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Existence of Triple Positive Solutions to a Four-Point Boundary Value Problem of a Fractional Differential Equations

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Abstract: In this paper, the existence result of at least triple positive solutions to a boundary value problem of a fractional differential equations is achieved by means of the Avery-Peterson fixed point theorem and the careful analysis of the associate Green's function in which the derivative of unknown function is involved in the nonlinear term explicitly. An example illustrating our main result is given. Our results complement the previous work in the area of the positive solutions of fractional differential equations.

Key words: fixed point; positive solution; cone; Avery-Peterson fixed point theorem

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0 Introduction

In this paper we consider the boundary value problem of nonlinear fractional differential equations

$$ {}^{C}D_{0+}^{\alpha}u(t)+f(t,\ u(t),\ u'(t))=0,\ t\in[0,\ 1] \quad (1)$$

$$ u'(0)-\beta u'(\xi)=0,\ u(1)+\gamma u'(\eta)=0 \quad (2)$$

where $f\in C\left([0,1]\times[0,+\infty)\times(-\infty,+\infty),(0,+\infty)\right)$ and $1<\alpha\leqslant 2$, $\beta\in(0,1)$, $\gamma>0$, $0\leqslant\xi\leqslant\eta\leqslant 1$ and ${}^{C}D_{0+}^{\alpha}$ denotes the Caputo's fractional derivative of order α.

Due to the development of the theory of fractional calculus and its applications, such as Bode's analysis of feedback amplifiers, aerodynamics and polymer rheology in the fields of physics, etc, many works on the basic theory of fractional calculus and fractional order differential equations have been done [1-7]. Recently, there have been many papers dealing with the solutions or positive solutions to boundary value problems for nonlinear fractional differential equations (FBVPs) with local boundary conditions[8-23] and nonlocal boundary conditions[24-35] and references along this line.

Zhang[13] proved the existence of positive solution to the boundary value problem of fractional order differential equation

$$ {}^{C}D_{0+}^{q}u(t)=f(t,\ u(t)),\ t\in(0,\ 1),$$

$$ u(0)+u'(0)=0,\ u(1)+u'(1)=0,$$

where $1<q\leqslant 2$ and $f\in C\left([0,1]\times[0,+\infty)\to[0,+\infty)\right)$.

By using the fixed point theorem, Goodrich[22] considered the following class of nonlinear fractional differential equations with the given boundary conditions for multiplicity of positive solutions as

$$D_{0+}^{\alpha}u(t)+f(t,\ u(t))=0,\ t\in(0,\ 1),\ n-1<\alpha\leqslant n$$

$$u^{(i)}(0)=0,\ 0\leqslant i\leqslant n-2 \text{ and } D_{0+}^{\beta}u(1)=0$$

Specially, there are a few researches concerning four-point boundary value problems for fractional differential equations. For examples, in Ref.[28], the authors considered a class of four-point fractional boundary value problem of the form

$$D_{0+}^{\alpha}u(t)+f(t,\ u(t))=0,\ t\in(0,\ 1),$$

$$u'(0)-\mu_1 u(\xi)=0,\ u'(1)+\mu_2 u(\eta)=0$$

where u' denotes the first order derivative of function u and $1<\alpha\leqslant 2,\ 0\leqslant\xi\leqslant\eta\leqslant 1,\ 0\leqslant\mu_1,\mu_2\leqslant 1,\ f\colon[0,1]\times\mathbf{R}^+\to\mathbf{R}^+$ is continuous and D_{0+}^{α} is the Caputo's fractional derivative of order α.

In Ref.[29], the following four-point nonlinear boundary value problem

$${}^{C}D_{0+}^{\alpha}u(t)+f(t,\ u(t),\ (Ku)(t),\ (Hu)(t))=0,\ t\in(0,\ 1),$$

$$a_1u(0)-b_1u'(0)=d_1u(\xi_1),\ a_2u(1)+b_2u'(1)=d_2u(\xi_2)$$

was considered. The existence of solutions of the problem were established.

Ji and Ge [30] studied the following four-point nonlocal boundary value problems of fractional order

$${}^{C}D_{0+}^{\alpha}u(t)+f(t,\ u(t))=0,\ t\in[0,\ 1],$$

$$u'(0)-\beta u'(\xi)=0,\ u(1)+\gamma u'(\eta)=0$$

where $1<\alpha\leqslant 2,\ \xi\leqslant\eta\leqslant 1$ and ${}^{C}D_{0+}^{\alpha}$ is Caputo's fractional derivative. By using the fixed point theorem, multiplicity results of positive solutions are obtained.

