\mathbb{T}^n} |u(\cdot,0) - v(\cdot,0)|.$$

Theorem 3[18] Assume (C), (CER) and (MON). Let $f \in C(\mathbb{T}^n)$. Then there exists a (unique) solution $u \in C(\mathbb{T}^n \times [0,\infty))$ of (CP).

Theorem 4 [18] Assume (C), (CER), (MON) and (Z). Let $u \in C(\mathbb{T}^n \times [0,\infty))$ be a visc. solution of (CP). Then u is bounded and uniformly continuous on $\mathbb{T}^n \times [0,\infty)$.

We will give some stability results concerning viscosity solutions.

Theorem 5[18] Let Ω be locally compact. $\mathcal{F}$ is a family of viscosity subsolutions of (EP). Assume that $\sup\mathcal{F}$ is locally bounded in Ω, then $\sup\mathcal{F}$ is also a visc. subsolution of (EP).

The theorems above are classical results in viscosity solution theory. We can find the proof in Refs. [19-22].

2 The Main Result

In this section, we want to prove our main result.

Theorem 6 Assume (C), (CER), (MON), (Z) and (DSTC^+) ((DSTC^-)). Let $f \in C(\mathbb{T}^n)$, and let $u \in C(\mathbb{T}^n \times [0,\infty))$ be a visc. solution of (CP). Then there exists $u_\infty \in C(\mathbb{T}^n)$ such that

$\lim_{t\to\infty} u(x,t) = u_\infty(x)$ uniformly on $\mathbb{T}^n$.

Moreover, u_∞ is a visc. solution of (EP), with $c = 0$.

First, we reduce the result to the case $f \in \mathrm{Lip}(\mathbb{T}^n)$. Indeed, we have

Lemma 1 If the result of Theorem 6 holds for any $f \in \mathrm{Lip}(\mathbb{T}^n)$, then it holds for any $f \in C(\mathbb{T}^n)$.

This is an easy consequence of Theorem 2 and the reader can find a proof of the lemma above in Ref. [7].

Lemma 2 There exists a viscosity subsolution $v_0 \in \mathrm{Lip}(\mathbb{T}^n)$ of (EP), with $c = 0$, such that

$0 \leqslant u(x,t) - v_0(x) \leqslant C_0$ for all $(x,t) \in \mathbb{T}^n \times [0,\infty)$.

Proof Due to (Z), there exists a solution $\omega \in \mathrm{Lip}(\mathbb{T}^n)$ of (EP), with $c = 0$. Since the function $\omega(x,t) := \omega_0(x)$ is a solution of (CP), by Theorem 2 we obtain

$$|u(x,t) - \omega_0(x)| \leqslant \max_{\mathbb{T}^n} |(u(\cdot,0) - \omega_0|$$

for all $(x,t) \in \mathbb{T}^n \times (0,\infty)$, which can be written as

$-C \leqslant u(x,t) - \omega_0(x) \leqslant C$ for all $(x,t) \in \mathbb{T}^n \times [0,\infty)$

with $C = \max_{\mathbb{T}^n} |(u(\cdot,0) - \omega_0|$. If we set $v_0(x) = \omega_0(x) - C$ and $C_0 = 2C$, then we have

$0 \leqslant u(x,t) - \omega_0(x) \leqslant C_0$ for all $(x,t) \in \mathbb{T}^n \times [0,\infty)$,

and, by (MON), the function v_0 is a subsolution of (EP), with $c = 0$.

For $(\eta,\theta) \in (0,\eta_0) \times (1,\theta_0)$, we define the function ω on $\bar{Q}$, $Q := \mathbb{T}^n \times (0,\infty)$ by

$$\omega(x,t) = \sup_{s \geqslant t} [u(x,t) - v_0(x) - \theta(u(x,s) - v_0(x) + \eta(s-t))] \quad (1)$$

where $v_0(x)$ is the function given by Lemma 2. We define the functions $\omega_{H,R}$, with $R > 0$, by

$$\omega_{H,R}(r) = \sup\{|H(x,u,p) - H(x,u,q)| : x \in \mathbb{T}^n, p, q \in \bar{B}_R, |p - q| \leqslant r\}.$$

Lemma 3 We have

$-C_0(\theta - 1) \leqslant \omega(x,t) \leqslant C_0$ for all $(x,t) \in \mathbb{T}^n \times (0,\infty)$.

