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Some Properties of a Class of Addition Cayley Signed Graph

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Abstract: Let $G=\mathrm{Cay}^+(Z_{p_1}\times Z_{p_1^{\alpha_1}p_2^{\alpha_2}\cdots p_k^{\alpha_k}},\Phi)$ be a simple graph having vertex set $V(G)=Z_{p_1}\times Z_{p_1^{\alpha_1}p_2^{\alpha_2}\cdots p_k^{\alpha_k}}$ and edge set $E(G)=\{\{x,y\}: x+y\in\Phi\}$, where all $p_1,p_2,\cdots,p_k$ are distinct prime factors and Φ is the set of all units of the ring $Z_{p_1}\times Z_{p_1^{\alpha_1}p_2^{\alpha_2}\cdots p_k^{\alpha_k}}$. Let $\Sigma=(G,\sigma)$ be a signed graph whose underlying graph is G and signature function is $\sigma: E(G)\to\{+1,-1\}$ defined as $\sigma(\{x,y\})=\begin{cases}+1, & \text{if } x\in\varphi(p_1) \text{ or } y\in\varphi(p_1^{\alpha_1}p_2^{\alpha_2}\cdots p_k^{\alpha_k});\\ -1, & \text{otherwise.}\end{cases}$ In this paper, we characterize the balance of Σ and some graphs derived from it such as $\eta(\Sigma)$, $L(\Sigma)$ and $C_E(\Sigma)$. Moreover, we investigate the clusterability and sign-compativility of Σ.

Key words: signed graphs; derived graph; balance; clusterability; sign-compatibility

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0 Introduction

In this paper we only consider finite and undirected graphs without loops or multiple edges. Let Γ be a group and $S=S^{-1}$ is a non-empty subset which does not contain the identity element 1 of Γ. The Cayley graph[1] $\mathrm{Cay}(\Gamma,S)$ is a graph having vertex set Γ and edge set $\{\{v,vs\}: v\in\Gamma, s\in S\}$. For further information about Cayley graphs, one may refer to Refs.[2,3].

In 1990, unitary addition Cayley graph[4] $G_n=\mathrm{Cay}^+(Z_n,U_n)$ was first introduced. For a positive integer $n>1$, G_n is the graph whose vertex set is Z_n, and Z_n is the ring of integers modulo n. If U_n denotes set of all its units, then two vertices a and b are adjacent if and only if $a+b\in U_n$. Note that G_n is $|U_n|$-regular if $|Z_n|$ is even and $(|U_n|,|U_n|-1)$-semiregular if $|Z_n|$ is odd[5]. Some examples of $G_n=\mathrm{Cay}^+(Z_n,U_n)$ are illustrated in Fig. 1.

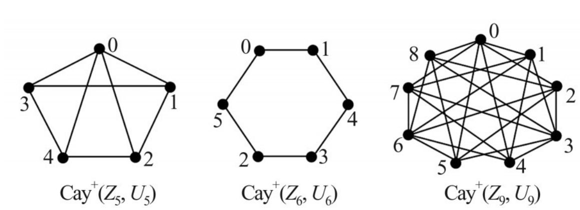

Fig. 1 Some examples of addition Cayley graphs

1 Preliminaries

1.1 Signed Graphs and Some Derived Graphs

Harary[6] introduced the definition of signed graphs in connection with the study of the theory of social balance, which can also be written as s-graphs[7] or sigraphs[8]. Signed graphs arose from the fact that psychologists used square matrices with elements −1, 0, and 1 to represent disliking, indifference and liking, respectively. It also can be depicted by a graph with positive and negative lines. Formally, a signed graph is an ordered pair $\Sigma=(G,\sigma)$, where $G=(V(G),E(G))$ is the underlying graph of Σ and σ: $E(G)\to\{+1,-1\}$ is a signature function from the edge set $E(G)$ into the set of $\{+1,-1\}$.