We noticed that in these work the existence results of positive solutions were all established under the assumption that the derivative of the unknown function was not involved in the nonlinear term explicitly. The main reason is that one can not derive the concavity or convexity of the function by the sign of its fractional derivative. On account of the practical meaning of $u'(t)$, it is interesting to consider the boundary value problem of fractional differential equations in which the derivative of the unknown function is involved in the nonlinear term explicitly.

In this paper, by using the careful analysis of the associated Green's function and defining the special cone in a suitable Banach space together with the Avery-Peterson fixed point theorem, we overcome the difficulties bringing by the lack of the concavity or convexity of the unknown function and show the existence of multiple positive solutions of problem (1)-(2).The results complete and extend the previous work on boundary value problem of fractional differential equations.

1 Preliminaries

Definition 1 The Riemann-Liouville fractional integral of order $\alpha>0$ of a function $u(t)$ is given by

$$I_{0+}^{\alpha}u(t)=\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}u(s)\mathrm{d}s$$

provided the right side is point-wise defined on $(0,+\infty)$.

Definition 2 The Caputo's fractional derivative of order $\alpha>0$ of a continuous function $u(t)$ is given by

$${}^{C}D_{0+}^{\alpha}u(t)=\frac{1}{\Gamma(n-\alpha)}\int_0^t \frac{u^{(n)}(s)}{(t-s)^{1+\alpha-n}}\mathrm{d}s$$

where $n=[\alpha]+1$, provided that the right side is point-wise defined on $(0,+\infty)$.

Lemma 1 Let $\alpha>0$. The fractional differential equation $D_{0+}^{\alpha}u(t)=0$ has solution

$$u(t)=C_1+C_2t+C_3t^2+\cdots+C_nt^{n-1},\ C_i\in\mathbf{R},\ i=1,2,\cdots,n.$$

Lemma 2 Assume that $u(t)$ is differentiable with a fractional derivative of order $\alpha>0$. Then

$$I_{0+}^{\alpha}D_{0+}^{\alpha}u(t)=u(t)+C_1+C_2t+C_3t^2+\cdots+C_nt^{n-1},\ C_i\in\mathbf{R},\ i=1,2,\cdots,n,$$

where n is the smallest integer greater than or equal to α.

Definition 3 The map ϕ is said to be a nonnegative continuous convex functional on cone P of a real Banach space E provided that ϕ: $P\to[0,+\infty)$ is continuous and

$$\phi(tx+(1-t)y)\leqslant t\phi(x)+(1-t)\phi(y),\ x,y\in P, t\in[0,1].$$

Definition 4 The map β is said to be a nonnegative continuous concave functional on cone P of a real Banach space E provided that β: $P\to[0,+\infty)$ is continuous and

$$\beta(tx+(1-t)y)\geqslant t\beta(x)+(1-t)\beta(y),\ x,y\in P, t\in[0,1].$$

Let γ,θ be nonnegative continuous convex functionals on P, ϕ be a nonnegative continuous concave functional on P and ψ be a nonnegative continuous functional on P. Then for positive numbers a, b, c and d, we define the following convex sets:

$$P(\gamma,d)=\{x\in P|\gamma(x)<d\},$$
$$P(\gamma,\phi,b,d)=\{x\in P|b\leqslant\phi(x),\gamma(x)\leqslant d\},$$
$$P(\gamma,\theta,\phi,b,c,d)=\{x\in P|b\leqslant\phi(x),\theta(x)\leqslant c,\gamma(x)\leqslant d\}$$

and a closed set

$$R(\gamma,\psi,a,d)=\{x\in P|a\leqslant\psi(x),\gamma(x)\leqslant d\}.$$

Lemma 3 [36] Let P be a cone in Banach space E. Let γ,θ be nonnegative continuous convex functionals on P, ϕ be a nonnegative continuous concave functional on P, and ψ be a nonnegative continuous functional on P satisfying

$$\psi(\lambda x)\leqslant\lambda\psi(x), 0\leqslant\lambda\leqslant 1,$$
$$\phi(x)\leqslant\psi(x),\|x\|\leqslant l\gamma(x) \text{ for } x\in\overline{P(\gamma,d)},$$

where $\overline{P(\gamma,d)}$ is the closure of the set $P(\gamma,d)$. Suppose $T:\overline{P(\gamma,d)}\to\overline{P(\gamma,d)}$ is completely continuous and there exist positive numbers a, b, c with $a<b$ such that

(S_1) $\{x\in P(\gamma,\theta,\phi,b,c,d)|\phi(x)>b\}\neq\varnothing$, $\phi(Tx)>b$ for $x\in P(\gamma,\theta,\phi,b,c,d)$;

(S_2) $\phi(Tx)>b, x\in P(\gamma,\phi,b,d)$ with $\theta(Tx)>c$;

(S_3) $0\notin R(\gamma,\psi,a,d)$ with $\psi(Tx)<a$ for $x\in R(\gamma,\psi,a,d)$ with $\psi(x)=a$.

