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Some Inequalities about the General L_p-Mixed Width-Integral of Convex Bodies

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Abstract: The Brunn-Minkowski type and the cyclic Brunn-Minkowski type inequalities for the i-th general L_p-mixed width-integral of convex bodies are established. Further, two cyclic inequalities for the differences of i-th general L_p-mixed width-integral of convex bodies are obtained.

Key words: Brunn-Minkowski type inequality; cyclic inequality; general L_p-mixed width-integral

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0 Introduction and Main Results

The setting for this paper is n-dimensional Euclidean spaces $\mathbb{R}^n$. Let $\mathcal{K}^n$ denote the set of convex bodies (compact, convex subsets with non-empty interiors) in Euclidean space $\mathbb{R}^n$. $\mathcal{K}_o^n$ denotes the set of convex bodies containing the origin in their interiors. $V(K)$ denotes the n-dimensional volume of a body K, B the standard unit ball, and $V(B)=\omega_n$. Let S^{n-1} denote the unit sphere in $\mathbb{R}^n$.

Blaschke[1] considered the classical width-integral of convex bodies first and Hadwiger[2] studied it further. In 1975 Lutwak[3] introduced the i-th width-integral of convex bodies. In 1977, Lutwak[4] generalized the i-th width-integral to the mixed width-integral of convex bodies. In 2016, Feng[5] gave the definitions of mixed width-integral and the general i-th width-integral of convex bodies. In 2017, Zhou[6] defined the general L_p-mixed width-integral of convex bodies. For the more results of the mixed width-integral of convex bodies, we refer the interested reader to Refs. [7-14].

In this paper, we first establish the Brunn-Minkowski type inequality for the i-th general L_p-mixed width-integral of convex bodies.

Theorem 1 Let $M, N, K, L\in\mathcal{K}_o^n$, $\tau\in[-1,1]$, $p>0, M$ and N have similar general L_p-width, for $K\subset M$, $L\subset N$, $n-p<i<n$ or $M\subset K$, $N\subset L$, $i>n$, we have

$$\left[B_{p,i}^{(\tau)}(M+_pN)-B_{p,i}^{(\tau)}(K+_pL)\right]^{\frac{p}{n-i}} \leqslant \left[B_{p,i}^{(\tau)}(M)-B_{p,i}^{(\tau)}(K)\right]^{\frac{p}{n-i}}+\left[B_{p,i}^{(\tau)}(N)-B_{p,i}^{(\tau)}(L)\right]^{\frac{p}{n-i}} \quad (1)$$

for $K\subset M, L\subset N$, $0\leqslant i<n-p$, we have

$$\left[B_{p,i}^{(\tau)}(M+_p N)-B_{p,i}^{(\tau)}(K+_p L)\right]^{\frac{p}{n-i}} \geqslant \left[B_{p,i}^{(\tau)}(M)-B_{p,i}^{(\tau)}(K)\right]^{\frac{p}{n-i}}+\left[B_{p,i}^{(\tau)}(N)-B_{p,i}^{(\tau)}(L)\right]^{\frac{p}{n-i}} \tag{2}$$

the equality holds in (1) or (2) if and only if K and L have similar general L_p-width, and $(B_{p,i}^{(\tau)}(M), B_{p,i}^{(\tau)}(K))= c(B_{p,i}^{(\tau)}(N), B_{p,i}^{(\tau)}(L))$, where c is a constant.

We also establish two cyclic inequalities for the differences of i-th general L_p-mixed width-integral of convex bodies.

Theorem 2 Let M, $K\in \mathcal{K}_o^n$, $K\subset M$, $\tau \in[-1, 1]$, $p>0$, M has constant general L_p-width, for $0\leqslant i<j<k\leqslant n$, $i,j,k \in \mathbb{R}$, we get

$$\left[B_{p,j}^{(\tau)}(M)-B_{p,j}^{(\tau)}(K)\right]^{k-i} \geqslant \left[B_{p,i}^{(\tau)}(M)-B_{p,i}^{(\tau)}(K)\right]^{k-j}\left[B_{p,k}^{(\tau)}(M)-B_{p,k}^{(\tau)}(K)\right]^{j-i} \tag{3}$$

with equality if and only if K has constant general L_p-width.