For an arbitrary edge e of Σ, if $\sigma(e)=+1$ (-1) then it is called positive (negative) edge. Moreover, Σ is said to be all-positive (all-negative) if all edges are positive (negative). Furthermore, Σ is also said to be homogeneous if it is either all-positive or all-negative and heterogeneous otherwise. The positive (negative) degree of a vertex $v\in V(G)$ denoted by $d^+(v)\,(d^-(v))$ which is the number of positive (negative) edges associates to v and the degree of v is $d(v)=d^+(v)+d^-(v)$. Based on these concepts, in what follows some derived graphs of Σ are introduced.

In 1957, Harary introduced the negation[9] $\eta(\Sigma)$ of signed graphs Σ which is obtained from Σ by negating the sign of every edge.

In 1969, Behzad and Chartrand defined line-sigraphs[7] $L(\Sigma)$ of signed graphs Σ as the sigraph whose points can be put in one-to-one correspondence with the lines of Σ in such a way that two points of $L(\Sigma)$ are joined by a negative line if and only if they correspond to two adjacent negative lines of Σ and are joined by a positive line if they correspond to some other two adjacent lines of Σ.

In 2006, Acharya and Sinha came up with common-edge sigraphs[8] $C_E(\Sigma)$. Given a sigraph Σ, its common-edge sigraph $C_E(\Sigma)$ is a sigraph whose vertex-set is the set of pairs of adjacent edges in Σ and two vertices of $C_E(\Sigma)$ are adjacent if and only if the corresponding pairs of adjacent edges of Σ have exactly one edge in common, with the same sign as that of their common edge.

An example of one signed graph and its three derived graphs is shown in Fig. 2. It should be noted that negative edges are represented by dash lines and positive edges by straight lines here and in the rest figures.

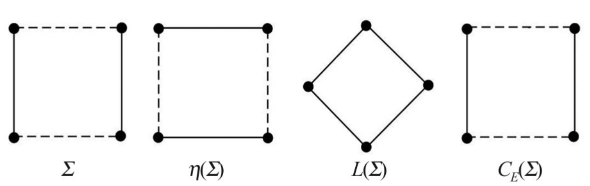

Fig. 2 The example of signed graphs and its derived graphs

Motivated by Iranmanesh and Moghaddami[10], in this paper, we will investigate some properties of an addition Cayley signed graph Σ which is defined as follows.

Definition 1 Let $G=\mathrm{Cay}^+(Z_{p_1}\times Z_{p_1^{\alpha_1}p_2^{\alpha_2}\cdots p_k^{\alpha_k}},\Phi)$ be a simple graph having vertex set $V(G)=Z_{p_1}\times Z_{p_1^{\alpha_1}p_2^{\alpha_2}\cdots p_k^{\alpha_k}}$ and edge set $E(G)=\{\{x,y\}:x+y\in\Phi\}$, where all $p_1,p_2,\cdots,p_k$ are distinct prime factors and Φ is the set of all units of the ring $Z_{p_1}\times Z_{p_1^{\alpha_1}p_2^{\alpha_2}\cdots p_k^{\alpha_k}}$. Let $\Sigma=(G,\sigma)$ be a signed graph whose underlying graph is G and signature function is σ: $E(G)\to\{+1,-1\}$ defined as[11]

$$\sigma(\{x,y\})=\begin{cases}+1, & \text{if } x\in\varphi(p_1) \text{ or } y\in\varphi(p_1^{\alpha_1}p_2^{\alpha_2}\cdots p_k^{\alpha_k});\\ -1, & \text{otherwise.}\end{cases}$$

Examples $\Sigma=(\mathrm{Cay}^+(Z_2\times Z_{2\times 3},\Phi),\sigma)$ and $\Sigma'=(\mathrm{Cay}^+(Z_3\times Z_{2\times 3},\Phi),\sigma)$ are displayed in Fig. 3 and 4, respectively.