Then T has at least three fixed points $x_1,x_2,x_3\in\overline{P(\gamma,d)}$ such that:

$$\gamma(x_i)\leqslant d, i=1,2,3;\ b<\phi(x_1);\ a<\psi(x_2),\phi(x_2)<b;\ \psi(x_3)<a.$$

2 Main Results

Lemma 4[30] Given $y(t)\in C[0,1]$, then boundary value problem

$$D_{0+}^{\alpha}u(t)+y(t)=0, t\in[0,1] \tag{3}$$
$$u'(0)-\beta u'(\xi)=0, u(1)+\gamma u'(\eta)=0 \tag{4}$$

is equivalent to

$$u(t)=\int_0^1 G(t,s)y(s)\mathrm{d}s,$$

where

$$G(t,s)=\begin{cases} -\dfrac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}+\dfrac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}+\dfrac{\beta(1+\gamma-t)(\xi-s)^{\alpha-2}}{\Gamma(\alpha-1)(1-\beta)}+\dfrac{\gamma(\eta-s)^{\alpha-2}}{\Gamma(\alpha-1)}, 0\leqslant s\leqslant\xi, s\leqslant t\\ \dfrac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}+\dfrac{\beta(1+\gamma-t)(\xi-s)^{\alpha-2}}{\Gamma(\alpha-1)(1-\beta)}+\dfrac{\gamma(\eta-s)^{\alpha-2}}{\Gamma(\alpha-1)}, 0\leqslant s\leqslant\xi, s\geqslant t\\ -\dfrac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}+\dfrac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}+\dfrac{\gamma(\eta-s)^{\alpha-2}}{\Gamma(\alpha-1)}, \xi\leqslant s\leqslant\eta,\ s\leqslant t\\ \dfrac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}+\dfrac{\gamma(\eta-s)^{\alpha-2}}{\Gamma(\alpha-1)}, \xi\leqslant s\leqslant\eta,\ s\geqslant t\\ -\dfrac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}+\dfrac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}, \eta\leqslant s\leqslant 1,\ s\leqslant t\\ \dfrac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}, \eta\leqslant s\leqslant 1,\ s\geqslant t \end{cases}$$

Furthermore, the function $G(t,s)$ satisfies that

$$G(t,s)\geqslant 0 \text{ for all } t,s\in[0,1] \text{ and } G(t,s)>0 \text{ for all } t,s\in(0,1)$$

Lemma 5 The function $G(t, s)$ satisfies the following properties:

1) $G(t, s)$ is decreasing with respect to t;

2) $\min\limits_{0\leqslant t\leqslant \eta} G(t,\ s)\geqslant \gamma_0 \max\limits_{0\leqslant t\leqslant 1} G(t,\ s),\ s\in[0,1]$, where

$$\gamma_0=\min\left\{\frac{\dfrac{1-\eta^{\alpha-1}}{\alpha-1}+\dfrac{\beta}{1-\beta}(1+\gamma-\eta)\xi^{\alpha-2}+\gamma\eta^{\alpha-2}}{\dfrac{1}{\alpha-1}+\dfrac{\beta}{1-\beta}(1+\gamma)\xi^{\alpha-2}+\gamma\eta^{\alpha-2}},1-\frac{(\eta-\xi)^{\alpha-1}}{(1-\xi)^{\alpha-1}+(\alpha-1)\gamma(\eta-\xi)^{\alpha-2}},1\right\}>0.$$

Proof 1) To prove that 1) is true, we begin with

$$\frac{\partial G(t,s)}{\partial t}=\begin{cases}-\dfrac{(t-s)^{\alpha-2}}{\Gamma(\alpha-1)}-\dfrac{\beta(\xi-s)^{\alpha-2}}{\Gamma(\alpha-1)(1-\beta)},0\leqslant s\leqslant\xi,s\leqslant t\\ -\dfrac{\beta(\xi-s)^{\alpha-2}}{\Gamma(\alpha-1)(1-\beta)},0\leqslant s\leqslant\xi,s\geqslant t\\ -\dfrac{(t-s)^{\alpha-2}}{\Gamma(\alpha-1)},\xi\leqslant s\leqslant 1,s\leqslant t\\ 0,\ \xi\leqslant s\leqslant 1,s\geqslant t\end{cases}$$

It is easy to find that $G(t, s)$ is decreasing with respect to t.