Theorem 3 Let $M,N,K,L\in \mathcal{K}_o^n$, $K\subset M$, $L\subset N$, $\tau \in[-1, 1]$, $p>0$, M and N have similar general L_p-width, for $0\leqslant i< j<k\leqslant n$, $i,j,k \in \mathbb{R}$, we can obtain

$$\left[B_{p,j}^{(\tau)}(M,N)-B_{p,j}^{(\tau)}(K,L)\right]^{k-i} \geqslant \left[B_{p,i}^{(\tau)}(M,N)-B_{p,i}^{(\tau)}(K,L)\right]^{k-j}\left[B_{p,k}^{(\tau)}(M,N)-B_{p,k}^{(\tau)}(K,L)\right]^{j-i} \tag{4}$$

with equality if and only if $(b_p^{(\tau)}(M,u), b_p^{(\tau)}(K,u))=c(b_p^{(\tau)}(N,u), b_p^{(\tau)}(L,u))$, where c is a constant, and K and L have similar general L_p -width.

Meanwhile, we establish a cyclic Brunn-Minkowski inequality for the i-th general L_p-mixed width-integral of convex bodies.

Theorem 4 Let $K, L\in \mathcal{K}_o^n$, $p>0$, $\tau \in[-1, 1]$, if $j<n-p$ and $i<j<k$, we can obtain

$$B_{p,j}^{(\tau)}(K+_p L)^{\frac{p}{n-j}} \leqslant B_{p,i}^{(\tau)}(K)^{\frac{p(k-j)}{(k-i)(n-j)}} B_{p,k}^{(\tau)}(K)^{\frac{p(j-i)}{(k-i)(n-j)}} + B_{p,i}^{(\tau)}(L)^{\frac{p(k-j)}{(k-i)(n-j)}} B_{p,k}^{(\tau)}(L)^{\frac{p(j-i)}{(k-i)(n-j)}} \tag{5}$$

with equality if and only if K and L both have constant general L_p-width. If $n-p<j<n$ and $j\leqslant i<k$, or $j>n$ and $i\leqslant j<k$, the inequality is reversed.

1 Preliminaries

1.1 Support Function and Firey L_p-Combination

If $K \in \mathcal{K}^n$, the support function, $h_K=h(K, \cdot)$: $\mathbb{R}^n \to (-\infty, \infty)$, is defined by[15,16]

$$h(K,x)=\max\{x\cdot y: y\in K\}, x\in \mathbb{R}^n$$

where $x \cdot y$ denotes the standard inner product of x and y.

For $K,L \in \mathcal{K}_o^n$, $p\geqslant 1$ and λ, $\mu\geqslant 0$ (not both zero), the Firey L_p-combination $\lambda\cdot K+_p\mu\cdot L \in K_o^n$ of K and L is defined by [17]

$$h(\lambda\cdot K+_p\mu\cdot L, \cdot)^p=\lambda h(K, \cdot)^p+\mu h(L, \cdot)^p \tag{6}$$

where the operation "$+_p$" is called Firey addition and $\lambda\cdot K$ denotes the Firey scalar multiplication.

1.2 General L_p-Mixed Width-Integral of Order i

For $\tau \in[-1, 1]$ and $p>0$, the general L_p-mixed width-integral $B_p^{(\tau)}(K_1, \cdots, K_n)$ of $K_1, \cdots, K_n \in \mathcal{K}_o^n$ is defined by [6]

$$B_p^{(\tau)}(K_1, \cdots, K_n)=\frac{1}{n}\int_{S^{n-1}} b_p^{(\tau)}(K_1,u)\cdots b_p^{(\tau)}(K_n,u)\mathrm{d}S(u) \tag{7}$$

where $b_p^{(\tau)}(K,u)=\left(f_1(\tau)h^p(K,u)+f_2(\tau)h^p(K,-u)\right)^{\frac{1}{p}}$ for any $u\in S^{n-1}$, and $f_1(\tau), f_2(\tau)$ are chosen as follows:

$$f_1(\tau)=\frac{(1+\tau)^{2p}}{(1+\tau)^{2p}+(1-\tau)^{2p}}, f_2(\tau)=\frac{(1-\tau)^{2p}}{(1+\tau)^{2p}+(1-\tau)^{2p}}$$

