Wuhan University Journal of Natural Sciences

2022, Vol.27 No.4, 273-280

Article ID 1007-1202(2022)04-0273-08

DOI https://doi.org/10.1051/wujns/2022274273

Differentiability of Functions on Spheres and Criterions of Convexity

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Abstract: Some basic concepts for functions defined on subsets of the unit sphere, such as the s-directional derivative, s-gradient and s-Gateaux and s-Frechet differentiability etc, are introduced and investigated. These concepts are different from the usual ones for functions defined on subsets of Euclidean spaces, however, the results obtained here are very similar. Then, as applications, we provide some criterions of s-convexity for functions defined on unit spheres which are improvements or refinements of some known results.

Key words: spherical convexity; spherically convex function; Gateaux and Frechet differentiability; criterion of convexity

CLC number: O 18

Received date: 2022-03-10

Foundation item: Supported by the National Natural Science Foundation of China (12071334, 11671293)

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0 Introduction

In the past two decades, the convexity theory on spherical spaces, emerging almost at the same time as that on Euclidean (linear) spaces and developing relatively slow in the last century (Refs.[1-7]), has attracted much attention in various mathematics areas, such as analysis, geometry and optimization theory etc. (Refs.[8-14]). Encouragingly, some efforts have been made to establish a systematic theory, parallel to that on Euclidean spaces, of convexity on spherical spaces (for details see Refs.[15-17] and the references therein).

Although some progresses have been made in the study on spherical convexity, the task to establish a systematic theory is far away from being completed, simply because many counterparts of concepts and definitions for convex sets in Euclidean spaces have not been found for spherical convexity, due to the lack of suitable compositions and operators on spheres. So, in this paper, we make an effort to study the so-called spherical differentiability of functions defined on spheres and the criterions of spherically convex functions.

The paper is organized as follows. In Section 1, we recall some notations, definitions and basic properties about spherically convex sets and spherically convex functions which will be used throughout the paper. In Section 2, we introduce the spherical Gateaux and spherical Frechet differentiability of functions defined merely on spheres. These differentiabilities are proper extensions of those defined in Ref. [15] where the authors have to assume that the functions are defined on some suitable Euclidean open sets. Section 3 is devoted to studying the criterions of spherical convexity of functions defined on spheres. The results obtained here generalize those in Ref.[15] etc. and will play some roles in the further study.

1 Preliminaries

$\mathbf{R}^n$, $\mathbf{S}^{n-1}$ denote the Euclidean n-space and the unit sphere in $\mathbf{R}^n$, respectively. As usual, $<\cdot, \cdot>$, $\|\cdot\|$ denote the standard inner product and the norm induced by $<\cdot, \cdot>$ on $\mathbf{R}^n$, respectively. Often we also view $\mathbf{R}^n$ as an affine space, so we will not distinguish vectors and points intentionally. The origin (or zero vector) of $\mathbf{R}^n$ is always denoted by the letter o.

A set of the form $\mathbf{S}_V := V \cap \mathbf{S}^{n-1}$, where V is a $(k+1)$-dimensional subspace of $\mathbf{R}^n$ $(0 \leqslant k \leqslant n-1)$, is called a k-sphere. If $V = <x>$ is a 1-dimensional subspace generated by a nonzero $x \in \mathbf{R}^n$, we write $\mathbf{S}_x$ and $\mathbf{S}_{x^\perp}$ simply instead of $\mathbf{S}_{<x>}$ and $\mathbf{S}_{<x>^\perp}$, respectively, where $<x>^\perp$ denotes the orthogonal complementary space of $<x>$. $\mathbf{S}_x$ is a 0-sphere and $\mathbf{S}_{x^\perp}$ is an $(n-2)$-sphere for each nonzero x. In geometric language, u and $-u$ in $\mathbf{S}^{n-1}$ are called (a pair of) antipodes. So a 0-sphere consists of a pair of antipodes. For the s-convexity of sets or functions on $\mathbf{S}^{n-1}$, there are several equivalent definitions, among which the one given in Refs.[16, 17] is the only analytic form. So, we follow the approach adopted in Refs.[16, 17].

The spherical addition, denoted by "$+_s$", in $\mathbf{R}^n$ (see Refs. [16, 18]) is defined by $x +_s y := \rho(x+y)$, $x, y \in \mathbf{R}^n$, where $\rho: \mathbf{R}^n \to \mathbf{S}^{n-1} \cup \{o\}$, called the radial projection, is defined by

$$\rho(x) := \begin{cases} \dfrac{x}{\|x\|}, x \neq o \\ o, x = o \end{cases}$$

which is clear of the properties:

i) $\rho \circ \rho = \rho$;

ii) $\rho(tx) = \rho(x), \rho(-x) = -\rho(x)$ for $x \in \mathbf{R}^n$ and $t > 0$;

iii) $\rho(x) = x$ if and only if $x \in \mathbf{S}^{n-1}$ or $x = o$.

The spherical addition is communicative but not associative, so the following composition is introduced in Refs.[16, 18]: for $x_1, x_2, \cdots, x_k \in \mathbf{R}^n$ $(k \geqslant 2)$, define

$$(s)\sum_{i=1}^{k} x_i := \rho\left(\sum_{i=1}^{k} x_i\right)$$

Naturally, when $k = 2$, we write $x +_s y$ instead of $(s)(x+y)$. In terms of the spherical addition, the so-called spherically convex combination (s-convex combination for brevity) of $x_1, x_2, \cdots, x_k \in \mathbf{R}^n$ and non-negative $\lambda_1, \lambda_2, \cdots, \lambda_k$ with $\sum_{i=1}^{k} \lambda_i = 1$ is defined as

$$(s)\sum_{i=1}^{k} \lambda_i x_i \ \left(= \rho\left(\sum_{i=1}^{k} \lambda_i x_i\right)\right).$$

Now, we introduce the definition of spherically convex sets given in Ref.[16] (for sets containing no antipodes) and in Ref.[17] (for general cases).

Definition 1 A subset $C \subset \mathbf{S}^{n-1}$ is called spherically convex (s-convex for brevity) if

$$\lambda u_1 +_s (1-\lambda) u_2 \in C$$

whenever $u_1, u_2 \in C$, $\lambda \in [0, 1]$ with $\lambda u_1 +_s (1-\lambda) u_2 \neq 0$.

If further C contains no antipodes, then C is called a proper s-convex set.