2) From the expression and monotonicity of function $G(t, s)$ with respect to t , we have

$$\min_{0\leqslant t\leqslant\eta} G(t,\ s)=G(\eta,\ s),\quad \max_{0\leqslant t\leqslant 1} G(t,\ s)=G(0,\ s).$$

Thus, for $0\leqslant s\leqslant\xi$,

$$\frac{G(\eta,s)}{G(0,s)}=\frac{-\dfrac{(\eta-s)^{\alpha-1}}{\Gamma(\alpha)}+\dfrac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}+\dfrac{\beta(1+\gamma-t)(\xi-s)^{\alpha-2}}{\Gamma(\alpha-1)(1-\beta)}+\dfrac{\gamma(\eta-s)^{\alpha-2}}{\Gamma(\alpha-1)}}{\dfrac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}+\dfrac{\beta(1+\gamma)(\xi-s)^{\alpha-2}}{\Gamma(\alpha-1)(1-\beta)}+\dfrac{\gamma(\eta-s)^{\alpha-2}}{\Gamma(\alpha-1)}}$$
$$\geqslant\frac{\dfrac{1-\eta^{\alpha-1}}{\alpha-1}+\dfrac{\beta}{1-\beta}(1+\gamma-\eta)\xi^{\alpha-2}+\gamma\eta^{\alpha-2}}{\dfrac{1}{\alpha-1}+\dfrac{\beta}{1-\beta}(1+\gamma)\xi^{\alpha-2}+\gamma\eta^{\alpha-2}}.$$

For $\xi\leqslant s\leqslant\eta$,

$$\frac{G(\eta,s)}{G(0,s)}=\frac{-\dfrac{(\eta-s)^{\alpha-1}}{\Gamma(\alpha)}+\dfrac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}+\dfrac{\gamma(\eta-s)^{\alpha-2}}{\Gamma(\alpha-1)}}{\dfrac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}+\dfrac{\gamma(\eta-s)^{\alpha-2}}{\Gamma(\alpha-1)}}\geqslant 1-\frac{(\eta-\xi)^{\alpha-1}}{(1-\xi)^{\alpha-1}+(\alpha-1)\gamma(\eta-\xi)^{\alpha-2}}>0.$$

For $\eta\leqslant s\leqslant 1,\ G(\eta,s)=G(0,s)$. Then, we conclude that

$$\min_{0\leqslant t\leqslant\eta} G(t,\ s)=G(\eta,\ s)\geqslant\gamma_0 G(0,\ s)=\gamma_0\max_{0\leqslant t\leqslant 1} G(t,\ s).$$

Lemma 6 Assume that $y(t)>0$ and $u(t)$ is a solution to problem (3)-(4). Then

$$\max_{0\leqslant t\leqslant 1}|u(t)|\leqslant(1+\gamma)\max_{0\leqslant t\leqslant 1}|u'(t)|.$$

Proof The fact that $u(0)=\max\limits_{0\leqslant t\leqslant 1}|u(t)|$ and

$$u(0)=u(1)-\int_0^1 u'(s)\mathrm{d}s=-\gamma u'(\eta)-\int_0^1 u'(s)\mathrm{d}s$$

ensure that

$$\max_{0\leqslant t\leqslant 1}|u(t)|\leqslant(1+\gamma)\max_{0\leqslant t\leqslant 1}|u'(t)|.$$

Let the space $X=C^1[0,1]$ endowed with the norm

$$\|u\|=\max\left\{\max_{0\leqslant t\leqslant 1}|u(t)|,\ \max_{0\leqslant t\leqslant 1}|u'(t)|\right\}.$$

It is well known that X is a Banach space . Define the cone $K\subset X$ by

$$K=\{u\in X\mid u(t)\geqslant 0,\ \min_{t\in[0,\eta]}u(t)\geqslant \gamma_0\max_{0\leqslant t\leqslant 1}u(t),\ \max_{0\leqslant t\leqslant 1}\left|u(t)\right|\leqslant(1+\gamma)\max_{0\leqslant t\leqslant 1}|u'(t)|\}.$$

Lemma 7 Let $T\colon K\to X$ be the operator defined by $(Tu)(t):=\int_0^1 G(t,\ s)f(s,\ u(s),\ u'(s))\mathrm{d}s$. Then $T\colon K\to K$ is completely continuous.

