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Lie Symmetry Theorem for Nonshifted Birkhoffian Systems on Time Scales

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Abstract: The Lie theorem for Birkhoffian systems with time-scale nonshifted variational problems are studied, including free Birkhoffian system, generalized Birkhoffian system and constrained Birkhoffian system. First, the time-scale nonshifted generalized Pfaff-Birkhoff principle is established, and the dynamical equations for three Birkhoffian systems under nonshifted variational problems are deduced. Afterwards, in the time-scale nonshifted variational problems, by introducing the infinitesimal transformations, Lie symmetry for free Birkhoffian system, generalized Birkhoffian system and constrained Birkhoffian system are defined respectively. Then Lie symmetry theorems for three kinds of Birkhoffian systems are deduced and proved. In the end, three examples are given to explain the applications for the results.

Key words: time scales; Lie symmetry; conserved quantity; nonshifted Birkhoffian system

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0 Introduction

Time scale provides effect mathematical tools for the research of complex dynamical systems. German scholar Hilger[1] first unified and extended the theory of continuity and discreteness — an analytical theory on time scales. Since then, many scholars[2-10] have conducted detailed studies on time scales. Time scale is also widely used in practical problems. For example, in a simple series circuit composed of resistance, capacitance and self-inductive coil, when the capacitor is periodically closed at a fixed frequency, the current change rate in the circuit can be precisely described by the derivative on time scales[11], nonlinear partially defined systems on an arbitrary unbounded time scale are studied[12,13].

The research for symmetry and conservation law is an important direction in the development of modern analysis mechanics. Noether[14] studied the invariance of action integrals in finite continuous groups and revealed the internal relationship between symmetries and conserved quantities in dynamic systems. Lutzky[15] introduced the Lie symmetry method of mechanical system, that is, the equation remains unchanged after the introduction of infinitesimal group transformation. Mei[16,17] studied the dynamics equations for holonomic systems, nonholonomic systems and Birkhoffian systems in detail, as well as the corresponding symmetries and conserved quantities for these systems. Afterwards, some progress has been made in the study of symmetry and conserved quantities for different mechanical systems[18-22].

The symmetries and conserved quantities for different dynamical systems under time-scale shifted variational problems are firstly studied. Take the Lagrange system as an example. In classical dynamical system, assuming that the configurations for the dynamics systems were determined by n generalized coordinates $q_s(s=1,2,\cdots,n)$, then Lagrangian functions were written as $L=L\left(t,q_s(t),\dot{q}_s(t)\right)$. On time scales, the Lagrangian functions were written as $L=L\left(t,q_s^{\sigma}(t),q_s^{\Delta}(t)\right)$. On the shifted variational problem, Bartosiewicz and Torres[23] derived Noether's theorem under the Δ-derivative by using the time reparameter method, and proved the variational problem of mechanical system. Bartosiewicz and his co-workers[24] studied the time-scale second Euler-Lagrange equation, and derived Noether's conserved quantities on time scales by using this equation. Subsequently, many scholars[25-34] began to carry out a series of studies on time-scale the variational principle for constrained mechanical systems and Noether's theorem of different dynamic systems. However, on a time scale, there are few studies on Lie symmetry in mechanical systems. Cai *et al*[35] preliminarily studied Lie symmetry and time-scale conserved quantity for Lagrangian systems. Zhang and his team[36-39] also discussed the time-scale for different dynamic systems and its Lie theorem.

Recently, Bourdin[40] studied the nonshifted problem on time scales. In the nonshifted variational principle, the classical Lagrangian function was written as $L=L\left(t,q_s(t),q_s^{\Delta}(t)\right)$ on time scales. They also found the nonshifted variational problem in discrete conditions related to the structure-preserving algorithm. Anerot *et al* [41] rederived time-scale Noether's theorem with shifted and nonshifted variational problems, and the correctness of the results was illustrated by numerical simulation. The calculation of the nonshifted variational principle is better than that of the shifted variational principle in the dynamic system. Song and Cheng[42] studied Noether's symmetry for free Birkhoffian systems and Hamiltonian systems about time-scale with nonshifted variational problems. Song[43] also studied the adiabatic invariants for three kinds of dynamical systems under the nonshifted variational problem. Then Chen and Zhang[44] researched Noether's quasi-symmetry theorem of three kinds of Birkhoffian systems about time-scale with nonshifted variational problems. Therefore, this paper will further research Lie symmetry theorem of three Birkhoffian systems about time-scale with nonshifted variational problems, including free Birkhoffian system, generalized Birkhoffian system and constrained Birkhoffian system.