Proof According to Lemma 2, for all $(x,t) \in \bar{Q}$,

$$\omega(x,t) \geqslant (1-\theta)(u(x,t) - v_0(x)) \geqslant -C_0(\theta - 1),$$

and

$$\omega(x,t) \leqslant \max_{s \geqslant t} (u(x,t) - v_0(x)) \leqslant C_0.$$

Theorem 7 The function ω is a subsolution of

$$\min\{\omega(x,t), \omega_t(x,t) - \omega_{H,R}(|D_x\omega(x,t)| + \psi)\} \leqslant 0 \text{ in } Q \quad (2)$$

where $\psi = \psi(\eta,\theta)$ is the constant from (DSTC^+), $R := (2\theta_0 + 1)L$ and $L := \max\{\|D_x u\|_\infty, \|D_x v_0\|_\infty\}$.

Proof Noting that $u \in \mathrm{Lip}(\mathbb{T}^n \times (0,\infty))$ and $v_0 \in \mathrm{Lip}(\mathbb{T}^n)$, then $\omega \in \mathrm{Lip}(\mathbb{T}^n \times (0,\infty))$.

Fix any $\phi_0 \in C^1(Q)$ and $(\hat{x},\hat{t}) \in Q$, and assume that

$$\max_Q (\omega - \phi_0) = (\omega - \phi_0)(\hat{x},\hat{t}).$$

If $\omega(\hat{x},\hat{t}) \leqslant 0$, then we have finished the proof. Therefore, we may assume that $\omega(\hat{x},\hat{t}) > 0$. We choose an $\hat{s} > \hat{t}$ so that

$$\omega(\hat{x},\hat{t}) = u(\hat{x},\hat{t}) - v_0(\hat{x}) - \theta(u(\hat{x},\hat{s}) - v_0(\hat{x}) + \eta(\hat{s} - \hat{t})).$$

If $\hat{s} = \hat{t}$, we get $\omega(\hat{x},\hat{t}) = (1-\theta)(u(\hat{x},\hat{t}) - v_0(\hat{x})) \leqslant 0$, and we are done. We may thus assume that $\hat{s} > \hat{t}$.

Define the function $\phi \in C^1(Q \times (0,\infty))$ by

$$\phi(x,t,s) = \phi_0(x,t) + (x - \hat{x})^2 + (t - \hat{t})^2 + (s - \hat{s})^2.$$

Note that the function

$$u(x,t)-v_0(x)-\theta(u(x,s)-v_0(x)+\eta(s-t))-\phi(x,t,s) \quad (3)$$

on $Q\times(0,\infty)$ attains a strict maximum at $(\hat{x},\hat{t},\hat{s})$, and that $D_x\phi(\hat{x},\hat{t},\hat{s})=D_x\phi_0(\hat{x},\hat{t})$, $\phi_t(\hat{x},\hat{t},\hat{s})=\phi_{0,t}(\hat{x},\ \hat{t})$ and $\phi_s(\hat{x},\hat{t},\hat{s})=0$.

Now, if B is an open ball of $\mathbb{T}^{3n+2}$ centered at $(\hat{x},\hat{x},\hat{x},\hat{t},\hat{s})$ with its closure $\bar{B}$ contained in $\mathbb{T}^{3n}\times(0,\infty)^2$, we consider the function Φ on $\bar{B}$ given by

$$\begin{aligned}&\Phi(x,y,z,t,s)\\&=u(x,t)-v_0(z)-\theta(u(y,s)-v_0(z)+\eta(s-t))-\phi(x,t,s)\\&-\alpha(|x-y|^2+|x-z|^2),\end{aligned}$$

where $\alpha>0$ is a large constant.