Remark 1 It is easy to check that all k-spheres are s-convex$(0 \leqslant k \leqslant n-1)$, and it was shown in Ref.[17] that if $C \neq \mathbf{S}^{n-1}$ is s-convex, then C is contained in some closed hemisphere.

Denote, for $u_1, u_2 \in \mathbf{S}^{n-1}$,

$$[u_1, u_2]_s := \{\lambda u_1 +_s (1-\lambda) u_2 \mid 0 \leqslant \lambda \leqslant 1\}$$

which is a subset of $\mathbf{S}^{n-1} \cup \{o\}$. When $u_1 \neq u_2$, $[u_1, u_2]_s$ is called the short arc connecting u_1 and u_2 (it can be checked easily that the short arc defined here coincides with the usual one defined in geometric languages), and $[u, -u]_s = \{o, u, -u\}$ for $u \in \mathbf{S}^{n-1}$. For $u_1 \neq u_2$, $(u_1, u_2)_s$, $(u_1, u_2]_s$ and $[u_1, u_2)_s$ are defined in a similar manner. From these notations, we see that the definition here is equivalent to the popular one adopted by other authors recently (see Refs.[16, 17] for the precise proof).

For $u \in \mathbf{S}^{n-1}$ and $v \in \mathbf{S}_u^\perp$, denote

$$[u, -u)_s(v) := [u, v]_s \cup [v, -u)_s$$
$$(u, -u)_s(v) := (u, v]_s \cup [v, -u)_s$$
$$[u, -u]_s(v) := [u, v]_s \cup [v, -u]_s$$

which are semicircles of various types passing through v.

Next, we recall the concept of spherically convex functions. In fact, before the "spherical convex combination" composition is introduced, it was not as simple as one may think to define spherically convex functions. Ferreira *et al* in Ref.[15] proposed a definition in terms of minimal geodesic segment (function): if $C \subset \mathbf{S}^{n-1}$ is an s-convex set and $f: C \to \mathbf{R}$ is a function, then f is called spherically convex if the function $f \circ \gamma: [a, b] \to \mathbf{R}$ is a univariate convex function for each minimal geodesic segment (function) $\gamma: [a, b] \to C$ (see Ref.[15] for more information). This definition is quite intuitive but not very convenient in application. Here, we adopt the one given in Refs. [16, 17] which is equivalent to Ferreira's in Ref.[15].

Definition 2 Let $C\subset \mathbf{S}^{n-1}$ be an s-convex set. A function $f: C\to \mathbf{R}$ is called spherically convex (s-convex for brevity) if

$$f(\lambda u_1 +_s (1-\lambda)u_2)\leqslant \lambda f(u_1)+(1-\lambda)f(u_2)$$

holds for $u_1,u_2\in C$, $\lambda\in[0,1]$ with $\lambda u_1 +_s(1-\lambda)u_2\in C$.

For an s-convex function on C, it was shown in Ref. [19] that it is continuous with respect to the both metrics mentioned below, and also that restricted on each k-sphere contained in $C(0\leqslant k\leqslant n-1)$, it is a constant, in particular, $f(u)=f(-u)$ if $u,-u\in C$.

The intrinsic metric $d_s(\cdot,\cdot)$ in $\mathbf{S}^{n-1}$, defined by $d_s(u_1,u_2):=\arccos\langle\cdot,\cdot\rangle$ for $u_1,u_2\in\mathbf{S}^{n-1}$, will be used in studying the continuity and the differentiability of functions on $\mathbf{S}^{n-1}$. An elementary geometric argument shows $\| u_1-u_2 \| =2\sin\dfrac{d_s(u_1,u_2)}{2}$, so the intrinsic metric and Euclidean metric are equivalent on $\mathbf{S}^{n-1}$.

For given $u\in\mathbf{S}^{n-1}$ and $\delta>0$, the set

$$B_s(u,\delta):=\{w\in\mathbf{S}^{n-1}\,|\,d_s(u,w)<\delta\}$$

is called an s-ball. A set $S\subset\mathbf{S}^{n-1}$ is called s-open if for each $u\in S$, there is $\delta>0$ such that $B_s(u,\delta)\subset S$. It is easy to see that S is s-open if and only if there is an open set $\Omega\subset\mathbf{R}^n$ such that $S=\Omega\cap\mathbf{S}^{n-1}$.

For $u_1,u_2\in\mathbf{S}^{n-1}$ with $u_1\neq u_2$, to describe the points in $[u_1,u_2]_s$, there are two popular geodesic segment functions connecting u_1,u_2 considered in Refs. [16, 17, 19] and Ref.[15], respectively:

$$u_\lambda=\gamma_{u_1,u_2}(\lambda):=\lambda u_1 +_s(1-\lambda)u_2,\ \lambda\in[0,1] \text{ and}$$

$$u_\theta^*=\gamma_{u_1,u_2}^*(\theta):=\frac{\sin(\alpha-\theta)}{\sin\alpha}u_1 +_s \frac{\sin\theta}{\sin\alpha}u_2,\ \theta\in[0,\alpha]$$

where $\alpha=d_s(u_1,u_2)$. It can be easily checked that $\theta=d_s(u_\theta^*,u_1)$ (see Ref.[15] for more information).

The following conclusion, cited as a lemma here (a proof is also included for convenience), was confirmed in Ref.[19].

Lemma 1 Let $u_1,u_2\in\mathbf{S}^{n-1}$ with $u_2\neq\pm u_1$. Then $u_\theta^*=u_\lambda$ if and only if

$$\lambda=\frac{\sin(\alpha-\theta)}{\sin\theta+\sin(\alpha-\theta)}$$

(or equivalently, $1-\lambda=\dfrac{\sin\theta}{\sin\theta+\sin(\alpha-\theta)}$).

Consequently,

$$\frac{\sin(\alpha-\theta)}{\sin\theta+\sin(\alpha-\theta)}u_1 +_s \frac{\sin\theta}{\sin\theta+\sin(\alpha-\theta)}u_2 = \frac{\sin(\alpha-\theta)}{\sin\alpha}u_1+\frac{\sin\theta}{\sin\alpha}u_2$$

for all $\theta\in[0,\alpha]$.