1 The Dynamical Equations

About time-scale calculus and its properties, please refer to Ref. [2].

1.1 Free Birkhoffian System

The time-scale nonshifted Pfaff-Birkhoff action is written as

$$A=\int_{t_1}^{t_2}\left[R_{\upsilon}\left(t,a_{\rho}\right)a_{\upsilon}^{\Delta}-B\left(t,a_{\rho}\right)\right]\Delta t \tag{1}$$

where a_{υ}^{Δ} is the delta derivative of a_{υ} with respect to t. The Birkhoffian $B{:}\mathbb{T}\times\mathbb{R}^{2n}\to\mathbb{R}$, Birkhoff's functions R_{υ}: $\mathbb{T}\times\mathbb{R}^{2n}\to\mathbb{R}$. Let all functions are differential and their derivatives are rd-continuous[2], $\rho,\upsilon=1,2,\cdots,2n$.

The isochronous variation principle

$$\delta A=0 \tag{2}$$

and exchange relationship

$$\delta a_{\upsilon}^{\Delta}=\left(\delta a_{\upsilon}\right)^{\Delta} \tag{3}$$

with the endpoint conditions

$$\delta a_{\upsilon}\big|_{t=t_1}=\delta a_{\upsilon}\big|_{t=t_2}=0 \tag{4}$$

are called the nonshifted Pfaff-Birkhoff principle.

We can obtain the time-scale nonshifted Birkhoff's equations [42]:

$$R_{\mu}^{\nabla}=\sigma^{\nabla}(t)\left[\frac{\partial R_{\upsilon}\left(t,a_{\rho}\right)}{\partial a_{\mu}}a_{\upsilon}^{\Delta}-\frac{\partial B\left(t,a_{\rho}\right)}{\partial a_{\mu}}\right],$$

$$\mu,\upsilon,\rho=1,2,\cdots,2n \tag{5}$$

1.2 Generalized Birkhoffian System

We extend the principle (2)-(4) to the time-scale nonshifted generalized Pfaff-Birkhoff principle

$$\int_{t_1}^{t_2}\left\{\delta\left(R_{\upsilon}a_{\upsilon}^{\Delta}-B\right)+\Lambda_{\upsilon}\left(t,a_{\rho}\right)\delta a_{\upsilon}\right\}\Delta t=0 \tag{6}$$

We have the time-scale nonshifted generalized Birkhoff's equations [44]

$$\sigma^{\nabla}(t)\left[\frac{\partial R_{\upsilon}}{\partial a_{\mu}}a_{\upsilon}^{\Delta}-\frac{\partial B}{\partial a_{\mu}}+\Lambda_{\mu}\right]=R_{\mu}^{\nabla},\ \mu,\upsilon=1,2,\cdots,2n \tag{7}$$

1.3 Constrained Birkhoffian System

If the constraint equations are shown as

$$f_{\beta}(t,a_{\rho})=0,\ \beta=1,2,\cdots,2n \tag{8}$$

To calculate the variation in equation (8)

$$\frac{\partial f_{\beta}}{\partial a_{\mu}}\delta a_{\mu}=0 \tag{9}$$

Form Eq. (9), we have

$$\int_{t_1}^{t_2} \lambda_\beta \frac{\partial f_\beta}{\partial a_\mu} \delta a_\mu \Delta t = 0 \tag{10}$$

Integration by parts with equation (10), we get

$$\int_{t_1}^{t_2} \left(\int_{t_1}^{\sigma(t)} \lambda_\beta \frac{\partial f_\beta}{\partial a_\mu} \nabla \tau \right) \left(\delta a_\mu \right)^\Delta \Delta t = 0 \tag{11}$$