$f_1(\tau)$ and $f_2(\tau)$ satisfy

$$f_1(\tau)+f_2(\tau)=1; f_1(-\tau)=f_2(\tau); f_2(-\tau)=f_1(\tau).$$

K and L are said to have similar general L_p-width if there exists a constant $\lambda>0$ such that $b_p^{(\tau)}(K,u)=\lambda b_p^{(\tau)}(L,u)$ for all $u\in S^{n-1}$. If $b_p^{(\tau)}(K,u)=b_p^{(\tau)}(L,u)$ for all $u\in S^{n-1}$, then we call K and L have the same general L_p-width. If $b_p^{(\tau)}(K,u)$ is a constant for all $u\in S^{n-1}$, we call K has the constant general L_p-width.

Taking $K_1=\cdots=K_{n-i}=K$ and $K_{n-i+1}=\cdots=K_n=L$ in (7), the general L_p-mixed width-integral $B_{p,i}^{(\tau)}(K,L)$ of $K,L\in\mathcal{K}_o^n$ is given by

$$B_{p,i}^{(\tau)}(K,L)=\frac{1}{n}\int_{S^{n-1}}b_p^{(\tau)}(K,u)^{n-i}b_p^{(\tau)}(L,u)^i\mathrm{d}S(u) \tag{8}$$

Further, let $L=B$ in (8), since $b_p^{(\tau)}(B,u)=1$, and write $B_{p,i}^{(\tau)}(K)$ for $B_{p,i}^{(\tau)}(K,B)$, we get

$$B_{p,i}^{(\tau)}(K)=\frac{1}{n}\int_{S^{n-1}}b_p^{(\tau)}(K,u)^{n-i}\mathrm{d}S(u) \tag{9}$$

where $B_{p,i}^{(\tau)}(K)$ is called the i-th general L_p-mixed width-integral of $K\in\mathcal{K}_o^n$. If $i=0$, we write $B_{p,0}^{(\tau)}(K)=B_p^{(\tau)}(K)$, $B_p^{(\tau)}(K)$ is called the general L_p-width-integral of $K\in\mathcal{K}_o^n$.

2 Proofs of Theorems

In this section, we give the proofs of the Theorems 1-4. The proof of Theorem 1 requires the following lemmas.

Lemma 1[6] If $K,L\in\mathcal{K}_o^n$, $\tau\in[-1,1]$ and $p>0$, for $i\leqslant n-p\leqslant j\leqslant n$ and $i\neq j$, then

$$\left(\frac{B_{p,i}^{(\tau)}(K+_pL)}{B_{p,j}^{(\tau)}(K+_pL)}\right)^{\frac{p}{j-i}}\leqslant\left(\frac{B_{p,i}^{(\tau)}(K)}{B_{p,j}^{(\tau)}(K)}\right)^{\frac{p}{j-i}}+\left(\frac{B_{p,i}^{(\tau)}(L)}{B_{p,j}^{(\tau)}(L)}\right)^{\frac{p}{j-i}} \tag{10}$$

for $j\geqslant n\geqslant i\geqslant n-p$ and $i\neq j$, then

$$\left(\frac{B_{p,i}^{(\tau)}(K+_pL)}{B_{p,j}^{(\tau)}(K+_pL)}\right)^{\frac{p}{j-i}}\geqslant\left(\frac{B_{p,i}^{(\tau)}(K)}{B_{p,j}^{(\tau)}(K)}\right)^{\frac{p}{j-i}}+\left(\frac{B_{p,i}^{(\tau)}(L)}{B_{p,j}^{(\tau)}(L)}\right)^{\frac{p}{j-i}} \tag{11}$$

with equality in every inequality if and only if K and L have similar general L_p-width.