Proof Clearly, $u_\theta^*=u_\lambda$ iff $d_s(u_\lambda,u_1)=d_s(u_\theta^*,u_1)(=\theta)$. For brevity, denote $\bar u_\lambda=\lambda u_1+(1-\lambda)u_2$. By an elementary geometric argument, we see that $d_s(u_\lambda,u_1)=\theta$ if and only if (noticing $\| u_1-u_2 \| =2\sin\dfrac{\alpha}{2}$),

$$\| u_2-\bar u_\lambda \| = \begin{cases} \| u_2-\bar u_{1/2} \| + \| \bar u_\lambda-\bar u_{1/2} \| \\ =\sin\dfrac{\alpha}{2}+\cos\dfrac{\alpha}{2}\tan(\dfrac{\alpha}{2}-\theta),\ \theta\in[0,\dfrac{\alpha}{2}] \\ \| u_2-\bar u_{1/2} \| - \| \bar u_\lambda-\bar u_{1/2} \| \\ =\sin\dfrac{\alpha}{2}-\cos\dfrac{\alpha}{2}\tan(\theta-\dfrac{\alpha}{2}),\ \theta\in[\dfrac{\alpha}{2},\alpha] \end{cases}$$

$$=\sin\frac{\alpha}{2}-\cos\frac{\alpha}{2}\tan(\theta-\frac{\alpha}{2})$$

and in turn if and only if

$$\lambda=\frac{\| u_2-\bar u_\lambda \|}{\| u-v \|}=\frac{\sin\frac{\alpha}{2}-\cos\frac{\alpha}{2}\tan(\theta-\frac{\alpha}{2})}{2\sin\frac{\alpha}{2}}$$

$$=\frac{\sin\frac{\alpha}{2}\cos(\theta-\frac{\alpha}{2})-\cos\frac{\alpha}{2}\sin(\theta-\frac{\alpha}{2})}{2\sin\frac{\alpha}{2}\cos(\theta-\frac{\alpha}{2})}$$

$$=\frac{\sin(\alpha-\theta)}{2\sin\frac{\alpha}{2}(\cos\frac{\alpha}{2}\cos\theta+\sin\frac{\alpha}{2}\sin\theta)}$$

$$=\frac{\sin(\alpha-\theta)}{\sin\alpha\cos\theta+2\sin^2\frac{\alpha}{2}\sin\theta}$$

$$=\frac{\sin(\alpha-\theta)}{\sin\alpha\cos\theta+\sin\theta-\cos\alpha\sin\theta}$$

$$=\frac{\sin(\alpha-\theta)}{\sin\theta+\sin(\alpha-\theta)}$$

Finally, another elementary geometric argument (drawing a picture to check) shows that

$$\| \bar u_{\lambda(\theta)} \| = \frac{\cos\frac{\alpha}{2}}{\cos(\frac{\alpha}{2}-\theta)}=\frac{\sin\alpha}{2\sin\frac{\alpha}{2}\cos(\frac{\alpha}{2}-\theta)}$$

$$=\frac{\sin\alpha}{\sin\theta+\sin(\alpha-\theta)}$$

Therefore,

$$\frac{\sin(\alpha-\theta)}{\sin\theta+\sin(\alpha-\theta)}u_1 +_s \frac{\sin\theta}{\sin\theta+\sin(\alpha-\theta)}u_2$$

$$=u_\lambda=\frac{\bar u_\lambda}{\|\bar u_\lambda\|}=\frac{\sin(\alpha-\theta)}{\sin\alpha}u_1+\frac{\sin\theta}{\sin\alpha}u_2$$

2 Differentiability of Functions on Sphere

Ferreira *et al* in Ref. [15] defined and studied the spherical Frechet differentiability only for the functions differentiable on an open set containing an s-convex set. More precisely, they simply took the orthogonal projection of gradients of the functions (to a suitable subspace) as the spherical gradients. Clearly, such an approach limits the applications of their results. In this section, we will define and study the Gateaux differentiability and Frechet differentiability of functions defined merely on a subset of $\mathbf{S}^{n-1}$.

First, we give the concepts of s-directional derivative and s-Gateaux differentiability.

Definition 3 Let $B_s(u_0,\delta)\subset\mathbf{S}^{n-1}$ and f: $B_s(u_0,\delta)\to\mathbf{R}$ be a function. If for some $v\in\mathbf{S}_{u_0}^{\perp}$, the limit

$$\lim_{\lambda\to 0^+}\frac{f(u_\lambda)-f(u_0)}{\sin d_s(u_\lambda,u_0)}=:D_s f(u_0,v)$$

exists, where $u_\lambda:=\lambda v+_s(1-\lambda)u_0$, then $D_s f(u_0,v)$ is called the s-directional derivative of f at u_0 along (direction) v.

If both $D_s f(u_0,v)$ and $D_s f(u_0,-v)$ exist and $D_s f(u_0,-v)=-D_s f(u_0,v)$, then $D_s f(u_0,v)$ is called the partial derivative of f at u_0 along (direction) v, denoted by $\frac{\partial f(u_0)}{\partial_s v}$.

If $D_s f(u_0,v)$ exists for all $v\in\mathbf{S}_{u_0}^{\perp}$, then f is called s-Gateaux differentiable (s-G-differentiable for brevity) at u_0.

Remark 2 i) $\sin d_s(u_\lambda,u_0)$ in Definition 1 can be replaced by $d_s(u_\lambda,u_0)$ since $\lim_{\lambda\to 0^+}\frac{\sin d_s(u_\lambda,u_0)}{d_s(u_\lambda,u_0)}=1$.

ii) Naturally, for each nonzero $w\in<u_0>^{\perp}$, one may define as well $D_s f(u_0,w)=\lim_{\lambda\to 0^+}\frac{f(u_\lambda)-f(u_0)}{\sin d_s(u_\lambda,u_0)}$(if exists), where $u_\lambda:=\lambda w+_s(1-\lambda)u_0$. However, we point out that such an extended definition is not essentially necessary: writing $w=tv$ for some $v\in\mathbf{S}_{u_0}^{\perp}$ and $t>0$, we have by the property of ρ,

$$\lambda w+_s(1-\lambda)u_0=\rho(\lambda tv+(1-\lambda)u_0)$$
$$=\rho\left(\frac{\lambda t}{\lambda(t-1)+1}v+\frac{1-\lambda}{\lambda(t-1)+1}u_0\right)=\lambda w+_s(1-\lambda)u_0$$

where $\mu=\frac{\lambda t}{\lambda(t-1)+1}$, and in turn $\lambda\to 0^+$ iff $\mu\to 0^+$.

Thus, $D_s f(u_0,w)$ exists iff $D_s f(u_0,v)$ exists, and in such a case $D_s f(u_0,w)=D_s f(u_0,v)$.