By the principle and Eq. (11), we have the time-scale nonshifted constrained Birkhoff's equations [44]

$$\sigma^\nabla(t) \left[\frac{\partial R_\upsilon}{\partial a_\mu} a_\upsilon^\Delta - \frac{\partial B}{\partial a_\mu} - \lambda_\beta \frac{\partial f_\beta}{\partial a_\mu} \right] - R_\mu^\nabla = 0,$$

$$\mu, \upsilon, \beta = 1, 2, \cdots, 2n \tag{12}$$

If the systems are nonsingular, by using Eq. (8) and (12), we can solve $\lambda_\beta = \lambda_\beta(t, a_\rho)$, the equation (12) can be written

$$\sigma^\nabla(t) \left[\frac{\partial R_\upsilon}{\partial a_\mu} a_\upsilon^\Delta - \frac{\partial B}{\partial a_\mu} - P_\mu \right] = R_\mu^\nabla \tag{13}$$

where

$$P_\mu = \lambda_\beta \frac{\partial f_\beta}{\partial a_\mu} \tag{14}$$

Equation (13) should be called as the corresponding nonshifted free Birkhoffian system (5).

2 Structural Equations and Conserved Quantities

The infinitesimal transformations

$$t^* = t + \varepsilon \xi_0(t, a_\rho), a_\mu^* = a_\mu + \varepsilon \xi_\mu(t, a_\rho) \tag{15}$$

where ε is an infinitesimal parameter, ξ_0 and ξ_μ are the infinitesimal generators.

Introduce the infinitesimal generated vectors [17]

$$\boldsymbol{X}^{(0)} = \xi_0 \frac{\partial}{\partial t} + \xi_\mu \frac{\partial}{\partial a_\mu} \tag{16}$$

And an extension of infinitesimal generator

$$\boldsymbol{X}^{(1)} = \boldsymbol{X}^{(0)} + \left(\xi_\mu^\Delta - a_\mu^\Delta \xi_0^\Delta \right) \frac{\partial}{\partial a_\mu^\Delta} \tag{17}$$

2.1 Free Birkhoffian System

Definition 1 If the infinitesimal transformations (15) satisfy deterministic equation

$$\sigma^\nabla \boldsymbol{X}^{(1)} \left[\left(\frac{\partial R_\upsilon}{\partial a_\mu} \right) a_\upsilon^\Delta - \frac{\partial B}{\partial a_\mu} \right] + \sigma^\nabla \left(\xi_\upsilon^\Delta - a_\upsilon^\Delta \right) \frac{\partial R_\upsilon}{\partial a_\mu} - \boldsymbol{X}^{(1)} \left(R_\mu^\nabla \right) = 0 \tag{18}$$

and corresponding symmetry is said the Lie symmetry for nonshifted free Birkhoffian systems (5).

Theorem 1 If ξ_0 and ξ_μ satisfy the deterministic equation (18), and there exists $G(t, a_\rho)$ satisfying structural equation

$$\left(R_\upsilon a_\upsilon^\Delta - B \right) \xi_0^\Delta + \boldsymbol{X}^{(1)} \left(R_\upsilon a_\upsilon^\Delta - B \right) + G^\Delta = 0 \tag{19}$$

This system (5) has the conserved quantity, that is

$$I = R_\upsilon \xi_\upsilon^\sigma + \int_{t_1}^{t} \left[\left(\frac{\partial R_\upsilon}{\partial \tau} a_\upsilon^\Delta - \frac{\partial B}{\partial \tau} \right) \xi_0 - B \xi_0^\Delta \right] \sigma^\nabla \nabla \tau + G^\sigma = \text{const}. \tag{20}$$

Proof Take the ∇- derivative for equation (20)

$$\begin{aligned} \frac{\nabla}{\nabla t} I &= \left(R_\upsilon \xi_\upsilon^\sigma \right)^\nabla - \left(B \xi_0^\sigma \right)^\nabla + \sigma^\nabla \xi_0 \frac{\partial R_\upsilon}{\partial t} a_\upsilon^\Delta - \sigma^\nabla \xi_0 \frac{\partial B}{\partial t} + G^\Delta \sigma^\nabla \\ &= \sigma^\nabla \xi_0 \left(\frac{\partial R_\upsilon}{\partial t} a_\upsilon^\Delta - \frac{\partial B}{\partial t} \right) + R_\upsilon^\nabla \xi_\upsilon + \sigma^\nabla R_\upsilon \xi_\upsilon^\Delta - \sigma^\nabla B \xi_0^\Delta + G^\Delta \sigma^\nabla \end{aligned} \tag{21}$$