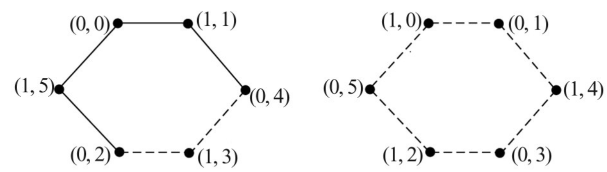

Fig. 3 $\Sigma=(\mathrm{Cay}^+(Z_2\times Z_{2\times 3},\Phi),\sigma)$

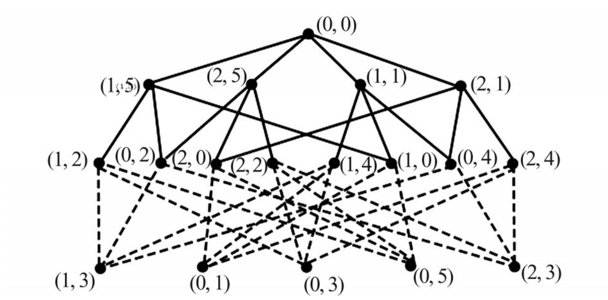

Fig. 4 $\Sigma'=(\mathrm{Cay}^+(Z_3\times Z_{2\times 3},\Phi),\sigma)$

1.2 The Notion of Balance in a Signed Graph

As we all known that a positive cycle of a signed graph is the one in which the number of negative lines is even, while a negative cycle is not positive[6]. In other words, a cycle in a signed graph is positive if it contains an even number of negative edges. An exact formula about signs for an arbitrary cycle C is given by $\sigma(C)=\prod_{e\in C}(\sigma(e))$, where $e\in C$[12].

Definition 2[6] A signed graph is said to be balanced if all its cycles are positive.

1.3 The Notion of Clustering in a Signed Graph

Definition 3[13] A clustering of a signed graph is a partition of the point set into subsets $P_1, P_2, \cdots, P_n$ (called plus-sets or Davis partitions) such that each positive line joins two points in the same subset and each negative line joins two points from different subsets.

Lemma 1[13] A signed graph has a clustering if and only if it contains no cycle having exactly one negative line.

1.4 The Notion of Sign-Compatibility in a Signed Graph

As known that a simple graph $G_1=(V_1,E_1)$ is said to be isomorphic to $G_2=(V_2,E_2)$ if and only if there is a bijective $f: V_1 \to V_2$ satisfied with $\{v_1,v_2\}\in E_1 \Leftrightarrow \{f(v_1),f(v_2)\}\in E_2$ for $v_1,v_2\in V_1$. Analogously, Σ_1 and Σ_2 are isomorphic if and only if there is an isomorphic mapping between the two underlying graphs that maintains edge signatures.

Lemma 2[14] A sigraph Σ is sign-compatible if and only if Σ does not contain a sub-sigraph isomorphic to either of the two sigraphs, S_1 formed by taking the path $P_4=(x,u,v,y)$ with both edges $\{x,u\}$ and $\{v,y\}$ negative and the edge $\{u,v\}$ positive and S_2 formed by taking S_1 and identifying the vertices x and y (see Fig. 5).

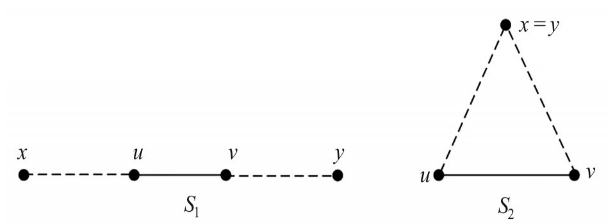

Fig. 5 Forbidden subsigraphs for a sign-compatible sigraph

The rest of this paper is organized as follows. In Section 2, we describe the balanced addition Cayley signed graphs. In Section 3, we prove some results about clusterability and sign-compatibility.

2 Balance in Σ and Derived Graphs

In this section, we discuss the balance of $\Sigma=(G,\sigma)$ and its derived graphs $\eta(\Sigma)$, $L(\Sigma)$ and $C_E(\Sigma)$, where $G=\mathrm{Cay}^+(Z_{p_1}\times Z_{p_1^{\alpha_1}p_2^{\alpha_2}\cdots p_k^{\alpha_k}}, \Phi)$ and $p_1,p_2,\cdots,p_k$ are distinct prime factors. Firstly, we give equivalent condition for the balanced addition Cayley signed graphs.