Consider the function f: $\mathbf{R}^n\to\mathbf{R}$ defined by $f(x):=d_s f(u_0,x)$ $(x\in\mathbf{S}^{n-1})$ or 0 (otherwise), where $u_0\in\mathbf{S}^{n-1}$ is fixed. It is easy to check that, for each $v\in\mathbf{S}_{u_0}^{\perp}$, the (usual) directional derivative

$$Df(u_0,v):=\lim_{\lambda\to 0^+}\frac{f(u_0+\lambda v)}{\lambda}=0$$

while the s-directional derivative $D_s f(u_0,v)=1$. Also, in a same manner, one may construct a function f such that $D_s f(u_0,v)$ (resp. $Df(u_0,v)$) exists, but $Df(u_0,v)$ (resp. $D_s f(u_0,v)$) does not. Therefore, the s-directional derivative and the (usual) directional derivative (along direction $v\in\mathbf{S}_{u_0}^{\perp}$) have nothing to do with each other in general. However, we have the following conclusion.

Proposition 1 Let $u_0\in\mathbf{S}^{n-1}$ and Ω be an open set containing u_0. If a function f: $\Omega\to\mathbf{R}$ is differentiable at u_0, then for each $v\in\mathbf{S}_{u_0}^{\perp}$, $D_s f(u_0,v)$ exists and

$$D_s f(u_0,v)=Df(u_0,v).$$

Proof We have $f(x)-f(u_0)=<\mathrm{grad}f(u_0),x-u_0>+o(\|x-u_0\|)$ for $x\in\Omega$ since f is differentiable at u_0, where grad$f$ denotes the usual gradient of f.

Thus, to calculate $D_s f(u_0,v)$ for $v\in\mathbf{S}_{u_0}^{\perp}$, we compute $\lim_{\lambda\to 0^+}\frac{u_\lambda-u_0}{\sin d_s(u_\lambda,u_0)}$ first. Denoting $\theta:=d_s(u_\lambda,u_0)$ (observing that $\theta\to 0^+$ iff $\lambda\to 0^+$), we have

$$u_\lambda=\sin\theta\cdot v+\cos\theta\cdot u_0$$

and in turn

$$\lim_{\lambda\to 0^+}\frac{u_\lambda-u_0}{\sin d_s(u_\lambda,u_0)}=\lim_{\theta\to 0^+}\frac{\sin\theta\cdot v+(\cos\theta-1)\cdot u_0}{\sin\theta}=v.$$

Therefore, we obtain

$$D_s f(u_0,v)=\lim_{\lambda\to 0^+}\frac{f(u_\lambda)-f(u_0)}{\sin d_s(u_\lambda,u_0)}$$
$$=\lim_{\lambda\to 0^+}\frac{<\mathrm{grad}f(u_0),u_\lambda-u_0>+o(\|u_\lambda-u_0\|)}{\sin d_s(u_\lambda,u_0)}$$
$$=<\mathrm{grad}f(u_0),\lim_{\lambda\to 0^+}\frac{u_\lambda-u_0}{\sin d_s(u_\lambda,u_0)}>+\lim_{\lambda\to 0^+}\frac{o(\|u_\lambda-u_0\|)}{\sin d_s(u_\lambda,u_0)}$$
$$=<\mathrm{grad}f(u_0),v>=D_s f(u_0,v)$$

where we used the fact $o(\|u_\lambda-u_0\|)=o(\sin d_s(u_\lambda,u_0))$ since $\lim_{\lambda\to 0}\frac{\sin d_s(u_\lambda,u_0)}{\|u_\lambda-u_0\|}=\lim_{\lambda\to 0}\frac{\sin d_s(u_\lambda,u_0)}{2\sin^2\frac{d_s(u_\lambda,u_0)}{2}}=1$. The proof completes.

Next, we define the s-gradient of functions defined on $\mathbf{S}^{n-1}$.

Definition 4 Let $B_s(u_0,\delta)\subset\mathbf{S}^{n-1}$ and f: $B_s(u_0,\delta)\to\mathbf{R}$

be a function s-G-differentiable at u_0. If $v^* \in \mathbf{S}_{u_0}^{\perp}$ satisfies $D_s f(u_0, v^*)=\max\{D_s f(u_0, v) \mid v \in \mathbf{S}_{u_0}^{\perp}\}$, then $\left| D_s f(u_0, v^*) \right| v^*$ is called an s-subgradient of f at u_0. If the s-subgradient is unique, then it is called the s-gradient of f at u_0, denoted by $\operatorname{grad}_s f(u_0)$.

Examples are easily found to show that, for a function defined in an open neighborhood of $u_0 \in \mathbf{S}^{n-1}$, its s-gradient and its usual gradient have nothing to do with each other in general. However, we have the following proposition.

Proposition 2 Let $u_0 \in \mathbf{S}^{n-1}$ and Ω be an open neighborhood of u_0. If a function $f: \Omega \to \mathbf{R}$ is differentiable at u_0, then $\operatorname{grad}_s f(u_0)$ exists and

$$\operatorname{grad}_s f(u_0) = P_{u_0^{\perp}}(\operatorname{grad} f(u_0))$$

where $P_{u_0^{\perp}}$ denotes the orthogonal projection from $\mathbf{R}^n$ to $<u_0>^{\perp}$. In turn, $D_s f(u_0, v) = <\operatorname{grad}_s f(u_0), v>$ for $v \in \mathbf{S}_{u_0}^{\perp}$.

Proof Writing

$$\operatorname{grad} f(u_0) = P_{u_0}(\operatorname{grad} f(u_0)) + P_{u_0^{\perp}}(\operatorname{grad} f(u_0)),$$

we have, as shown in the proof of Proposition 1, for each $v \in \mathbf{S}_{u_0}^{\perp}$,

$$\begin{aligned} D_s f(u_0, v) &= <\operatorname{grad} f(u_0), v> \\ &= <P_{u_0}(\operatorname{grad} f(u_0)) + P_{u_0^{\perp}}(\operatorname{grad} f(u_0)), v> \\ &= <P_{u_0^{\perp}}(\operatorname{grad} f(u_0)), v> \end{aligned}$$

since $v \in <u_0>^{\perp}$, which implies clearly $\operatorname{grad}_s f(u_0)$ exists and $\operatorname{grad}_s f(u_0) = P_{u_0^{\perp}}(\operatorname{grad} f(u_0))$, and in turn

$$D_s f(u_0, v) = <\operatorname{grad}_s f(u_0), v>$$

for all $v \in \mathbf{S}_{u_0}^{\perp}$.