According to

$$\boldsymbol{X}^{(1)} \left(R_\upsilon a_\upsilon^\Delta - B \right) = \xi_0 \left(\frac{\partial R_\upsilon}{\partial t} a_\upsilon^\Delta - \frac{\partial B}{\partial t} \right) + \xi_\mu \left(\frac{\partial R_\upsilon}{\partial a_\mu} a_\upsilon^\Delta - \frac{\partial B}{\partial a_\mu} \right) + \left(\xi_\mu^\Delta - a_\mu^\Delta \xi_0^\Delta \right) R_\mu \tag{22}$$

Substituting equation (22) into equation (21), we can get

$$\begin{aligned} \frac{\nabla}{\nabla t} I &= \sigma^\nabla \xi_0 \left(\frac{\partial R_\upsilon}{\partial t} a_\upsilon^\Delta - \frac{\partial B}{\partial t} \right) + R_\mu^\nabla \xi_\mu + \sigma^\nabla R_\upsilon \xi_\upsilon^\Delta \\ &\quad - \sigma^\nabla B \xi_0^\Delta - \sigma^\nabla \left(R_\upsilon a_\upsilon^\Delta - B \right) \xi_0^\Delta - \sigma^\nabla \xi_0 \left(\frac{\partial R_\upsilon}{\partial t} a_\upsilon^\Delta - \frac{\partial B}{\partial t} \right) \\ &\quad - \sigma^\nabla \xi_\mu \left(\frac{\partial R_\upsilon}{\partial a_\mu} a_\upsilon^\Delta - \frac{\partial B}{\partial a_\mu} \right) - \sigma \left(\xi_\mu^\Delta - a_\mu^\Delta \xi_0^\Delta \right) R_\mu \\ &= \left[R_\mu^\nabla - \sigma^\nabla \left(\frac{\partial R_\upsilon}{\partial a_\mu} a_\upsilon^\Delta - \frac{\partial B}{\partial a_\mu} \right) \right] \xi_\mu = 0 \end{aligned} \tag{23}$$

Therefore, the proof is completed.

2.2 Generalized Birkhoffian System

Definition 2 If ξ_0 and ξ_μ of the infinitesimal transformations (15) satisfy deterministic equation

$$\sigma^\nabla \boldsymbol{X}^{(1)} \left[\left(\frac{\partial R_\upsilon}{\partial a_\mu} \right) a_\upsilon^\Delta - \frac{\partial B}{\partial a_\mu} \right] + \sigma^\nabla \left(\xi_\upsilon^\Delta - a_\upsilon^\Delta \right) \frac{\partial R_\upsilon}{\partial a_\mu} - \boldsymbol{X}^{(1)} \left(R_\mu^\nabla \right) + \sigma^\nabla \boldsymbol{X}^{(1)} \left(\varLambda_\mu \right) = 0 \tag{24}$$

then corresponding symmetry is said the Lie symmetry for the nonshifted generalized Birkhoffian system (7).

Theorem 2 If ξ_0 and ξ_μ satisfy the deterministic equation (24), and there exists $G(t, a_\rho)$ satisfying the structural equation

$$\left(R_\upsilon a_\upsilon^\Delta - B \right) \xi_0^\Delta + \boldsymbol{X}^{(1)} \left(R_\upsilon a_\upsilon^\Delta - B \right) + G^\Delta + \left(\xi_\upsilon - a_\upsilon^\Delta \xi_0 \right) \varLambda_\upsilon = 0 \tag{25}$$

This system (7) has the conserved quantity, that is

$$I=R_{\upsilon}\xi_{\upsilon}^{\sigma}+\int_{t_1}^{t}\left[\left(\frac{\partial R_{\upsilon}}{\partial\tau}a_{\upsilon}^{\Delta}-\frac{\partial B}{\partial\tau}-\Lambda_{\upsilon}a_{\upsilon}^{\Delta}\right)\xi_0-B\xi_0^{\Delta}\right]\sigma^{\nabla}\nabla\tau$$
$$+G^{\sigma}=\text{const.} \tag{26}$$