Lemma 3[15] Two signed graphs on the same underlying graph are switching equivalent if and only if they have the same list of balanced circles.

For any positive integer n, let $\varphi(n)=\{l: 1\leqslant l<n, \gcd(l,n)=1\}$.

Lemma 4 Let $\Sigma=(G,\sigma)$ be an addition Cayley signed graph, where $G=\mathrm{Cay}^+(Z_{p_1}\times Z_{p_1^{\alpha_1}p_2^{\alpha_2}\cdots p_k^{\alpha_k}}, \Phi)$ and all $p_1,p_2,\cdots,p_k$ are distinct prime factors. If one of $p_1,p_2,\cdots,p_k$ is 2, then Σ has exactly $|\Phi|^2$ positive edges.

Proof Since one of the prime factors is 2 then there are two following cases.

Case 1 Suppose $p_1=2$. In this case, Σ is disconnected and has exactly two connected components $\Sigma_1=(G_1,\sigma)$ with vertex set $V(\Sigma_1)=\{(u,v): u=1$ and v is odd$\}\cup\{(u,v): u=0$ and v is even$\}$ and $\Sigma_2=(G_2, \sigma)$ with vertex set $V(\Sigma_2)=\{(u,v): u=0$ and v is odd$\}\cup\{(u,v): u=1$ and v is even$\}$.

Note that the underlying graph G of Σ is a $|\Phi|$ -regular graph. Hence,

$$|E(\Sigma_1)|=|E(\Sigma_2)|=\frac{1}{2}|E(\Sigma)|=\frac{1}{2}|E(G)|=\frac{1}{4}|\Phi|p_1^{\alpha_1+1}p_2^{\alpha_2}\cdots p_k^{\alpha_k}.$$

Assume $\{(u,v),(u',v')\}$ is an arbitrary edge in Σ_1, where $(u,v)\in\{(u,v): u=1$ and v is odd$\}$, $(u',v')\in\{(u,v): u=0$ and v is even$\}$ and $v+v'\in\varphi(p_1^{\alpha_1}p_2^{\alpha_2}\cdots p_k^{\alpha_k})$. By Definition 1, $\{(u,v),(u',v')\}$ is positive edge if and only if $(u,v)\in\Phi$ since $\Phi\subseteq\{(u,v): u=1$ and v is odd$\}$ and $\{(u,v): u=0$ and v is even$\}\cap\Phi=\varnothing$. Moreover, if $(u,v)\in\{(u,v): u=1$ and v is odd$\}\backslash\Phi$ then this edge is negative. Therefore Σ_1 has exactly $|\Phi|^2$ positive edges.

For any edge $\{(u,v),(u',v')\}\in E(\Sigma_2)$, where $(u,v)\in\{(u,v): u=0$ and v is odd$\}$, $(u',v')\in\{(u,v): u=1$ and v is even$\}$ and $v+v'\in\varphi(p_1^{\alpha_1}p_2^{\alpha_2}\cdots p_k^{\alpha_k})$. Obviously, all members of $V(\Sigma_2)$ are not in Φ and all edges in Σ_2 are negative.

Case 2 Suppose $p_1\geqslant 3$ and $\exists\, p_i=2$ for $i\in\{2,3,\cdots,k\}$. It is not difficult to see that Σ is connected. The vertex $(0,0)$ is adjacent to all members of Φ, and each member of Φ is adjacent to other $|\Phi|-1$ vertices of Σ. Moreover, those edges have at least one vertex in Φ. So there are $|\Phi|+|\Phi|(|\Phi|-1)=|\Phi|^2$ positive edges. Let

$$V_1=\{(u,v): u=0 \text{ and } v \text{ is even}\}\cup\{(u,v): u\in\{1,2,\cdots,p_1-1\} \text{ and } v \text{ is not odd-multiple of prime factors}\};$$

$$V_2=\{(u,v): u=0 \text{ and } v \text{ is odd}\}\cup\{(u,v): u\in\{1,2,\cdots,p_1-1\} \text{ and } v \text{ is odd-multiple of prime factors}\}.$$