Ferreira *et al* in Ref.[15] took $P_{u_0}(\operatorname{grad} f)$ as the definition of s-gradient directly. Obviously, such a definition may make sense only for functions satisfying the conditions as in Proposition 2. To illustrate our definition as a useful proper extension of Ferreira's, we introduce the following concept.

Definition 5 Let $u_0 \in \mathbf{S}^{n-1}$ and $f: U \to \mathbf{R}$ be a function, where U is an s-open set containing u_0. If there is $v^{\circ} \in <u_0>^{\perp}$, such that

$$f(u) - f(u_0) = <v^{\circ}, \sin d_s(u, u_0)> + o(\sin d_s(u, u_0))$$

whenever $v \in \mathbf{S}_{u_0}^{\perp}$ and $u \in [u_0, -u_0)_s(v) \cap U$, then f is called s-Frechet differentiable (s-F-differentiable for brevity or s-differentiable simply) at u_0, and $f_s'(u_0) := v^{\circ}$ is called the s-F-derivative of f at u_0.

Remark 3 It is easy to check that if f is differentiable at u_0, then f is s-differentiable at u_0. However, the inverse is not true even if the function is defined in an open neighborhood of u_0, as shown by the following example: consider the function $f: \mathbf{R}^n \to \mathbf{R}$ defined by

$$f(x) := d_s^2(u_0, x)\ (x \in \mathbf{S}^{n-1}) \text{ or } 1 \text{ (otherwise)},$$

where $u_0 \in \mathbf{S}^{n-1}$ is fixed. It is easy to see that f is s-differentiable but not differentiable at u_0. Also, it is easy to check that the s-differentiability implies the s-G-differentiability (however, the inverse is not true). More precisely, we have the following improvement of Proposition 2.

Theorem 1 If a function f is defined on an s-open set U containing $u_0 \in \mathbf{S}^{n-1}$ and s-differentiable at u_0, then $\operatorname{grad}_s f(u_0) = f_s'(u_0)$ and

$$D_s f(u_0, v) = <\operatorname{grad}_s f(u_0), v> \text{ for all } v \in \mathbf{S}_{u_0}^{\perp}.$$

Proof Since

$$f(u_\lambda) - f(u_0) = <f_s'(u_0), \sin d_s(u_\lambda, u_0) \cdot v> + o(\sin d_s(u_\lambda, u_0))$$

whenever $v \in \mathbf{S}_{u_0}^{\perp}$ and $u_\lambda := \lambda v +_s (1-\lambda) u_0 \in U$, we have

$$\begin{aligned} D_s f(u_0, v) &= \lim_{\lambda \to 0^+} \frac{f(u_\lambda) - f(u_0)}{\sin d_s(u_\lambda, u_0)} \\ &= <f_s'(u_0), v> + \lim_{\lambda \to 0^+} \frac{o(\sin d_s(u_\lambda, u_0))}{\sin d_s(u_\lambda, u_0)} \\ &= <f_s'(u_0), v> \end{aligned}$$

from which the conclusions follow.

The following proposition provides the expressions of s-gradients for s-differentiable functions, which has its own significance clearly even if it will not be used in this paper.

Proposition 3 If a function f is defined on an s-open U containing $u_0 \in \mathbf{S}^{n-1}$ and s-differentiable at u_0, then for arbitrary pairly orthogonal $e_1, e_2, \cdots, e_{n-1} \in \mathbf{S}_{u_0}^{\perp}$ (that is, $e_1, e_2, \cdots, e_{n-1}$ form a standard orthogonal basis of $<u_0>^{\perp}$), we have

$$\operatorname{grad}_s f(u_0) = \frac{\partial f(u_0)}{\partial_s e_1} e_1 + \frac{\partial f(u_0)}{\partial_s e_2} e_2 + \cdots + \frac{\partial f(u_0)}{\partial_s e_{n-1}} e_{n-1}.$$

Proof Since f is s-differentiable at u_0, it is easy to check that $\frac{\partial f(u_0)}{\partial_s v}$ exists and $\frac{\partial f(u_0)}{\partial_s v} = <f_s'(u_0), v>$ for each $v \in \mathbf{S}_{u_0}^{\perp}$, in particular, $\frac{\partial f(u_0)}{\partial_s e_i} = <f_s'(u_0), e_i>$. So,

$$\begin{aligned} D_s f(u_0, v) &= \frac{\partial f(u_0)}{\partial_s v} = <f_s'(u_0), v> = <f_s'(u_0), \sum_{i=1}^{n-1} \alpha_i e_i> \\ &= \sum_{i=1}^{n-1} \alpha_i <f_s'(u_0), e_i> = \sum_{i=1}^{n-1} <e_i, v> \frac{\partial f(u_0)}{\partial_s e_i} \\ &= <\sum_{i=1}^{n-1} \frac{\partial f(u_0)}{\partial_s e_i} e_i, v> \end{aligned}$$

which implies clearly $\operatorname{grad}_s f(u_0) = \sum_{i=1}^{n-1} \frac{\partial f(u_0)}{\partial_s e_i} e_i$.

3 Criterions of s-Convex Functions

In this section, we discuss the criterions of s-convexity for functions defined on $\mathbf{S}^{n-1}$ in terms of s-gradients. The results obtained here are improvement of several criterions of s-convexity in Ref. [15] since the functions considered there have to be differentiable on some open set.

The first one is an improvement of Proposition 7 in Ref.[15].

Theorem 2 Let $C\subset\mathbf{S}^{n-1}$ be an s-open s-convex set and $f\colon C\to\mathbf{R}$ be s-differentiable. Then f is s-convex on C if and only if, for any $u_0\in C$ and $v\in\mathbf{S}_{u_0}^{\perp}$,

$$f(u)\geqslant f(u_0)+<\mathrm{grad}_s f(u_0),\sin d_s(u,u_0)\cdot v>$$

holds for any $u\in C\cap[u_0,-u_0)_s(v)$.

Proof ($\Rightarrow$) Suppose f is s-convex on C. By the s-differentiability of f, for any $u_0\in C$ and $v\in\mathbf{S}_{u_0}^{\perp}$, we have

$$f(u)-f(u_0)=<\mathrm{grad}_s f(u_0),\sin d_s(u,u_0)\cdot v>+o(\sin d_s(u,u_0))$$

whenever $u\in C\cap[u_0,-u_0)_s(v)$.