Proof Take the ∇- derivative for equation (26)

$$\frac{\nabla}{\nabla t}I=\left(R_{\upsilon}\xi_{\upsilon}^{\sigma}\right)^{\nabla}-\left(B\xi_0^{\sigma}\right)^{\nabla}+\sigma^{\nabla}\xi_0\frac{\partial R_{\upsilon}}{\partial t}a_{\upsilon}^{\Delta}-\sigma^{\nabla}\xi_0\frac{\partial B}{\partial t}$$
$$-\sigma^{\nabla}\xi_0\Lambda_{\upsilon}a_{\upsilon}^{\Delta}+G^{\Delta}\sigma^{\nabla}$$
$$=\sigma^{\nabla}\xi_0\left(\frac{\partial R_{\upsilon}}{\partial t}a_{\upsilon}^{\Delta}-\frac{\partial B}{\partial t}\right)+R_{\upsilon}^{\nabla}\xi_{\upsilon}+\sigma^{\nabla}R_{\upsilon}\xi_{\upsilon}^{\Delta}$$
$$-\sigma^{\nabla}B\xi_0^{\Delta}-\sigma^{\nabla}\xi_0\Lambda_{\upsilon}a_{\upsilon}^{\Delta}+G^{\Delta}\sigma^{\nabla} \tag{27}$$

And because

$$\boldsymbol{X}^{(1)}\left(R_{\upsilon}a_{\upsilon}^{\Delta}-B\right)=\xi_0\left(\frac{\partial R_{\upsilon}}{\partial t}a_{\upsilon}^{\Delta}-\frac{\partial B}{\partial t}\right)+\xi_{\mu}\left(\frac{\partial R_{\upsilon}}{\partial a_{\mu}}a_{\upsilon}^{\Delta}-\frac{\partial B}{\partial a_{\mu}}\right)$$
$$+\left(\xi_{\mu}^{\Delta}-a_{\mu}^{\Delta}\xi_0^{\Delta}\right)R_{\mu} \tag{28}$$

Substituting Eqs. (7) and (28) into equation (27), we can get

$$\frac{\nabla}{\nabla t}I=\sigma^{\nabla}\xi_0\left(\frac{\partial R_{\upsilon}}{\partial t}a_{\upsilon}^{\Delta}-\frac{\partial B}{\partial t}\right)+R_{\mu}^{\nabla}\xi_{\mu}+\sigma^{\nabla}R_{\upsilon}\xi_{\upsilon}^{\Delta}-\sigma^{\nabla}B\xi_0^{\Delta}$$
$$-\sigma^{\nabla}\xi_0\Lambda_{\upsilon}a_{\upsilon}^{\Delta}-\sigma^{\nabla}\left(R_{\upsilon}a_{\upsilon}^{\Delta}-B\right)\xi_0^{\Delta}$$
$$-\sigma^{\nabla}\xi_0\left(\frac{\partial R_{\upsilon}}{\partial t}a_{\upsilon}^{\Delta}-\frac{\partial B}{\partial t}\right)-\sigma^{\nabla}\xi_{\mu}\left(\frac{\partial R_{\upsilon}}{\partial a_{\mu}}a_{\upsilon}^{\Delta}-\frac{\partial B}{\partial a_{\mu}}\right)$$
$$-\sigma^{\nabla}\left(\xi_{\mu}^{\Delta}-a_{\mu}^{\Delta}\xi_0^{\Delta}\right)R_{\mu}-\sigma^{\nabla}\Lambda_{\upsilon}\left(\xi_{\upsilon}-a_{\upsilon}^{\Delta}\xi_0\right)$$
$$=\left[R_{\mu}^{\nabla}-\sigma^{\nabla}\left(\frac{\partial R_{\upsilon}}{\partial a_{\mu}}a_{\upsilon}^{\Delta}-\frac{\partial B}{\partial a_{\mu}}+\Lambda_{\mu}\right)\right]\xi_{\mu}=0 \tag{29}$$

Therefore, the proof is completed.