Proof First, we will show that the operator $T\colon K\to X$ is continuous. For any $u_n, u\in K, n=1,2,\cdots,$ with $\lim\limits_{n\to\infty}\|u_n-n\|=0$, we have

$$\lim_{n\to\infty}u_n=u,\ \lim_{n\to\infty}u_n'=u',\ t\in[0,1].$$

From the continuity of function f, we obtain

$$\lim_{n\to+\infty}f(t,u_n(t),u_n'(t))=f(t,u(t),u'(t)),\ t\in[0,1].$$

Thus

$$\sup_{t\in[0,1]}\left|f(t,u_n(t),u_n'(t))-f(t,u(t),u'(t))\right|\to 0,\ n\to+\infty.$$

Therefore,

$$\left|(Tu_n)(t)-(Tu)(t)\right|=\left|\int_0^1 G(t,\ s)(f(s,\ u_n(s),\ u_n'(s))-f(s,\ u(s),\ u'(s))\mathrm{d}s\right|$$

$$=\left(\frac{1}{\Gamma(\alpha+1)}+\frac{\gamma\eta^{\alpha-1}}{\Gamma(\alpha)}+\frac{\beta(1+\gamma)\xi^{\alpha-1}}{(1-\beta)\Gamma(\alpha)}\right)\sup_{t\in[0,1]}\left|f(t,u_n(t),u_n'(t))-f(t,u(t),u'(t))\right|,$$

$$\left|(Tu_n)'(t)-(Tu)'(t)\right|$$

$$=\left|-\int_0^t\frac{(t-s)^{\alpha-2}}{\Gamma(\alpha-1)}|f(s,\ u_n,\ u_n')-f(s,\ u,\ u')|\mathrm{d}s-\int_0^\xi\frac{\beta(\xi-s)^{\alpha-2}}{(1-\beta)\Gamma(\alpha-1)}|f(s,\ u_n,\ u_n')-f(s,\ u,\ u'|\mathrm{d}s\right|$$

$$\leqslant\left(\frac{1}{\Gamma(\alpha)}+\frac{\beta\xi^{\alpha-1}}{(1-\beta)\Gamma(\alpha)}\right)\sup_{t\in[0,1]}\left|f(t,u_n(t),u_n'(t))-f(t,u(t),u'(t))\right|,$$

which implies that $\|Tu_n-Tu\|\to 0, n\to\infty$. These ensure that T is continuous. Second, we will show that T is completely continuous.

Let $\Omega\subset K$ be bounded. Then there exists a positive constant $R_1>0$ such that $\|u\|\leqslant R_1, u\in\Omega$. Denote

$$R=\max_{0\leqslant t\leqslant 1,\ u\in\Omega}|f(t,\ u(t),\ u'(t))|+1.$$

Then for $u\in\Omega$, we have

$$|Tu|\leqslant\int_0^1 G(0,\ s)f(s,\ u(s),\ u'(s))\mathrm{d}s$$

$$\leqslant\left(\int_0^\xi\left(\frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}+\frac{\beta(1+\gamma)(\xi-s)^{\alpha-2}}{\Gamma(\alpha-1)(1-\beta)}+\frac{\gamma(\eta-s)^{\alpha-2}}{\Gamma(\alpha-1)}\right)\mathrm{d}s+\int_\xi^\eta\left(\frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}+\frac{\gamma(\eta-s)^{\alpha-2}}{\Gamma(\alpha-1)}\right)\mathrm{d}s+\int_\eta^1\frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}\mathrm{d}s\right)\times R$$

$$=\left(\frac{1}{\Gamma(\alpha+1)}+\frac{\gamma\eta^{\alpha-1}}{\Gamma(\alpha)}+\frac{\beta(1+\gamma)\xi^{\alpha-1}}{\Gamma(\alpha)(1-\beta)}\right)\times R,$$

$$|T'u|=\left|-\int_0^t\frac{(t-s)^{\alpha-2}}{\Gamma(\alpha-1)}f(s,\ u(s),\ u'(s))\mathrm{d}s-\frac{\beta}{\Gamma(\alpha-1)(1-\beta)}\int_0^\xi(\xi-s)^{\alpha-2}f(s,\ u(s),\ u'(s))\mathrm{d}s\right|$$

$$\leqslant\left(\frac{1}{\Gamma(\alpha)}+\frac{\beta\xi^{\alpha-1}}{\Gamma(\alpha)(1-\beta)}\right)\times R.$$

Hence $T(\Omega)$ is bounded. For $u\in\Omega, t_1, t_2\in[0,1]$, one has

$$|Tu(t_2)-Tu(t_1)|\leqslant\left|\frac{1}{\Gamma(\alpha)}\left(\int_0^{t_2}(t_2-s)^{\alpha-1}f(s,\ u(s),\ u'(s))\mathrm{d}s-\int_0^{t_1}(t_1-s)^{\alpha-1}f(s,\ u(s),\ u'(s))\mathrm{d}s\right)\right|$$

$$+\frac{\beta|t_2-t_1|}{\Gamma(\alpha-1)(1-\beta)}\int_0^\xi(\xi-s)^{\alpha-2}f(s,\ u(s),\ u'(s))\mathrm{d}s$$

$$\leqslant\frac{R}{\Gamma(\alpha+1)}\times|t_2^\alpha-t_1^\alpha|+\frac{\beta R}{\Gamma(\alpha-1)(1-\beta)}\times|t_2-t_1|,$$

$$|Tu'(t_2)-Tu'(t_1)|\leqslant\left|\frac{1}{\Gamma(\alpha-1)}\left(\int_0^{t_1}(t_1-s)^{\alpha-2}f(s,\ u(s),\ u'(s))\mathrm{d}s-\int_0^{t_2}(t_2-s)^{\alpha-2}f(s,\ u(s),\ u'(s))\mathrm{d}s\right)\right|$$
$$\leqslant\frac{R}{\Gamma(\alpha)}\times|t_2^{\alpha-1}-t_1^{\alpha-1}|.$$