Let $V=V_1\cup V_2$. Assume (u,v) and (u',v') are arbitrary vertices of V_2, the sum of v and v' is even since v and v' are both odd, hence $(u,v)+(u',v')\notin\Phi$. Thus none of V_2 are adjacent to each other. Each vertex of V_2 is adjacent to one of the vertices of V_1 since Σ is connected.

Let $(u,v)\in V_1$ be adjacent to $(u',v')\in V_2$. v should be even since v' is odd. We have $(u,v)\notin \Phi$ and all vertices of V_2 are not in Φ. All edges incident with the vertices of V_2 are negative and the number is

$$|V_2|\times|\Phi|=\frac{|V(\Sigma)|}{2}|\Phi|-|\Phi|^2,$$

where $|V(\Sigma)|=p_1^{\alpha_1+1}p_2^{\alpha_2}\cdots p_k^{\alpha_k}$.

The following theorem gives an equivalent characterization for Σ being balanced.

Theorem 1 Let $\Sigma=(G,\sigma)$ be an addition Cayley signed graph, where $G=\mathrm{Cay}^+(Z_{p_1}\times Z_{p_1^{\alpha_1}p_2^{\alpha_2}\cdots p_k^{\alpha_k}},\Phi)$ and all $p_1,p_2,\cdots,p_k$ are distinct prime factors. Then Σ is balanced if and only if one of the prime factors is 2.

Proof

Necessity: Contrarily, assume $p_i\geqslant 3$ for any $i=1,2,\cdots,k$. In this case, Σ is connected. Moreover, the numbers 1 and 2 are both in $\varphi(p_1)$ and $\varphi(p_1^{\alpha_1}p_2^{\alpha_2}\cdots p_k^{\alpha_k})$. Let us consider a cycle $C=((0,1),(1,0),(1,1),(0,1))$ in Σ. In fact, $(0,1)+(1,0)=(1,1)\in\Phi$, the vertex $(0,1)$ is adjacent to $(1,0)$. Also, $(1,0)$ is adjacent to $(1,1)$ since $(1,0)+(1,1)=(2,1)\in\Phi$. The vertex $(1,1)$ is adjacent to $(0,1)$ since $(1,1)+(0,1)=(1,2)\in\Phi$. Note that $(0,1)\notin\Phi$, $[(1,0)\notin\Phi]$ and $(1,1)\in\Phi$, we have $\sigma(\{(0,1),(1,0)\})=-1$, $\sigma(\{(1,0),(1,1)\})=+1$, $\sigma(\{(1,1),(0,1)\})=+1$. Then the cycle C is not positive and thus Σ is unbalanced. This is a contradiction.

Sufficiency: Suppose one of the prime factors is 2.

Case 1 Assume $p_1=2$. Then by the proof of Lemma 4 it is known that Σ is disconnected and has exactly two connected components $\Sigma_1=(G_1,\sigma)$ with vertex set $V(\Sigma_1)=\{(u,v)$: $u=1$ and v is odd$\}\cup\{(u,v)$: $u=0$ and v is even$\}$ and $\Sigma_2=(G_2,\sigma)$ with vertex set $V(\Sigma_2)=\{(u,v)$: $u=0$ and v is odd$\}\cup\{(u,v)$: $u=1$ and v is even$\}$.