Let $(x_\alpha,y_\alpha,z_\alpha,t_\alpha,s_\alpha)\in\bar{B}$ be a maximum point of Φ. We can get

$$\lim_{\alpha\to\infty}(x_\alpha,y_\alpha,z_\alpha,t_\alpha,s_\alpha)=(\hat{x},\hat{x},\hat{x},\hat{t},\hat{s}) \quad (4)$$

Next, set

$p_\alpha=2(\theta-1)^{-1}\alpha(z_\alpha-x_\alpha)$ and $q_\alpha=2\theta^{-1}\alpha(x_\alpha-y_\alpha)$.

We observe that

$$\begin{aligned}&p_\alpha\in D^+v_0(z_\alpha),\\&(q_\alpha,\ -\theta^{-1}\phi_s(x_\alpha,t_\alpha,s_\alpha)-\eta)\in D^-u(y_\alpha,s_\alpha),\\&(D_x\phi(x_\alpha,t_\alpha,s_\alpha)+\theta q_\alpha-(\theta-1)p_\alpha,\phi_t(x_\alpha,y_\alpha,s_\alpha)-\theta\eta)\\&\in D^+u(x_\alpha,t_\alpha).\end{aligned}$$

We have $\max\{|p_\alpha|,|q_\alpha|\}\leqslant L$ by the definition of L, and by sending $\alpha\to+\infty$ along an appropriate sequence, we can find points $\hat{p},\hat{q}\in B_L$ such that

$$\hat{p}\in\bar{D}^+v_0(\hat{x}) \quad (5)$$

$$(\hat{q},\ -\theta^{-1}\phi_s(\hat{x},\hat{t},\hat{s})-\eta)\in\bar{D}^-u(\hat{x},\hat{s}) \quad (6)$$

$$(D_x\phi(\hat{x},\hat{t},\hat{s})+\theta\hat{q}-(\theta-1)\hat{p},\phi_t(\hat{x},\hat{y},\hat{s})-\theta\eta)\in\bar{D}^+u(\hat{x},\hat{t}) \quad (7)$$

where $\bar{D}^\pm$ stands for the closure of $D^\pm$, for instance, $\bar{D}^+u(\hat{x},\hat{s})$ denotes the set of points $(q,b)\in\mathbb{T}^n\times\mathbb{R}$ for which there are sequences $\{(q_j,b_j)\}_j\subset\mathbb{T}^n\times\mathbb{R}$ and $\{(x_j,s_j)\}_j\subset Q$ such that $\lim_{j\to\infty}(q_j,b_j,x_j,s_j)=(q,b,\hat{x},\hat{s})$ and $(q_j,b_j)\in D^+u(x_j,s_j)$ for all j. Recall that $\phi_s(\hat{x},\hat{t},\hat{s})=\phi_{0,t}(\hat{x},\hat{t})$ and $D_x\phi(\hat{x},\hat{t},\hat{s})=D_x\phi_0(\hat{x},\ \hat{t})$, so that we have

$$\begin{aligned}&H(\hat{x},v_0(\hat{x}),\hat{p})\leqslant 0,\\&-\eta+H(\hat{x},u(\hat{x},\hat{s}),\hat{q})\geqslant 0\end{aligned}$$

from (5) and (6). Therefore, (DSTC[+]) ensures

$$H(\hat{x},v_0(\hat{x})+\theta(u(\hat{x},\hat{s})-v_0(\hat{x})),\hat{p}+\theta(\hat{q}-\hat{p}))>\eta\theta+\psi \quad (8)$$

Since $\omega(\hat{x},\hat{t})>0$, we have

$$u(\hat{x},\hat{t})>\theta u(\hat{x},\hat{s})+(1-\theta)v_0(\hat{x})+\theta\eta(\hat{s}-\hat{t}) \quad (9)$$

Because of the assumption (MON), (8) and (9),

$$\begin{aligned}&0\geqslant\phi_t(\hat{x},\hat{y},\hat{s})-\theta\eta+H(\hat{x},u(\hat{x},\hat{t}),D_x\phi(\hat{x},\hat{t},\hat{s})\\&+\theta\hat{q}+(1-\theta)\hat{p})\\&\geqslant\phi_t(\hat{x},\hat{y},\hat{s})-\theta\eta+H(\hat{x},\theta u(\hat{x},\hat{s})+(1-\theta)v_0(\hat{x}),\\&\theta\hat{q}+(1-\theta)\hat{p})-\omega_{H,R}\left(\left|D_x\phi\right|\right)\\&\geqslant\phi_t(\hat{x},\hat{y},\hat{s})+\psi-\omega_{H,R}\left(\left|D_x\phi\right|\right).\end{aligned}$$