Denote $u_\lambda:=\lambda u+_s(1-\lambda)u_0$ for $u\in C\cap[u_0,-u_0)_s(v)$ and $0<\lambda<1$. Then $u_\lambda\in C\cap[u_0,-u_0)_s(v)$ clearly, so by the s-convexity of f, we have

$$f(u_\lambda)\leqslant\lambda f(u)+(1-\lambda)f(u_0)$$

which leads to

$$\lambda(f(u)-f(u_0))\geqslant f(u_\lambda)-f(u_0)$$
$$=<\mathrm{grad}_s f(u_0),\sin d_s(u_\lambda,u_0)\cdot v>+o(\sin d_s(u_\lambda,u_0))$$

that is

$$f(u)-f(u_0)=<\mathrm{grad}_s f(u_0),\lambda^{-1}\sin d_s(u_\lambda,u_0)\cdot v>+\lambda^{-1}o(\sin d_s(u_\lambda,u_0)) \quad (1)$$

Since $\lambda=\sin d_s(u_\lambda,u_0)(\sin d_s(u_\lambda,u_0)+\sin d_s(u,u_\lambda))^{-1}$ by Lemma 1, we have

$$\lim_{\lambda\to0^+}\lambda^{-1}\sin d_s(u_\lambda,u_0)=\sin d_s(u,u_0);$$
$$\lim_{\lambda\to0^+}\lambda^{-1}o(\sin d_s(u_\lambda,u_0))=0.$$

Thus, letting $\lambda\to0^+$ in (1), we obtain

$$f(u)\geqslant f(u_0)+<\mathrm{grad}_s f(u_0),\sin d_s(u,u_0)\cdot v>.$$

($\Leftarrow$) Suppose for any $u_0\in C$ and $v\in\mathbf{S}_{u_0}^{\perp}$,

$$f(u)\geqslant f(u_0)+<\mathrm{grad}_s f(u_0),\sin d_s(u,u_0)\cdot v>$$

hold for all $u\in C\cap[u_0,-u_0)_s(v)$.

If C contains antipodes, then $C=\mathbf{S}^{n-1}$ since C is s-open (one may find the argument easily). Thus, for any $u,v\in\mathbf{S}^{n-1}$ with $<u,v>=0$, i.e. $v\in\mathbf{S}_u^{\perp}$ and $u\in\mathbf{S}_v^{\perp}$, we have

$$f(v)-f(u)\geqslant<\mathrm{grad}_s f(u),\sin d_s(v,u)\cdot v>$$
$$=<\mathrm{grad}_s f(u),v>,$$
$$f(-v)-f(u)\geqslant<\mathrm{grad}_s f(u),\sin d_s(-v,u)\cdot(-v)>$$
$$=-<\mathrm{grad}_s f(u),v>$$

since $v\in C\cap[u,-u)_s(v)$ and $-v\in C\cap[u,-u)_s(-v)$. Therefore

$$f(v)\leqslant f(u)\ \text{iff}\ f(u)\leqslant f(-v) \quad (2)$$

With the same arguments on the pairs $(-v,u),(-u,-v)$ and $(v,-u)$, respectively, we have as well

$$f(u)\leqslant f(-v)\ \text{iff}\ f(-v)\leqslant f(-u)$$
$$f(-v)\leqslant f(-u)\ \text{iff}\ f(-u)\leqslant f(v) \quad (3)$$
$$f(-u)\leqslant f(v)\ \text{iff}\ f(v)\leqslant f(u)$$

Clearly, (2) and (3) lead to $f(u)=f(-u)=f(v)=f(-v)$. By the arbitrariness of u and v, f is a constant on $\mathbf{S}^{n-1}$ (noticing that any two circles have intersection) and so is convex.

If C contains no antipodes, for any distinct $u_1,u_2\in C$ and $0<\lambda<1$ (nothing to prove for the case $u_1=u_2$), denote $u_\lambda:=\lambda u_1+_s(1-\lambda)u_2$, we have

$$f(u_1)\geqslant f(u_\lambda)+<\mathrm{grad}_s f(u_\lambda),\sin d_s(u_1,u_\lambda)\cdot v>,$$
$$f(u_2)\geqslant f(u_\lambda)+<\mathrm{grad}_s f(u_\lambda),\sin d_s(u_2,u_\lambda)\cdot(-v)>$$

where (unique) $v\in\mathbf{S}_{u_\lambda}^{\perp}$ such that $u_1\in C\cap[u_\lambda,-u_\lambda)_s(v)$ (naturally, $u_2\in C\cap[u_\lambda,-u_\lambda)_s(-v)$). Thus

$$\lambda f(u_1)+(1-\lambda)f(u_2)$$
$$\geqslant\lambda(f(u_\lambda)+<\mathrm{grad}_s f(u_\lambda),\sin d_s(u_1,u_\lambda)\cdot v>)$$
$$+(1-\lambda)(f(u_\lambda)+<\mathrm{grad}_s f(u_\lambda),\sin d_s(u_2,u_\lambda)\cdot(-v)>)$$
$$=f(u_\lambda)+<\mathrm{grad}_s f(u_\lambda),(\lambda\sin d_s(u_1,u_\lambda)$$
$$-(1-\lambda)\sin d_s(u_2,u_\lambda))v>=f(u_\lambda)$$

since

$$\lambda=\frac{\sin d_s(u_2,u_\lambda)}{\sin d_s(u_1,u_\lambda)+\sin d_s(u_2,u_\lambda)},$$
$$1-\lambda=\frac{\sin d_s(u_1,u_\lambda)}{\sin d_s(u_1,u_\lambda)+\sin d_s(u_2,u_\lambda)}.$$

by Lemma 1 and so

$$\lambda\sin d_s(u_1,u_\lambda)-(1-\lambda)\sin d_s(u_2,u_\lambda)=0.$$

To give another criterion of s-convexity, we introduce the following concept.

Definition 6 Let $C\subset\mathbf{S}^{n-1}$ be an s-open s-convex set and $F\colon C\to\mathbf{R}^n$ be a map. Then, F is called s-monotone on C if for any $u_1,u_2\in C$ with $u_2\neq\pm u_1$,

$$<F(u_1),v_1>+<F(u_2),v_2>\leqslant0$$

holds, where $v_1\in\mathbf{S}_{u_1}^{\perp},v_2\in\mathbf{S}_{u_2}^{\perp}$ such that

$$u_2\in C\cap[u_1,-u_1)_s(v_1)\ \text{and}\ u_1\in C\cap[u_2,-u_2)_s(v_2).$$

In terms of the s-monotonicity we have the following improvement of Proposition 8 in Ref.[15].