For Σ_2, all edges are negative since $V(\Sigma_2)\cap\Phi=\varnothing$. Next we prove that every cycle in Σ_2 is positive. Suppose $C'=(x_1,x_2,x_3,\cdots,x_m,x_1)$ is an arbitrary cycle in Σ_2. Without loss of generality, assume that $x_1\in\{(u,v)$: $u=0$ and v is odd$\}$. By Definition 1, we have x_2 and x_m both in $\{(u,v)$: $u=1$ and v is even$\}$. Similarly, the vertices that are adjacent to x_2 and x_m are in $\{(u,v)$: $u=0$ and v is odd$\}$. By continuing this process, it is easy to see that C' contains an even number of negative edges. Thus Σ_2 is balanced.

For Σ_1, let us consider an arbitrary cycle $C''=(x_1',x_2',x_3',\cdots,x_m',x_1')$. Note that every edge in Σ_1 has one vertex in $\{(u,v)$: $u=0$ and v is even$\}$ and the other in $\{(u,v)$: $u=1$ and v is odd$\}$. Without loss of generality, assume that $x_1'\in\{(u,v)$: $u=0$ and v is even$\}$ then $x_2'\in\{(u,v)$: $u=1$ and v is odd$\}$ and $x_3'\in\{(u,v)$: $u=0$ and v is even$\}$. Obviously, $\{(u,v)$: $u=0$ and v is even$\}\cap\Phi=\varnothing$. There are two following cases. If $x_2'\in\Phi$, then $\{x_1',x_2'\}$ and $\{x_2',x_3'\}$ will be positive edges. If $x_2'\notin\Phi$, then $\{x_1',x_2'\}$ and $\{x_2',x_3'\}$ will be negative, which implies that the number of negative edges is even in C''. Thus C'' is positive and Σ_1 is balanced.

Case 2 Assume $p_1\geqslant 3$ and $\exists\, p_i=2$ for $i\in\{2,3,\cdots,k\}$. Now Σ is connected. Let $V(\Sigma)=V_1\cup V_2$, where

$V_1=\{(u,v)$: $u=0$ and v is even$\}\cup\{(u,v)$: $u\in\{1,2,\cdots,p_1-1\}$ and v is not odd-multiple of prime factors$\}$;

$V_2=\{(u,v)$: $u=0$ and v is odd$\}\cup\{(u,v)$: $u\in\{1,2,\cdots,p_1-1\}$ and v is odd-multiple of prime factors$\}$.

Suppose C''' is an arbitrary cycle in Σ. If all edges of C''' are positive, then C''' is positive. If C''' contains a negative edge $\{(u,v),(u',v')\}$, then this edge has one vertex in V_1 and the other in V_2 from the proof of Lemma 4. Without loss of generality, assume that $(u,v)\in V_1$ and $(u',v')\in V_2$. Next, we will prove that another negative edge which associates to (u',v') is required. Let above edge be $\{(u',v'),(u'',v'')\}$. Neither of the vertices of V_2 is adjacent to each other, $(u'',v'')\in V_1$ since Σ is connected. $V(\Sigma_2)\cap\Phi=\varnothing$ and v' is odd. v'' is even and (u'',v'') is not in Φ. Therefore $\{(u',v'),(u'',v'')\}$ is negative, and the negative occurs in pairs. It is easy to conclude that C''' is positive. Thus Σ is balanced.

Remark 1 If $G=\mathrm{Cay}^+(Z_2\times Z_2,\Phi)$, then $\Sigma=(G,\sigma)$ does not contain cycles. However, Σ is balanced. (In fact, Σ is disconnected and has exactly two connected components, say Σ_1 having vertex set $V(\Sigma_1)=\{(0,0),(1,1)\}$ and Σ_2 having vertex set $V(\Sigma_2)=\{(0,1),(1,0)\}$. For Σ_1, by definition of signature function σ, we have $\sigma(\{(0,0),(1,1)\})=+1$ since $(1,1)\in\Phi$, $(0,0)\notin\Phi$ and $(0,0)$ is adjacent to $(1,1)$. For Σ_2, neither of $V(\Sigma_2)$ is in Φ and $(0,1)$ is adjacent to $(1,0)$, hence $\sigma(\{(0,1),(1,0)\})=-1$. The tree has no cycle and we can know that each tree is balanced according to the proof of Lemma 3. Σ is balanced since it is the union of two balanced trees.)