The second inequality holds since $|\theta\hat{q}+(1-\theta)\hat{p}|\leqslant(1+2\theta)L\leqslant R$ and $|D_x\phi_0(\hat{x},\hat{t})+\theta\hat{q}+(1-\theta)\hat{p}|\leqslant L$ because of (7). Therefore, we get

$$\phi_t(\hat{x},\hat{t},\hat{s})-\omega_{H,R}\left(\left|D_x\phi\right|\right)+\psi\leqslant 0,$$

i.e.,

$$\phi_{0,t}(\hat{x},\hat{t})-\omega_{H,R}\left(\left|D_x\phi_0\right|\right)+\psi\leqslant 0.$$

We set

$\omega_\infty(x)=\limsup_{t\to\infty}\omega(x,t)$ for all $x\in\mathbb{T}^n$.

Lemma 4 We have

$\omega_\infty(x)\leqslant 0$ for all $x\in\mathbb{T}^n$.

Moreover, the convergence

$$\lim_{t\to\infty}\max\{\omega(x,t),0\}=0$$

is uniform in $x\in\mathbb{T}^n$.

Proof If the convergence does not hold uniformly in $x\in\mathbb{T}^n$, we can choose a sequence (x_j,t_j) such that $\lim_{j\to\infty}t_j=\infty$ and $\omega(x_j,t_j)\geqslant\delta$ for all $j\in\mathbb{N}$ and some constant $\delta>0$. We may assume that $\lim_{j\to\infty}x_j=y$ for some $y\in\mathbb{T}^n$. In view of the Ascoli-Arzela theorem, we may assume by passing to a subsequence of (x_j,t_j) if needed that

$\lim_{j\to\infty}\omega(x,t+t_j)=g(x,t)$ uniformly in $\mathbb{T}^n\times(-\infty,+\infty)$,

for some bounded function $g\in\mathrm{Lip}(\mathbb{T}^n\times\mathbb{R})$ and $g(y,0)\geqslant\delta$.

By the stability of the subsolution property under uniform convergence, we see that g is a subsolution of

$$\min\{g(x,t),g_t(x,t)-\omega_{H,R}(|D_xg(x,t)|)+\psi\}\leqslant 0$$

in $\mathbb{T}^{n+1}$. Since $g\in\mathrm{Lip}(\mathbb{T}^n\times\mathbb{R})$ and g is bounded on $\mathbb{R}^{n+1}$, for every $\varepsilon>0$, the function $g(x,t)-\varepsilon t^2$ attains a maximum at a point $(x_\varepsilon,t_\varepsilon)$, then we have

$$g(x_\varepsilon,t_\varepsilon)-\varepsilon t_\varepsilon^2\geqslant g(y,0)\geqslant\delta.$$

Therefore, we know that

$$g(x_\varepsilon,t_\varepsilon)>\delta \text{ and } \varepsilon|t_\varepsilon|\leqslant(\varepsilon\|g\|_\infty)^{1/2}.$$

In particular, we have $\lim_{\varepsilon\to 0^+}\varepsilon t_\varepsilon=0$. Then, as usual in the viscosity solutions theory, we get

$$2\varepsilon t_\varepsilon-\omega_{H,R}(0)+\psi\leqslant 0,$$

which, in the limits as $\varepsilon\to 0^+$, yields $\psi\leqslant 0$, a contradiction.

Proof of Theorem 6 under condition (DSTC[+])

Let ω be the function defined by (1), with arbitrary $(\eta,\theta)\in(0,\eta_0)\times(1,\theta_0)$.

Fix any $\varepsilon>0$. Because of (1), we may choose a constant T_ε so that for any $t > T_\varepsilon$, $\omega(x,t) \leqslant \varepsilon$ for all $x \in \mathbb{T}^n$.