Thus,

$$\left\|Tu(t_2)-Tu(t_1)\right\|\to 0 \text{ for } t_1\to t_2,\ u\in\Omega.$$

By means of the Arzela-Ascoli theorem, we claim that T is completely continuous. Finally, we see that

$$\min_{0\leqslant t\leqslant\eta}|Tu(t)|=\min_{0\leqslant t\leqslant\eta}\int_0^1 G(t,\ s)f(s,\ u(s),\ u'(s))\mathrm{d}s\geqslant\gamma_0\int_0^1 G(0\ s)f(s,\ u(s),\ u'(s))\mathrm{d}s\geqslant\gamma_0\max_{0\leqslant t\leqslant 1}(Tu)(t)$$

Considering the definition of the operator T together with Lemma 6, one can find that

$$\max_{0\leqslant t\leqslant 1}|Tu(t)|\leqslant(1+\gamma)\max_{0\leqslant t\leqslant 1}|Tu'(t)|.$$

Thus, we conclude that $T: K\to K$ is a completely continuous operator.

Let the nonnegative continuous concave functional ϕ, the nonnegative continuous convex functionals γ,θ and the nonnegative continuous functional ψ be defined on the cone by

$$\gamma(u)=\max_{0\leqslant t\leqslant 1}|u'(t)|,\ \theta(u)=\psi(u)=\max_{0\leqslant t\leqslant 1}|u(t)|,\ \phi(u)=\min_{0\leqslant t\leqslant\eta}|u(t)|.$$

By Lemmas 5 and 6, the functionals defined above satisfy that

$$\gamma_0\theta(u)\leqslant\phi(u)\leqslant\theta(u)=\psi(u),\ \ \|u\|\leqslant(1+\gamma)\gamma(u),\ u\in K.$$

Therefore condition of Lemma 3 is satisfied.

Assume that there exist constants $0<a,b,d$ with $a<b<d, c=\dfrac{b}{\gamma_0}$ and

$$d>\frac{\alpha(1-\beta+\beta\xi^{\alpha-1})}{\gamma_0(1-\beta+\alpha\gamma(1-\beta)\eta^{\alpha-1}+\alpha\beta(1+\gamma)\xi^{\alpha-1})}b$$

such that

(A_1) $f(t,u,v)\leqslant\dfrac{(1-\beta)\Gamma(\alpha)}{1-\beta+\beta\xi^{\alpha-1}}d,\ (t,\ u,\ v)\in[0,\ 1]\times[0,\ (1+\gamma)d]\times[-d,\ d];$

(A_2) $f(t,u,v)>\dfrac{(1-\beta)\Gamma(\alpha+1)}{\gamma_0(1-\beta+\alpha\gamma(1-\beta)\eta^{\alpha-1}+\alpha\beta(1+\gamma)\xi^{\alpha-1})}b,\ (t,\ u,\ v)\in[0,\ \eta]\times[b,\ b/\gamma_0]\times[-d,\ d];$

(A_3) $f(t,u,v)<\dfrac{(1-\beta)\Gamma(\alpha+1)}{1-\beta+\alpha\gamma(1-\beta)\eta^{\alpha-1}+\alpha\beta(1+\gamma)\xi^{\alpha-1}}a,\ (t,\ u,\ v)\in[0,\ 1]\times[0,\ a]\times[-d,\ d].$

Theorem 1 Under assumptions (A_1)-(A_3), problem (1-2) has at least three positive solutions $u_1,\ u_2,\ u_3$ satisfying

$$\max_{0\leqslant t\leqslant 1}|u_i'(t)|\leqslant d,\ i=1,\ 2,\ 3;\ b<\min_{0\leqslant t\leqslant\eta}|u_1(t)|;\ a<\max_{0\leqslant t\leqslant 1}|u_2(t)|,\ \min_{0\leqslant t\leqslant\eta}|u_2(t)|<b,\ \max_{0\leqslant t\leqslant 1}|u_3(t)|\leqslant a.$$