Theorem 3 Let $C\subset\mathbf{S}^{n-1}$ be an s-open s-convex set and $f\colon C\to\mathbf{R}^n$ be s-differentiable. Then f is s-convex if and only if $\mathrm{grad}_s f$ is s-monotone on C.

Proof (⇒) Suppose f is s-convex on C. Then for any $u_1, u_2 \in C$ with $u_2 \neq \pm u_1$ and $v_1 \in \mathbf{S}^{\perp}_{u_1}$, $v_2 \in \mathbf{S}^{\perp}_{u_2}$ with $u_2 \in [u_1, -u_1)_s(v_1)$, $u_1 \in [u_2, -u_2)_s(v_2)$, we have by Theorem 2

$$f(u_2)-f(u_1) \geqslant <\mathrm{grad}_s f(u_1), \sin d_s(u_2,u_1)\cdot v_1>,$$
$$f(u_1)-f(u_2) \geqslant <\mathrm{grad}_s f(u_2), \sin d_s(u_1,u_2)\cdot v_2>$$

which leads clearly to

$$<\mathrm{grad}_s f(u_1), \sin d_s(u_2,u_1)\cdot v_1> + <\mathrm{grad}_s f(u_2), \sin d_s(u_1,u_2)\cdot v_2> \leqslant 0,$$

and in turn

$$<\mathrm{grad}_s f(u_1), v_1> + <\mathrm{grad}_s f(u_2), v_2> \leqslant 0,$$

since $\sin d_s(u_2,u_1) > 0$ when $u_2 \neq \pm u_1$, i. e. $\mathrm{grad}_s f$ is s-monotone on C.

(⇐) Conversely, suppose $\mathrm{grad}_s f$ is s-monotone on C. If C contains antipodes, then, as explained in the proof of Theorem 2, $C = \mathbf{S}^{n-1}$. Thus, for any $u \in \mathbf{S}^{n-1}$ and $v \in \mathbf{S}^{\perp}_u$, we have

$$<\mathrm{grad}_s f(u), v> + <\mathrm{grad}_s f(v), u> \leqslant 0,$$
$$<\mathrm{grad}_s f(-u), v> + <\mathrm{grad}_s f(v), -u> \leqslant 0 \qquad (4)$$

which leads to $<\mathrm{grad}_s f(u) + \mathrm{grad}_s f(-u), v> \leqslant 0$. Hence, since $\mathrm{grad}_s f(u) + \mathrm{grad}_s f(-u) \in <u>^{\perp}$ and $v \in \mathbf{S}^{\perp}_u \subset <u>^{\perp}$ is arbitrary, we obtain

$$\mathrm{grad}_s f(-u) = -\mathrm{grad}_s f(u).$$

Thus, by the s-differentiability of f, for $w \in (u, -u)_s(v)$, we have

$$f(w)-f(u) = <\mathrm{grad}_s f(u), \sin d_s(w,u)\cdot v> + o(\sin d_s(w,u)) \qquad (5)$$

$$\begin{aligned} & f(w)-f(-u) \\ &= <\mathrm{grad}_s f(-u), \sin d_s(w,-u)\cdot v> + o(\sin d_s(w,-u)) \\ &= <-\mathrm{grad}_s f(u), \sin d_s(w,u)\cdot v> + o(\sin d_s(w,u)) \end{aligned} \qquad (6)$$

where we used the fact that $\mathrm{grad}_s f(-u) = -\mathrm{grad}_s f(u)$ and $\sin d_s(w,-u) = \sin d_s(w,u)$. Now, (6) subtracted from (5) gives

$$f(-u)-f(u) = 2<\mathrm{grad}_s f(u), \sin d_s(w,u)\cdot v> + o(\sin d_s(w,u))$$

which implies $f(-u)=f(u)$ (letting $w \to u$), and in turn, for any $w \in (u, -u)_s(v)$,

$$2<\mathrm{grad}_s f(u), \sin d_s(w,u)\cdot v> + o(\sin d_s(w,u)) = 0.$$

Dividing both sides by $\sin d_s(w,u)$ and then letting $w \to u$, we have

$$<\mathrm{grad}_s f(u), v> = 0$$

By the arbitrariness of $v \in \mathbf{S}^{\perp}_u$, we have $\mathrm{grad}_s f(u) = o$. Hence $\mathrm{grad}_s f \equiv o$ on $\mathbf{S}^{n-1}$ by the arbitrariness of u. Now, for any distinct $u_1, u_2 \in \mathbf{S}^{n-1}$ with $u_2 \neq -u_1$, denoting $\varphi(\lambda) := f(u_\lambda)$ where $u_\lambda := \lambda u_2 +_s (1-\lambda)u_1$, $0 \leqslant \lambda \leqslant 1$, we have

$$\frac{\mathrm{d}}{\mathrm{d}\lambda}\varphi(\lambda) = <\mathrm{grad}_s f(u_\lambda), \frac{\mathrm{d}u_\lambda}{\mathrm{d}\lambda}> = 0$$

Therefore φ is a constant on [0, 1], in particular, $f(u_1) = \varphi(0) = \varphi(1) = f(u_2)$. By the arbitrariness of u_1 and u_2, f is a constant and in turn s-convex on $\mathbf{S}^{n-1}$.

If C contains no antipodes, then for any distinct $u_1, u_2 \in C$ and $0 < \lambda < 1$, denote $u_\lambda := \lambda u_2 +_s (1-\lambda)u_1$, $\alpha := d_s(u_1,u_2)$, $\theta := d_s(u_\lambda, u_1)$ for brevity first.

Then, define $\varphi(\lambda) := f(u_\lambda) = f(\lambda u_2 +_s (1-\lambda)u_1)$, $0 \leqslant \lambda \leqslant 1$, and $\lambda = \lambda(\theta) = \dfrac{\sin\theta}{\sin\theta + \sin(\alpha-\theta)}$, $0 \leqslant \theta \leqslant \alpha$. Thus, by Lemma 1,

$$\varphi(\lambda(\theta)) = f\left(\frac{\sin(\alpha-\theta)}{\sin\alpha}u_1 + \frac{\sin\theta}{\sin\alpha}u_2\right), 0 \leqslant \theta \leqslant \alpha$$

Therefore,

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}\theta}\varphi(\lambda(\theta)) &= <\mathrm{grad}_s f(u_{\lambda(\theta)}), \frac{\mathrm{d}}{\mathrm{d}\theta}\left(\frac{\sin(\alpha-\theta)}{\sin\alpha}u_1 + \frac{\sin\theta}{\sin\alpha}u_2\right)> \\ &= <\mathrm{grad}_s f(u_{\lambda(\theta)}), \frac{-\cos(\alpha-\theta)}{\sin\alpha}u_1 + \frac{\cos\theta}{\sin\alpha}u_2> \end{aligned}$$