From the above, for any $s > t$, we have

$$u(x,t) - v_0(x) \leqslant \varepsilon + \theta(u(x,s) - v_0(x)) + \theta\eta(s-t)$$
$$\leqslant \varepsilon + u(x,s) - v_0(x) + (\theta - 1)C_0 + \theta\eta(s-t).$$

Thus, for any $0 \leqslant s \leqslant 1$, we have

$$u(x,t) \leqslant u(x,t+s) + (\theta-1)C_0 + \theta\eta + \varepsilon \quad (10)$$

Now, since u is bounded and Lipschitz continuous in $\bar{Q}$, in view of the Ascoli-Arzela theorem, we may choose a sequence $\tau_j \to \infty$ and a bounded function $z \in \mathrm{Lip}(\mathbb{T}^n \times (-\infty, +\infty))$ so that

$$\lim_{j\to\infty} u(x, t+\tau_j) = z(x,t) \text{ locally uniformly on } \mathbb{T}^{n+1} \quad (11)$$

By (10) we get

$$z(x,t) \leqslant z(x,t+s) + (\theta-1)C_0 + \theta\eta + \varepsilon \quad (12)$$

for all $(x,t,s) \in \mathbb{R}^{n+1} \times [0,1]$. This is valid for all $(\eta, \theta) \in (0,\eta_0) \times (1,\theta_0)$. Hence we obtain

$$z(x,t) \leqslant z(x,t+s) \text{ for all } (x,t,s) \in \mathbb{T}^n \times \mathbb{R} \times [0,1] \quad (13)$$

Thus we find that the function $z(x,t)$ is nondecreasing in $t \in \mathbb{R}$ for all $x \in \mathbb{T}^n$.

From this we conclude that

$$\lim_{t\to\infty} z(x,t) = u_\infty(x) \text{ uniformly on } \mathbb{T}^n \quad (14)$$

for some function $u_\infty \in \mathrm{Lip}(\mathbb{T}^n)$. Since $u(x,t)$ is a viscosity solution of (CP), and $u(x,t)$ is bounded on $\mathbb{T}^n \times [0,\infty)$, we get from Theorem 5 that $z(x,t)$ is a solution of (CP), and moreover, $u_\infty(x)$ is a solution of (EP).

Fix any $\delta > 0$. By (14) there is a constant $\tau > 0$ such that

$$\|z(\cdot,\tau) - u_\infty\|_\infty < \delta,$$

and by (11) there is a $j \in \mathbb{N}$ such that

$$\|z(\cdot,\tau+\tau_j) - u(\cdot,\tau+\tau_j)\|_\infty < \delta.$$

Therefore,

$$\|u(\cdot,\tau+\tau_j) - u_\infty\|_\infty < 2\delta.$$

By the contraction property, we see that for any $t \geqslant \tau + \tau_j$,

$$\|u(\cdot,t) - u_\infty\|_\infty \leqslant \|u(\cdot,\tau+\tau_j) - u_\infty\|_\infty < 2\delta,$$

which completes the proof.

Proof of Theorem 6 under condition (DSTC$^-$)

We adjust $\omega(x,t)$ to the form

$$\omega(x,t) = \sup_{0\leqslant s\leqslant t} [u(x,t) - v_0(x) - \theta(u(x,s) - v_0(x) - \eta(s-t))]$$

where (η,θ) is chosen arbitrarily in $(0,\eta_0) \times (1,\theta_0)$ and the constants η_0 and θ_0 are from (DSTC$^-$).

Theorem 8 The function ω is a subsolution of

$$\min\{\omega(x,t), \omega_t(x,t) - \omega_{H,R}(|D_x\omega(x,t)|) + \psi\} \leqslant 0,$$
$$(x,t) \in \mathbb{T}^n \times (T,\infty),$$

where $\psi = \psi(\eta,\theta) > 0$ is the constant from (DSTC$^-$), $T := C_0/\eta$ and $R := (2\theta_0 + 1)L$.