Remark 2 If $G=\mathrm{Cay}^+(Z_2\times Z_2,\Phi)$, then it follows from Remark 1 that $\eta(\Sigma)$ is balanced. Also, $L(\Sigma)$ is an empty graph and $C_E(\Sigma)$ is a null graph, therefore they all can be considered as balanced (see Fig. 6). Hence in the below proof, this case will be omitted and not considered.

Next, the balance of three derived graphs $\eta(\Sigma)$,

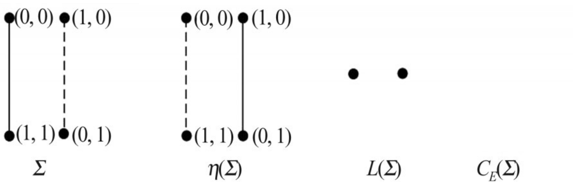

Fig. 6 $\boldsymbol{\Sigma}=(\mathbf{Cay}^{+}(\boldsymbol{Z}_2\times\boldsymbol{Z}_2,\boldsymbol{\Phi}),\sigma)$ **and its three derived graphs**

$L(\Sigma)$ and $C_E(\Sigma)$ is discussed.

Lemma 5[5] G_n is bipartite if and only if either $n=3$ or n is even.

It is known that the bipartite graph contains no odd cycle. So we have

Corollary 1 The negation of balanced bipartite signed graph is always balanced.

Lemma 6[16] For an integer n, if $i\in\varphi(n)$, then $(n-i)\in\varphi(n)$ and if $i\notin\varphi(n)$ then $(n-i)\notin\varphi(n)$.

Theorem 2 Let $\Sigma=(G,\sigma)$ be an addition Cayley signed graph, where $G=\mathrm{Cay}^{+}(Z_{p_1}\times Z_{p_1^{\alpha_1}p_2^{\alpha_2}\cdots p_k^{\alpha_k}},\ \Phi)$ and all $p_1,p_2,\cdots,p_k$ are distinct prime factors. Then $\eta(\Sigma)$ is balanced if and only if one of the following conditions holds:

(i) one of the prime factors is 2;

(ii) $G=\mathrm{Cay}^{+}(Z_3\times Z_3,\Phi)$.

Proof

Necessity: Contrarily, assume that $p_i\geqslant 3$ for any $i=1,2,\cdots,k$ and $\alpha_i\geqslant 1$ except for $G=\mathrm{Cay}^{+}(Z_3\times Z_3,\Phi)$. The numbers 1 and 2 are both in $\varphi(p_1)$ and $\varphi(p_1^{\alpha_1}p_2^{\alpha_2}\cdots p_k^{\alpha_k})$. Let $n_1=p_1^{\alpha_1}p_2^{\alpha_2}\cdots p_k^{\alpha_k}$, then we have n_1-1, $n_1-2\in\varphi(n_1)$ using Lemma 6. Now let us consider a cycle $C=((0,0),(1,n_1-2),(1,1),(0,0))$ in Σ. It is not difficult to see that C is all-positive and $\eta(C)$ is all-negative. Thus $\eta(\Sigma)$ is unbalanced. This is a contradiction.

Sufficiency:

Case 1 Suppose one of the prime factors is 2. In this case, Σ is balanced according to Theorem 1. The order of Σ is even and Σ is bipartite using Lemma 5. Applying Corollary 1, $\eta(\Sigma)$ is balanced.

Case 2 Suppose $G=\mathrm{Cay}^{+}(Z_3\times Z_3,\Phi)$. Obviously, $\eta(\Sigma)$ is balanced (see Fig. 7).

Lemma 7[17] Σ is balanced if and only if it switches to an all-positive signature, and it is unbalanced if and only if it switches to an all-negative signature.