Proof Problem (1-2) has a solution $u=u(t)$ if and only if u solves the operator equation

$$u(t)=\int_0^1 G(t,\ s)f(s,\ u(s),\ u'(s))\mathrm{d}s=(Tu)(t).$$

For $u\in\overline{K(\gamma,d)}$, we have $\gamma(u)=\max_{0\leqslant t\leqslant 1}|u'(t)|<d$. Then

$$f(t,u(t),u'(t))\leqslant\frac{(1-\beta)\Gamma(\alpha)}{1-\beta+\beta\xi^{\alpha-1}}d.$$

Thus

$$|\gamma(Tu)|=\max_{0\leqslant t\leqslant 1}\left|-\int_0^t\frac{(t-s)^{\alpha-2}}{\Gamma(\alpha-1)}f(s,\ u(s),\ u'(s))\mathrm{d}s-\frac{\beta}{\Gamma(\alpha-1)(1-\beta)}\int_0^{\xi}(\xi-s)^{\alpha-2}f(s,\ u(s),\ u'(s))\mathrm{d}s\right|$$
$$\leqslant\left(\frac{1}{\Gamma(\alpha)}+\frac{\beta\xi^{\alpha-1}}{\Gamma(\alpha)(1-\beta)}\right)\times\frac{(1-\beta)\Gamma(\alpha)}{1-\beta+\beta\xi^{\alpha-1}}d=d$$

Hence, T: $\overline{K(\gamma,d)}\to\overline{K(\gamma,d)}$.

The fact that the constant function $u(t)=b/\gamma_0\in K(\gamma,\ \theta,\ \phi,\ b,\ c,d)$ and $\phi(b/\gamma_0)>b$ implies that $\{u\in K(\gamma,\ \theta,\ \phi,\ b,\ c,\ d|\phi(u)>b)\}\neq\varnothing$. For $u\in K(\gamma,\ \theta,\ \phi,\ b,\ c,\ d)$, we have $b\leqslant u(t)\leqslant\dfrac{b}{\gamma_0},|u'(t)|<d$ for $0\leqslant t\leqslant\eta$. From

assumption (A_2), we see

$$f\left(t,u(t),u'(t)\right) > \frac{(1-\beta)\Gamma(\alpha+1)}{\gamma_0(1-\beta+\alpha\gamma(1-\beta)\eta^{\alpha-1}+\alpha\beta(1+\gamma)\xi^{\alpha-1})}b$$

Thus

$$\phi(Tu)=\min_{0\leqslant t\leqslant \eta}\int_0^1 G(t,\ s)f(s,\ u(s),\ u'(s))\mathrm{d}s \geqslant \gamma_0\int_0^1 G(0,\ s)f(s,\ u(s),\ u'(s))\mathrm{d}s$$

$$=\gamma_0\left(\frac{1}{\Gamma(\alpha+1)}+\frac{\gamma\eta^{\alpha-1}}{\Gamma(\alpha)}+\frac{\beta(1+\gamma)\xi^{\alpha-1}}{\Gamma(\alpha)(1-\beta)}\right)\times\frac{(1-\beta)\Gamma(\alpha+1)}{\gamma_0(1-\beta+\alpha\gamma(1-\beta)\eta^{\alpha-1}+\alpha\beta(1+\gamma)\xi^{\alpha-1})}b=b$$

which means $\phi(Tu)>b,\ \forall u\in K(\gamma,\ \theta,\ \phi,\ b,\ \frac{b}{\gamma_0},\ d)$. These ensure that condition (S_1) of Lemma 3 is satisfied. Secondly, for all, $u\in K(\gamma,\ \phi,\ b,\ d), \theta(Tu)>c$,

$$\phi(Tu)\geqslant\gamma_0\theta(Tu)>\gamma_0 c=\gamma_0\frac{b}{\gamma_0}=b$$

Thus, condition (S_2) of Lemma 3 holds. Finally we show that (S_3) also holds. We see that $\psi(0)=0<a$ and $0\notin R(\gamma,\psi,a,d)$. Suppose that $x\in R(\gamma,\ \psi,\ a,\ d), \psi(x)=a$. Then by assumption ($A_3$)

$$\psi(Tu)=\max_{0\leqslant t\leqslant\eta}\left|\int_0^1 G(t,\ s)f(s,\ u(s),\ u'(s))\mathrm{d}s\right|=\int_0^1 G(0,\ s)f(s,\ u(s),\ u'(s))\mathrm{d}s$$

$$\leqslant\left(\frac{1}{\Gamma(\alpha+1)}+\frac{\gamma\eta^{\alpha-1}}{\Gamma(\alpha)}+\frac{\beta(1+\gamma)\xi^{\alpha-1}}{\Gamma(\alpha)(1-\beta)}\right)\times\frac{(1-\beta)\Gamma(\alpha+1)}{1-\beta+\alpha\gamma(1-\beta)\eta^{\alpha-1}+\alpha\beta(1+\gamma)\xi^{\alpha-1}}a=a$$