We can prove

$$-C_0(\theta-1) \leqslant \omega(x,t) \leqslant C_0 \text{ for all } (x,t) \in \bar{Q}.$$

On the other hand, for any $(x,t) \in \mathbb{T}^n \times (T,\infty)$ and $s \in [0, t-T)$,

$$u(x,t) - v_0(x) - \theta(u(x,s) - v_0(x) - \eta(s-t))$$
$$\leqslant C_0 - \theta\eta(t-s) < C_0 - \theta\eta T = -(\theta-1)C_0.$$

Hence, for any $(x,t) \in \mathbb{T}^n \times (T,\infty)$ we have

$$\omega(x,t) = \max_{t-T\leqslant s\leqslant t} [u(x,t) - v_0(x) - \theta(u(x,s) - v_0(x) - \eta(s-t))]$$
$$= \max_{-T\leqslant s\leqslant 0} [u(x,t) - v_0(x) - \theta(u(x,s+t) - v_0(x) - \eta s)]$$

We fix any test function $\phi_0 \in C^1(\mathbb{T}^n \times (T,\infty))$ and assume that $\omega - \phi_0$ attains a strict maximum at a point $(\hat{x},\hat{t})$.

We can get the following conclusion like the proof of (DSTC$^+$):

$$\hat{p} \in \bar{D}^+ v_0(\hat{x}),$$
$$(\hat{q},\eta) \in \bar{D}^- u(\hat{x},\hat{s}),$$
$$(D_x\phi_0(\hat{x},\hat{t}) + \theta\hat{q} - (\theta-1)\hat{p}, \phi_{0,t}(\hat{x},\hat{t}) + \theta\eta \in \bar{D}^+ u(\hat{x},\hat{t}),$$

for some $\hat{p}, \hat{q} \in \mathbb{T}^n$.

Similarly, we can assume $\omega(\hat{x},\hat{t}) > 0$. According to assumption (MON), we have

$$0 \geqslant \phi_t(\hat{x},\hat{y},\hat{s}) + \theta\eta + H(\hat{x}, u(\hat{x},\hat{t}), D_x\phi(\hat{x},\hat{t},\hat{s}) + \theta\hat{q} + (1-\theta)\hat{p})$$
$$\geqslant \phi_t(\hat{x},\hat{y},\hat{s}) + \theta\eta + H(\hat{x}, u(\hat{x},\hat{t}), \theta\hat{q} - (\theta-1)\hat{p})$$
$$-\omega_{H,R}\left(\left|D_x(\phi)\right|\right)$$
$$\geqslant \phi_t(\hat{x},\hat{y},\hat{s}) + \theta\eta - \omega_{H,R}\left(\left|D_x(\phi)\right|\right)$$
$$+H(\hat{x}, \theta u(\hat{x},\hat{s}) + (1-\theta)v_0(\hat{x}), \theta\hat{q} + (1-\theta)\hat{p})$$

Since $-\eta \leqslant H(x, u(\hat{x},\hat{s}), \hat{q}), H(\hat{x}, v_0(\hat{x}), \hat{p}) \leqslant 0$, we get

$$\phi_t(\hat{x},\hat{t},\hat{s}) - \omega_{H,R}\left(\left|D_x(\phi)\right|\right) + \psi < 0$$

by the condition (DSTC$^-$).

Also, we can prove

$$\lim_{t\to\infty} \max\{\omega(x,t), 0\} = 0 \text{ uniformly in } \mathbb{T}^n \quad (15)$$

We can get $u(x,t)$ is nonincreasing for all $x \in \mathbb{T}^n$ and enough large t from (15). Since u is bounded and Lipschitz continuous in $\mathbb{T}^n \times (-\infty,\infty)$, we can choose a sequence $\tau_j \to \infty$ such that for some function $z \in \mathrm{Lip}(\mathbb{T}^n \times \mathbb{R})$,

$$\lim_{j\to\infty} u(x,t+\tau_j) = z(x,t) \text{ locally uniformly on } \mathbb{T}^n \times \mathbb{R}.$$

$z(x,t)$ is nonincreasing in t for all $x \in \mathbb{T}^n$, then there exists the function $u_\infty \in C(\mathbb{T}^n)$ such that

$$\lim_{t\to\infty} z(x,t) = u_\infty(x) \text{ uniformly on } \mathbb{T}^n.$$

Furthermore

$$\lim_{t\to\infty} u(x,t) = u_\infty(x) \text{ uniformly on } \mathbb{T}^n.$$

Thus we complete the proof.

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