2.3 Constrained Birkhoffian System

Definition 3 If ξ_0 and ξ_{μ} of the infinitesimal transformations (15) satisfy the Lie symmetry deterministic equation

$$\sigma^{\nabla}\boldsymbol{X}^{(1)}\left[\left(\frac{\partial R_{\upsilon}}{\partial a_{\mu}}\right)a_{\upsilon}^{\Delta}-\frac{\partial B}{\partial a_{\mu}}\right]+\sigma^{\nabla}\left(\xi_{\upsilon}^{\Delta}-a_{\upsilon}^{\Delta}\right)\frac{\partial R_{\upsilon}}{\partial a_{\mu}}-\boldsymbol{X}^{(1)}\left(R_{\mu}^{\nabla}\right)$$
$$-\sigma^{\nabla}\boldsymbol{X}^{(1)}\left(P_{\mu}\right)=0 \tag{30}$$

then corresponding symmetry is said the Lie symmetry for nonshifted constrained Birkhoffian system (13).

Theorem 3 If ξ_0 and ξ_{μ} satisfy the deterministic equation (30), and exist $G(t,a_{\rho})$ satisfying the structural equation

$$\left(R_{\upsilon}a_{\upsilon}^{\Delta}-B\right)\xi_0^{\Delta}+\boldsymbol{X}^{(1)}\left(R_{\upsilon}a_{\upsilon}^{\Delta}-B\right)+G^{\Delta}-\left(\xi_{\upsilon}-a_{\upsilon}^{\Delta}\xi_0\right)P_{\upsilon}=0 \tag{31}$$

then system (13) has the conserved quantity, that is

$$I=R_{\upsilon}\xi_{\upsilon}^{\sigma}+\int_{t_1}^{t}\left[\left(\frac{\partial R_{\upsilon}}{\partial\tau}a_{\upsilon}^{\Delta}-\frac{\partial B}{\partial\tau}+P_{\upsilon}a_{\upsilon}^{\Delta}\right)\xi_0-B\xi_0^{\Delta}\right]\sigma^{\nabla}\nabla\tau+G^{\sigma}$$
$$=\text{const.} \tag{32}$$

3 Example

Example 1 On time scales, the Hojman-Urrutia problem can be written

$$\begin{cases}B=\dfrac{1}{2}\left\{\left(a_3\right)^2+2a_2a_3-\left(a_4\right)^2\right\}\\ R_1=a_2+a_3,R_2=0,R_3=a_4,R_4=0\end{cases} \tag{33}$$

According to the study of Santilli[45], the Hojman-Urrutia problem admits a Birkhoffian representation. But since the equation itself is not self-adjoint, it has no Hamiltonian representation.

By the nonshifted Birkhoff's equation (5), we have

$$\begin{cases}\left(a_2+a_3\right)^{\nabla}=0\\ \sigma^{\nabla}\left(a_1^{\Delta}-a_3\right)=0\\ \sigma^{\nabla}\left(a_1^{\Delta}-a_3-a_2\right)=a_4^{\nabla}\\ \sigma^{\nabla}\left(a_3^{\Delta}+a_4\right)=0\end{cases} \tag{34}$$

We take

$$\xi_0=0,\xi_1=t,\xi_2=0,\xi_3=1,\xi_4=0,\boldsymbol{X}^{(1)}=t\frac{\partial}{\partial a_1}+\frac{\partial}{\partial a_3}+\frac{\partial}{\partial a_1^{\Delta}} \tag{35}$$

The generator (35) satisfies determination equation (18), so the generator (35) corresponds to Lie symmetry in the system (5).

Form the structural equation (19), we can get

$$\left\{\left(a_2+a_3\right)a_1^{\Delta}+a_4a_3^{\Delta}-\frac{1}{2}\left[\left(a_3\right)^2+2a_2a_3-\left(a_4\right)^2\right]\right\}\xi_0^{\Delta}$$
$$+\xi_2a_1^{\Delta}-\xi_2a_3+\xi_3\left(a_1^{\Delta}-a_2-a_3\right)+\xi_4\left(a_3^{\Delta}+a_4\right)$$
$$+\left(\xi_1^{\Delta}-a_1^{\Delta}\xi_0^{\Delta}\right)\left(a_2+a_3\right)+\left(\xi_3^{\Delta}-a_3^{\Delta}\xi_0^{\Delta}\right)a_4+G^{\Delta}=0 \tag{36}$$

Substituting (35) into (36), we have $G=-a_1$.