Lemma 2[18,19] Let $x=(x_1,\cdots,x_n)$ and $y=(y_1,\cdots,y_n)$ be two series of non-negative real numbers, and $x_1{}^p\geqslant\sum_{i=2}^n x_i^p$, $y_1{}^p\geqslant\sum_{i=2}^n y_i^p$, for $p>1$, then

$$\left(x_1{}^p-\sum_{i=2}^n x_i^p\right)^{\frac{1}{p}}+\left(y_1{}^p-\sum_{i=2}^n y_i^p\right)^{\frac{1}{p}}\leqslant\left((x_1+y_1)^p-\sum_{i=2}^n(x_i+y_i)^p\right)^{\frac{1}{p}} \tag{12}$$

for $p<0$ or $0<p<1$,

$$\left(x_1{}^p-\sum_{i=2}^n x_i^p\right)^{\frac{1}{p}}+\left(y_1{}^p-\sum_{i=2}^n y_i^p\right)^{\frac{1}{p}}\geqslant\left((x_1+y_1)^p-\sum_{i=2}^n(x_i+y_i)^p\right)^{\frac{1}{p}} \tag{13}$$

with equality in every inequality if and only if x and y are proportional.

Proof of Theorem 1 Let $j=n$ in Lemma 1, since $B_{p,n}^{(\tau)}(K+_pL)=\omega_n$ is a constant, for $i\leqslant n-p$, we immediately obtain

$$B_{p,i}^{(\tau)}(K+_pL)^{\frac{p}{n-i}}\leqslant B_{p,i}^{(\tau)}(K)^{\frac{p}{n-i}}+B_{p,L}^{(\tau)}(L)^{\frac{p}{n-i}} \tag{14}$$

and for $n-p<i<n$,

$$B_{p,i}^{(\tau)}(K+_pL)^{\frac{p}{n-i}}\geqslant B_{p,i}^{(\tau)}(K)^{\frac{p}{n-i}}+B_{p,L}^{(\tau)}(L)^{\frac{p}{n-i}} \tag{15}$$

the equality holds if and only if K and L have similar general L_p-width, which is just Corollary 3.

Since $M,N,K,L\in\mathcal{K}_o^n$, M and N have similar general L_p-width, for $n-p<i<n$, we have

$$B_{p,i}^{(\tau)}(K+_pL)^{\frac{p}{n-i}}\geqslant B_{p,i}^{(\tau)}(K)^{\frac{p}{n-i}}+B_{p,i}^{(\tau)}(L)^{\frac{p}{n-i}} \tag{16}$$

$$B_{p,i}^{(\tau)}(M+_pN)^{\frac{p}{n-i}}=B_{p,i}^{(\tau)}(M)^{\frac{p}{n-i}}+B_{p,i}^{(\tau)}(N)^{\frac{p}{n-i}} \tag{17}$$

By the definition of the i-th general L_p-mixed width-integral of convex bodies, we know that $B_{p,i}^{(\tau)}(K)\leqslant B_{p,i}^{(\tau)}(M)$ and $B_{p,i}^{(\tau)}(L)\leqslant B_{p,i}^{(\tau)}(N)$ if $K\subset M$, $L\subset N$ for $n-p<i<n$; $B_{p,i}^{(\tau)}(M)>B_{p,i}^{(\tau)}(K)$ and $B_{p,i}^{(\tau)}(N)>B_{p,i}^{(\tau)}(L)$ if $M\subset K$, $N\subset L$ for $i>n$. According to (13), combining (16) with (17), we can obtain

$$\left[B_{p,i}^{(\tau)}(M+_pN)-B_{p,i}^{(\tau)}(K+_pL)\right]^{\frac{p}{n-i}}\leqslant\left[\left(B_{p,i}^{(\tau)}(M)^{\frac{p}{n-i}}+B_{p,i}^{(\tau)}(N)^{\frac{p}{n-i}}\right)^{\frac{n-i}{p}}-\left(B_{p,i}^{(\tau)}(K)^{\frac{p}{n-i}}+B_{p,i}^{(\tau)}(L)^{\frac{p}{n-i}}\right)^{\frac{n-i}{p}}\right]^{\frac{p}{n-i}}$$

$$\leqslant\left[B_{p,i}^{(\tau)}(M)-B_{p,i}^{(\tau)}(K)\right]^{\frac{p}{n-i}}+\left[B_{p,i}^{(\tau)}(N)-B_{p,i}^{(\tau)}(L)\right]^{\frac{p}{n-i}} \tag{18}$$

the equality holds if and only if $(B_{p,i}^{(\tau)}(M),B_{p,i}^{(\tau)}(K))$ is proportional to $(B_{p,i}^{(\tau)}(N),B_{p,i}^{(\tau)}(L))$, and K and L have similar general L_p-width. The proof of inequality (2) is similar. This proves the theorem.