Let (unique) $v_1 \in \mathbf{S}^{\perp}_{u_{\lambda(\theta)}}$ such that $u_2 \in [u_{\lambda(\theta)}, -u_{\lambda(\theta)})_s(v_1)$, then clearly,

$$P_{u^{\perp}_{\lambda(\theta)}}(u_1) = \sin\theta\cdot(-v_1) \text{ and } P_{u^{\perp}_{\lambda(\theta)}}(u_2) = \sin(\alpha-\theta)\cdot v_1$$

where $P_{u^{\perp}_{\lambda(\theta)}}$ denotes the orthogonal projection from $\mathbf{R}^n$ to $<u_{\lambda(\theta)}>^{\perp}$. Observing $\mathrm{grad}_s f(u_{\lambda(\theta)}) \in <u_{\lambda(\theta)}>^{\perp}$ which implies $<\mathrm{grad}_s f(u_{\lambda(\theta)}), u_i> = <\mathrm{grad}_s f(u_{\lambda(\theta)}), P_{u^{\perp}_{\lambda(\theta)}}(u_i)>$, $i = 1, 2$, we obtain

$$\begin{aligned} & \frac{\mathrm{d}}{\mathrm{d}\theta}\varphi(\lambda(\theta)) \\ &= <\mathrm{grad}_s f(u_{\lambda(\theta)}), \frac{-\cos(\alpha-\theta)}{\sin\alpha}P_{u^{\perp}_{\lambda(\theta)}}(u_1) + \frac{\cos\theta}{\sin\alpha}P_{u^{\perp}_{\lambda(\theta)}}(u_2)> \\ &= <\mathrm{grad}_s f(u_{\lambda(\theta)}), \frac{-\cos(\alpha-\theta)}{\sin\alpha}\sin\theta\cdot(-v_1) \\ &\quad + \frac{\cos\theta}{\sin\alpha}\sin(\alpha-\theta)\cdot v_1> \\ &= <\mathrm{grad}_s f(u_{\lambda(\theta)}), v_1> \end{aligned}$$

Now, for any $0 \leqslant \theta_1 < \theta_2 \leqslant \alpha$ (observing

$$u_{\lambda(\theta_1)} \in C \cap [u_1, u_{\lambda(\theta_2)})_s, \; u_{\lambda(\theta_2)} \in C \cap [u_2, u_{\lambda(\theta_1)})_s$$

in such a case), choosing $v_1 \in \mathbf{S}^{\perp}_{u_{\lambda(\theta_1)}}$ such that $u_2 \in [u_{\lambda(\theta_1)}, -u_{\lambda(\theta_1)})_s(v_1)$ and $v_2 \in \mathbf{S}^{\perp}_{u_{\lambda(\theta_2)}}$ such that $u_1 \in [u_{\lambda(\theta_2)}, -u_{\lambda(\theta_2)})_s(v_2)$, we have, by the monotonicity of $\mathrm{grad}_s f$,

$$\begin{aligned} & \frac{\mathrm{d}}{\mathrm{d}\theta}\varphi(\lambda(\theta_1)) - \frac{\mathrm{d}}{\mathrm{d}\theta}\varphi(\lambda(\theta_2)) \\ &= <\mathrm{grad}_s f(u_{\lambda(\theta_1)}), v_1> - <\mathrm{grad}_s f(u_{\lambda(\theta_2)}), -v_2> \\ &= <\mathrm{grad}_s f(u_{\lambda(\theta_1)}), v_1> + <\mathrm{grad}_s f(u_{\lambda(\theta_2)}), v_2> \leqslant 0 \end{aligned}$$

where $\dfrac{\mathrm{d}}{\mathrm{d}\theta}\varphi(\lambda(\theta_2)) = <\mathrm{grad}_s f(u_{\lambda(\theta_2)}), -v_2>$ simply because $u_{\lambda(\theta)} \in [u_{\lambda(\theta_2)}, -u_{\lambda(\theta_2)})_s(-v_2)$ when $\theta \geqslant \theta_2$.

So, $\frac{\mathrm{d}}{\mathrm{d}\theta}\varphi(\lambda(\theta))$ is increasing and in turn $\varphi(\lambda(\theta))$ is convex on $[0,\alpha]$, in particular,

$$\begin{aligned} f(\frac{1}{2}u_1 +_s \frac{1}{2}u_2) &= \varphi(\lambda(\frac{\alpha}{2})) = \varphi(\lambda(\frac{1}{2}0 + \frac{1}{2}\alpha)) \\ &\leqslant \frac{1}{2}\varphi(\lambda(0)) + \frac{1}{2}\varphi(\lambda(\alpha)) \\ &= \frac{1}{2}f(u_1) + \frac{1}{2}f(u_2) \end{aligned}$$

which, together with the arbitrariness of u_1 and u_2, implies clearly that $\varphi(\cdot)$ is middle-point convex (i. e. $\varphi(\frac{\lambda_1+\lambda_2}{2}) \leqslant \frac{1}{2}\varphi(\lambda_1) + \frac{1}{2}\varphi(\lambda_2)$) and in turn convex since it is clearly continuous on $[0,1]$. Thus, for any $0 \leqslant \lambda \leqslant 1$, we have

$$\begin{aligned} &f(\lambda u_2 +_s (1-\lambda)u_1) = \varphi(\lambda) = \varphi((1-\lambda)\cdot 0 + \lambda \cdot 1) \\ &\leqslant (1-\lambda)\varphi(0) + \lambda\varphi(1) = (1-\lambda)f(u_1) + \lambda f(u_2) \end{aligned}$$

By the arbitrariness of u_1 and u_2, f is s-convex.

Final Remark In this paper, we introduce first the concepts of s-directional derivative, s-gradient and s-Gateaux differentiability and s-Frechet differentiability of functions defined on subsets of the unit sphere, which are different from the usual ones for functions defined on subsets of Euclidean spaces. Then, as applications, we provide some criterions of s-convexity for functions on spheres which are improvements of results in Ref. [15]. Also, compared with the work in Ref. [15], our work has some advantages in applications since the functions considered here are defined merely on subsets of spheres.

Acknowledgements: The authors thank the referees very much for their carefully reading the first version of the manuscript and pointing out typos and mistakes.

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