By Theorem 1, we can get conserved quantity

$$I=\sigma(t)\left(a_2+a_3\right)+a_4-a_1^{\sigma}=\text{const} \tag{37}$$

Let $\mathbb{T}=\left\{2^n:n\in\mathbb{N}\cup\{0\}\right\}$, we have $\sigma(t)=2t,\mu(t)=\sigma(t)-t=t$, then the conserved quantity

$$I=2t\left(a_2+a_3\right)+a_4-a_1-ta_1^{\Delta}=\text{const.} \tag{38}$$

If the initial condition are $a_1(1)=1$, $a_2(1)=1,a_2(2)=2,a_3(1)=-2,a_4(1)=1$, let $\mathbb{T}=\left\{2^n:n\in\mathbb{N}\cup\{0\}\right\}$ the trajectory of motion a_1,a_2,a_3,a_4 and the conserved quantity I are calculated, the results are shown in Fig. 1.

Let $\mathbb{T}=\mathbb{R}$, we have $\sigma(t)=t,\mu(t)=\sigma(t)-t=0$, then the conserved quantity

$$I=t\left(a_2+a_3\right)+a_4-a_1=\text{const} \tag{39}$$

Example 2 The nonshifted generalized Birkhoffian and Birkhoff's functions on time scales are

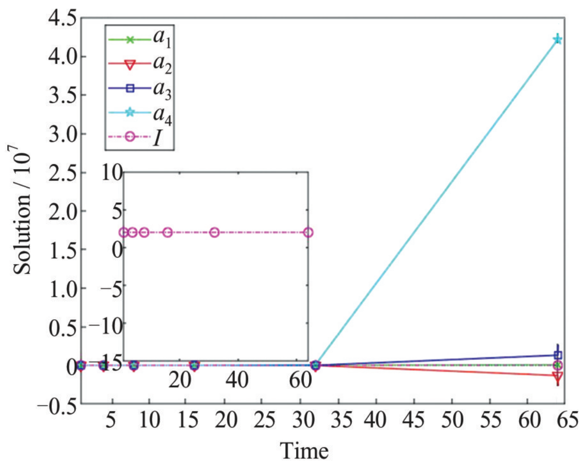

Fig. 1 Simulations of a_1, a_2, a_3, a_4, I on time scale $\mathbb{T}=\{2^n: n\in\mathbb{N}\cup\{0\}\}$

$$\begin{cases} B=\frac{1}{2}\left(a_3\right)^2+\frac{1}{2}\left(a_4\right)^2 \\ R_1=0, R_2=0, R_3=a_1, R_4=a_2 \\ \Lambda_1=-t, \Lambda_2=-t, \Lambda_3=a_3, \Lambda_4=a_4 \end{cases} \tag{40}$$

By the nonshifted generalized Birkhoff's equation (7), we have

$$a_3^{\Delta}=t, a_4^{\Delta}=t, a_1^{\nabla}=a_2^{\nabla}=0 \tag{41}$$

We take

$$\xi_0=0, \xi_1=0, \xi_2=0, \xi_3=t, \xi_4=t, \boldsymbol{X}^{(1)}=t\frac{\partial}{\partial a_3}+t\frac{\partial}{\partial a_4} \tag{42}$$

The generator (42) satisfies determination equation (24), so the generator (42) corresponds to Lie symmetry in this system (7).

According to the structural equation (25), we can get

$$\begin{aligned} &\left(a_1a_3^{\Delta}+a_2a_4^{\Delta}-B\right)\xi_0^{\Delta}+\left(ta_1^{\Delta}+ta_2^{\Delta}-a_3a_3^{\Delta}-a_4a_4^{\Delta}\right)\xi_0 \\ &+\left(a_3^{\Delta}-t\right)\xi_1+\left(a_4^{\Delta}-t\right)\xi_2+G^{\Delta}=0 \end{aligned} \tag{43}$$

Substituting (42) into (43), we have $G=0$.