The proof of Theorem 2-3 requires the following lemma.

Lemma 3[20] Suppose that f_i , g_i (i= 1,2) are non-negative continuous functions on S^{n-1} such that $\int_{S^{n-1}}f_1^p(\xi)\mathrm{d}\xi\geqslant\int_{S^{n-1}}f_2^p(\xi)\mathrm{d}\xi$ and $\int_{S^{n-1}}g_1^q(\xi)\mathrm{d}\xi\geqslant\int_{S^{n-1}}g_2^q(\xi)\mathrm{d}\xi$ for $p>1$, $\frac{1}{p}+\frac{1}{q}=1$, and for all $\xi\in S^{n-1}$, $f_1^p(\xi)=\lambda g_1^q(\xi)$ where λ is a constant, then

$$\left(\int_{S^{n-1}}\left(f_1^p(\xi)-f_2^p(\xi)\right)\mathrm{d}\xi\right)^{\frac{1}{p}}\left(\int_{S^{n-1}}\left(g_1^q(\xi)-g_2^q(\xi)\right)\mathrm{d}\xi\right)^{\frac{1}{q}}\leqslant\int_{S^{n-1}}\left(f_1(\xi)g_1(\xi)-f_2(\xi)g_2(\xi)\right)\mathrm{d}\xi \tag{19}$$

with equality if and only if $f_2^p(\xi)=\lambda g_2^q(\xi)$ for any $\xi\in S^{n-1}$.

Proof of Theorem 2 Suppose that $0\leqslant i<j<k\leqslant n$, let

$$f_1=\left(b_p^{(\tau)}(M,u)^{n-k}\right)^{\frac{1}{\lambda}},f_2=\left(b_p^{(\tau)}(K,u)^{n-k}\right)^{\frac{1}{\lambda}},\ g_1=\left(b_p^{(\tau)}(M,u)^{n-i}\right)^{\frac{1}{\mu}},g_2=\left(b_p^{(\tau)}(K,u)^{n-i}\right)^{\frac{1}{\mu}},\ \lambda=(k-i)/(j-i),\mu=(k-i)/(k-j)$$

in Lemma 3, since $K\subset M$, M has constant general L_p-width, $f_1^\lambda/g_1^\mu=b_p^{(\tau)}(M,u))^{i-k}$ is a constant, according to Lemma 3, we can get the inequality (3).

By Lemma 3, the equality in (3) holds if and only if $b_p^{(\tau)}(K,u))^{i-k}=b_p^{(\tau)}(M,u))^{i-k}$ for all $u\in S^{n-1}$, that means K has constant general L_p-width. This proves the theorem.

Proof of Theorem 3 We can prove Theorem 3 by Lemma 3 as well. Suppose that $0\leqslant i<j<k\leqslant n,\lambda=(k-i)/(j-i)$ and $\mu=(k-i)/(k-j)$, let

$$f_1=\left(b_p^{(\tau)}(M,u)^{n-k}b_p^{(\tau)}(N,u)^k\right)^{\frac{1}{\lambda}},f_2=\left(b_p^{(\tau)}(K,u)^{n-k}b_p^{(\tau)}(L,u)^k\right)^{\frac{1}{\lambda}},\ g_1=\left(b_p^{(\tau)}(M,u)^{n-i}b_p^{(\tau)}(N,u)^i\right)^{\frac{1}{\mu}},\ g_2=\left(b_p^{(\tau)}(K,u)^{n-i}b_p^{(\tau)}(L,u)^i\right)^{\frac{1}{\mu}}$$

in Lemma 3, since $K\subset M$, $L\subset N$, M and N have similar general L_p-width, $f_1^\lambda/g_1^\mu=(b_p^{(\tau)}(N,u)/b_p^{(\tau)}(M,u))^{k-i}$ is a constant. We get the consequence.