Thus, all conditions of Lemma 3 are satisfied. Hence problem (1-2) has at least three positive solutions u_1,u_2,u_3 satisfying

$$\max_{0\leqslant t\leqslant 1}|u_i'(t)|\leqslant d,\ i=1,\ 2,\ 3;\ b<\min_{0\leqslant t\leqslant\eta}|u_1(t)|;\ a<\max_{0\leqslant t\leqslant 1}|u_2(t)|,\ \min_{0\leqslant t\leqslant\eta}|u_2(t)|<b, \max_{0\leqslant t\leqslant 1}|u_3(t)|\leqslant a.$$

3 Example

Here we present an example to illustrate the main theorem. Consider the boundary value problem

$$D_{0+}^{1.7}u(t)+f(t,\ u(t),\ u'(t))=0,\ t\in[0,\ 1] \tag{5}$$

$$u'(0)-0.6u'\left(\frac{1}{4}\right)=0,\ u(1)+u'\left(\frac{1}{2}\right)=0 \tag{6}$$

where $\alpha=1.7,\ \beta=0.6, \xi=\frac{1}{4},\ \eta=\frac{1}{2}, \gamma=1$ and

$$f(t,u,v)=\begin{cases}\frac{1}{20}\mathrm{e}^t+\frac{1}{\pi^3}\left(u+\frac{1}{2}\right)^3+\frac{1}{100}\sin\left(\frac{v}{1\,000}\right), 0\leqslant u\leqslant 12\\ \frac{1}{20}\mathrm{e}^t+\frac{15625}{8\pi^3}+\frac{1}{100}\sin\left(\frac{v}{1\,000}\right), u>12\end{cases}$$

By a straightforward calculation, we see that

$$\gamma_0=\min\left\{\frac{\frac{1-\eta^{\alpha-1}}{\alpha-1}+\frac{\beta}{1-\beta}(1+\gamma-\eta)\xi^{\alpha-2}+\gamma\eta^{\alpha-2}}{\frac{1}{\alpha-1}+\frac{\beta}{1-\beta}(1+\gamma)\xi^{\alpha-2}+\gamma\eta^{\alpha-2}}, 1-\frac{(\eta-\xi)^{\alpha-1}}{(1-\xi)^{\alpha-1}+(\alpha-1)\gamma(\eta-\xi)^{\alpha-2}}, 1\right\}\approx 0.720\,2.$$

We choose positive constants a=1, b=5, d=1 000 and check that the nonlinear term $f(t,\ u,v)$ satisfies

1) $f(t,u,v)\leqslant\frac{(1-\beta)\Gamma(\alpha)}{1-\beta+\beta\xi^{\alpha-1}}d\approx 579.3,\ (t,\ u,\ v)\in[0,1]\times[0,\ 2\,000]\times[-1\,000,\ 1\,000]$;

2) $f(t,u,v)>\frac{(1-\beta)\Gamma(\alpha+1)}{\gamma_0(1-\beta+\alpha\gamma(1-\beta)\eta^{\alpha-1}+\alpha\beta(1+\gamma)\xi^{\alpha-1})}b\approx 2.6951,\ (t,\ u,\ v)\in[0,\ 0.5]\times[5,\ 6.942\,5]\times[-1\,000,\ 1\,000]$;

3) $f(t,u,v)<\frac{(1-\beta)\Gamma(\alpha+1)}{1-\beta+\alpha\gamma(1-\beta)\eta^{\alpha-1}+\alpha\beta(1+\gamma)\xi^{\alpha-1}}a\approx 0.388\,2,\ (t,\ u,\ v)\in[0,\ 1]\times[0,\ 1]\times[-1\,000,\ 1\,000]$.

Then all assumptions of Theorem 1 are satisfied. Thus problem (5-6) has at least three positive solutions $u_1(t), u_2(t), u_3(t)$ satisfying

$$\max_{0\leqslant t\leqslant 1}|u_i'(t)|\leqslant 1000,\ i=1,\ 2,\ 3;\ 5<\min_{0\leqslant t\leqslant \frac{1}{2}}|u_1(t)|;\ 1<\max_{0\leqslant t\leqslant 1}|u_2(t)|,\ \ \min_{0\leqslant t\leqslant \frac{1}{2}}|u_2(t)|<5, \max_{0\leqslant t\leqslant 1}|u_3(t)|\leqslant 1.$$

Remark We see that the first order derivative of function $u(t)$ is involved in the nonlinear term of the problem (5-6) explicitly. The early results for positive solutions to this kind of fractional differential equations are not applicable to this problem.

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