By Theorem 2, we can get conserved quantity

$$I=\sigma(t)\left(a_1+a_2\right)=\text{const.} \tag{44}$$

Let $\mathbb{T}=\{3^n: n\in\mathbb{N}\cup\{0\}\}$, we get $\sigma(t)=3t, \mu(t)=\sigma(t)-t=2t$, then the conserved quantity

$$I=3t\left(a_1+a_2\right)=\text{const.} \tag{45}$$

Example 3 On time scales, let $\mathbb{T}=\{2^n: n\in\mathbb{N}\cup\{0\}\}$, the nonshifted constrained Birkhoffian and Birkhoff's functions are

$$B=\frac{1}{2}\left[\left(a_1\right)^2+\left(a_3\right)^2+\left(a_4\right)^2\right], R_1=a_3, R_2=a_4, R_3=R_4=0 \tag{46}$$

The constraints equations are

$$f_1=a_1a_3-\left(c_1\right)^2=0, f_2=a_1+a_4-c_2=0 \tag{47}$$

By Eq. (13), we can get

$$\begin{cases} a_3^{\nabla}=2\left(-\lambda_1a_3-\lambda_2-a_1\right) \\ a_4^{\nabla}=0 \\ a_1^{\Delta}-a_3=\lambda_1a_1 \\ a_2^{\Delta}-a_4=\lambda_2 \end{cases} \tag{48}$$

From Eqs. (47) and (48), we have

$$\lambda_1=-\frac{a_3}{a_1}, \lambda_2=-a_1+\frac{\left(a_3\right)^2}{a_1} \tag{49}$$

Hence, we have

$$P_1=-a_1, P_2=0, P_3=-a_3, P_4=-a_1+\frac{\left(a_3\right)^2}{a_1} \tag{50}$$

Let

$$\xi_0=0, \xi_1=1, \xi_2=0, \xi_3=0, \boldsymbol{X}^{(1)}=\frac{\partial}{\partial a_1} \tag{51}$$

The generator (51) satisfies determination equation (30), so the generator (51) corresponds to Lie symmetry in this system (13).

According to the structural equation (31), we can get

$$\begin{aligned} &\left\{a_3a_1^{\Delta}+a_4a_2^{\Delta}-\frac{1}{2}\left[\left(a_1\right)^2+\left(a_2\right)^2+\left(a_3\right)^2+\left(a_4\right)^2\right]\right\}\xi_0^{\Delta} \\ &-\xi_1a_1-\xi_2a_2+\xi_3\left(a_1^{\Delta}-a_3\right) \\ &+\xi_4\left(a_2^{\Delta}-a_4\right)+a_1\left(\xi_1-a_1^{\Delta}\xi_0\right)+a_3\left(\xi_3-a_3^{\Delta}\xi_0\right) \\ &+\left(a_1-\frac{\left(a_3\right)^2}{a_1}\right)\left(\xi_4-a_4^{\Delta}\xi_0\right)+G^{\Delta}=0 \end{aligned} \tag{52}$$

Substituting (51) into (52), we have $G=0$.

By Theorem 3, we can get conserved quantity

$$I=a_3=\text{const.} \tag{53}$$

4 Conclusion

Time-scale theory has been generally used in many fields. When $\mathbb{T}=\mathbb{R}$, the dynamic equation on time scale can be degraded into a continuous differential equation. When $\mathbb{T}=\mathbb{Z}$, the dynamic equation on time scale can be degraded into discrete difference equation. This paper mainly studies the time-scale with nonshifted variational problem which Lie theorem for three kinds of Birkhoffian systems. In this paper, nonshifted generalized Birkhoff's equations and nonshifted constrained Birkhoff's equations are derived by using principle (6). By introducing infinitesimal transformation, the structural equation of Lie symmetry is derived, and then the Noether conserved quantities caused by Lie symmetries is deduced. Since the study about nonshifted variational problem on a time scale is just beginning, the research of paper has great theoretical and practical significance.

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