By Lemma 3, the equality in (4) holds if and only if $(b_p^{(\tau)}(L,u)/b_p^{(\tau)}(K,u))^{k-i}=(b_p^{(\tau)}(N,u)/b_p^{(\tau)}(M,u))^{k-i}$ for all $u\in S^{n-1}$, that means K and L have similar general L_p-width. This proves the theorem.

Taking $i=0, j=1, k=n$ in Theorem 3, we obtain

Corollary 1 Let $M,N,K,L\in\mathcal{K}_o^n$, $\tau\in[-1,1]$, $p>0$, M and N have similar general L_p-width, for $K\subset M$, $L\subset N$, one gets

$$\left[B_{p,1}^{(\tau)}(M,N)-B_{p,1}^{(\tau)}(K,L)\right]^n\geqslant\left[B_p^{(\tau)}(M)-B_p^{(\tau)}(K)\right]^{n-1}\left[B_p^{(\tau)}(N)-B_p^{(\tau)}(L)\right] \tag{20}$$

with equality if and only if $(b_p^{(\tau)}(M,u),b_p^{(\tau)}(K,u))=c(b_p^{(\tau)}(N,u),b_p^{(\tau)}(L,u))$, where c is a constant.

Proof of Theorem 4 If $j<n-p$, combined (6) with (9), according to the Minkowski's inequality[21], it follows that

$$B_{p,j}^{(\tau)}(K+_pL)=\frac{1}{n}\int_{S^{n-1}}b_p^{(\tau)}(K+_pL,u)^{n-j}\mathrm{d}S(u)=\frac{1}{n}\int_{S^{n-1}}\left(f_1(\tau)h^p(K+_pL,u)+f_2(\tau)h^p(K+_pL,-u)\right)^{\frac{n-j}{p}}\mathrm{d}S(u)$$

$$=\frac{1}{n}\int_{S^{n-1}}\left(f_1(\tau)h^p(K,u)+f_2(\tau)h^p(K,-u)+f_1(\tau)h^p(L,u)+f_2(\tau)h^p(L,-u)\right)^{\frac{n-j}{p}}\mathrm{d}S(u)$$

$$=\frac{1}{n}\int_{S^{n-1}}\left(b_p^{(\tau)}(K,u)^p+b_p^{(\tau)}(L,u)^p\right)^{\frac{n-j}{p}}\mathrm{d}S(u)$$

$$=\frac{1}{n}\int_{S^{n-1}}\left(b_p^{(\tau)}(K,u)^{\frac{p(k-j)(n-i)}{(k-i)(n-j)}}b_p^{(\tau)}(K,u)^{\frac{p(j-i)(n-k)}{(k-i)(n-j)}}+b_p^{(\tau)}(L,u)^{\frac{p(k-j)(n-i)}{(k-i)(n-j)}}b_p^{(\tau)}(L,u)^{\frac{p(j-i)(n-k)}{(k-i)(n-j)}}\right)^{\frac{n-j}{p}}\mathrm{d}S(u)$$

$$\leqslant\left[\left(\frac{1}{n}\int_{S^{n-1}}b_p^{(\tau)}(K,u)^{\frac{(k-j)(n-i)}{k-i}}b_p^{(\tau)}(K,u)^{\frac{(j-i)(n-k)}{k-i}}\mathrm{d}S(u)\right)^{\frac{p}{n-j}}+\left(\frac{1}{n}\int_{S^{n-1}}b_p^{(\tau)}(L,u)^{\frac{(k-j)(n-i)}{k-i}}b_p^{(\tau)}(L,u)^{\frac{(j-i)(n-k)}{k-i}}\mathrm{d}S(u)\right)^{\frac{p}{n-j}}\right]^{\frac{n-j}{p}} \quad (21)$$

Since $i<j<k$ means $\dfrac{k-i}{k-j}>1$, using Hölder's inequality [22] we have

$$\frac{1}{n}\int_{S^{n-1}}b_p^{(\tau)}(K,u)^{\frac{(k-j)(n-i)}{k-i}}b_p^{(\tau)}(K,u)^{\frac{(j-i)(n-k)}{k-i}}\mathrm{d}S(u)$$

$$\leqslant\left(\frac{1}{n}\int_{S^{n-1}}\left(b_p^{(\tau)}(K,u)^{\frac{(k-j)(n-i)}{k-i}}\right)^{\frac{k-i}{k-j}}\mathrm{d}S(u)\right)^{\frac{k-j}{k-i}}\times\left(\frac{1}{n}\int_{S^{n-1}}\left(b_p^{(\tau)}(K,u)^{\frac{(j-i)(n-k)}{k-i}}\right)^{\frac{k-i}{j-i}}\mathrm{d}S(u)\right)^{\frac{j-i}{k-i}}=B_{p,i}^{(\tau)}(K)^{\frac{k-j}{k-i}}B_{p,k}^{(\tau)}(K)^{\frac{j-i}{k-i}} \quad (22)$$

Hence, we can get the following inequality

$$\left[\frac{1}{n}\int_{S^{n-1}}b_p^{(\tau)}(K,u)^{\frac{(k-j)(n-i)}{k-i}}b_p^{(\tau)}(K,u)^{\frac{(j-i)(n-k)}{k-i}}\mathrm{d}S(u)\right]^{\frac{p}{n-j}}\leqslant B_{p,i}^{(\tau)}(K)^{\frac{p(k-j)}{(n-j)(k-i)}}B_{p,k}^{(\tau)}(K)^{\frac{p(j-i)}{(n-j)(k-i)}} \quad (23)$$

Similarly, we can also obtain

$$\left[\frac{1}{n}\int_{S^{n-1}}b_p^{(\tau)}(L,u)^{\frac{(k-j)(n-i)}{k-i}}b_p^{(\tau)}(L,u)^{\frac{(j-i)(n-k)}{k-i}}\mathrm{d}S(u)\right]^{\frac{p}{n-j}}\leqslant B_{p,i}^{(\tau)}(L)^{\frac{p(k-j)}{(n-j)(k-i)}}B_{p,k}^{(\tau)}(L)^{\frac{p(j-i)}{(n-j)(k-i)}} \quad (24)$$

By (21), (23) and (24), we get

$$B_{p,j}^{(\tau)}(K+_pL)^{\frac{p}{n-j}}\leqslant B_{p,i}^{(\tau)}(K)^{\frac{p(k-j)}{(k-i)(n-j)}}B_{p,k}^{(\tau)}(K)^{\frac{p(j-i)}{(k-i)(n-j)}}+B_{p,i}^{(\tau)}(L)^{\frac{p(k-j)}{(k-i)(n-j)}}B_{p,k}^{(\tau)}(L)^{\frac{p(j-i)}{(k-i)(n-j)}} \quad (25)$$

From the equality condition of the Minkowski's inequality, we see that the equality (21) holds if and only if K and L have similar general L_p-width. By the equality conditions of Hölder's inequality, equality holds in (22) if and only if K has constant general L_p-width. Similary, the equality holds in (24) if and only if L has constant general L_p-width. Thus, the equality holds in (5) or its reverse if and only if both K and L have constant general L_p-width.

In particular, take $L=\{o\}$ in Theorem 4. Since $K+\{o\}=K$, and notice that $B_{p,i}^{(\tau)}(\{o\})=0$, by inequality (5), we can obtain the following Corollary.

Corollary 2 Let $K\in\mathcal{K}_o^n$, $p>0$, $\tau\in[-1,1]$, for $i<j<k$, then

$$B_{p,j}^{(\tau)}(K)^{k-i}\leqslant B_{p,i}^{(\tau)}(K)^{k-j}B_{p,k}^{(\tau)}(K)^{j-i} \quad (26)$$

with equality if and only if K has constant general L_p-width.

Let $i=j$ in Theorem 4, we may obtain the following Brunn-Minkowski inequality for the i-th general L_p-mixed width-integral.

Corollary 3 For $K,L\in\mathcal{K}_o^n$, $p>0$, $\tau\in[-1,1]$, $i\neq n$, if $i\leqslant n-p$, then

$$B_{p,i}^{(\tau)}(K+_pL)^{\frac{p}{n-i}}\leqslant B_{p,i}^{(\tau)}(K)^{\frac{p}{n-i}}+B_{p,L}^{(\tau)}(L)^{\frac{p}{n-i}} \quad (27)$$

with equality if and only if K and L have simliar general L_p-width. If $n-p<i<n$, or $i>n$, inequality (27) is reversed.

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