Lemma 8[18] For a sigraph Σ, $L(\Sigma)$ is balanced if and only if the following conditions hold:

(i) for any cycle Z in Σ,

(a) if Z is all-negative, then Z has even length;

(b) if Z is heterogeneous, then Z has an even number of negative sections with even length;

(ii) for $v\in V(\Sigma)$, if $d(v)>2$, then there is at most one negative edge incident at v in Σ.

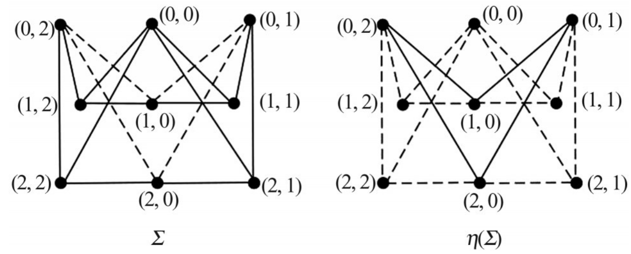

Fig. 7 $\boldsymbol{\Sigma}=(\mathbf{Cay}^{+}(\boldsymbol{Z}_3\times\boldsymbol{Z}_3,\boldsymbol{\Phi}),\sigma)$ **and its derived graph** $\boldsymbol{\eta}(\boldsymbol{\Sigma})$

Theorem 3 Let $\Sigma=(G,\sigma)$ be an addition Cayley signed graph, where $G=\mathrm{Cay}^{+}(Z_{p_1}\times Z_{p_1^{\alpha_1}p_2^{\alpha_2}\cdots p_k^{\alpha_k}},\Phi)$ and all $p_1,p_2,\cdots,p_k$ are distinct prime factors. Then $L(\Sigma)$ is balanced if and only if Σ is balanced.

Proof

Necessity: Contrarily, assume that Σ is unbalanced. By Theorem 1, we have $p_i\geqslant 3$ for any $i=1,2,\cdots,k$. In this case, we have $d(v)\geqslant|\Phi|-1$ which is greater than 2 for all vertices. Note that $(0,1)$ is adjacent to vertices $(1,0),(2,0),\cdots,(p_1-1,0)$ respectively. None of these p_1 vertices is in Φ, hence $d^{-}((0,1))\geqslant p_1-1$. $p_1\geqslant 3$. It follows from Lemma 8 that $L(\Sigma)$ is unbalanced. This is a contradiction.

Sufficiency: Assume that Σ is balanced. According to Lemma 7, Σ can be considered all-positive. Therefore $L(\Sigma)$ is all-positive and balanced.

Lemma 9[8] For any sigraph Σ, $C_E(\Sigma)$ is balanced if and only if Σ is a balanced sigraph such that for every vertex $v\in V(\Sigma)$ with $d(v)\geqslant 3$,

(i) if $d(v)>3$ then $d^{-}(v)=0$;

(ii) if $d(v)=3$ then $d^{-}(v)=0$ or $d^{-}(v)=2$; and

(iii) for every path $P_4=(x,v,w,y)$ of length three, $\{v,w\}$ is a positive edge in Σ.

Theorem 4 Let $\Sigma=(G,\sigma)$ be an addition Cayley signed graph, where $G=\mathrm{Cay}^{+}(Z_{p_1}\times Z_{p_1^{\alpha_1}p_2^{\alpha_2}\cdots p_k^{\alpha_k}},\Phi)$ and all $p_1,p_2,\cdots,p_k$ are distinct prime factors. Then $C_E(\Sigma)$ is balanced if and only if one of the following conditions holds:

(i) $|\Phi|=1$, i.e. $G=\mathrm{Cay}^{+}(Z_2\times Z_2,\Phi)$;

(ii) $|\Phi|=2$, i. e. $G=\mathrm{Cay}^{+}(Z_2\times Z_{2^2},\Phi)$ or $G=\mathrm{Cay}^{+}(Z_2\times Z_{2\times 3},\Phi)$.

Proof

Necessity: Contrarily, assume that $G=\mathrm{Cay}^{+}(Z_{p_1}\times Z_{p_1^{\alpha_1}p_2^{\alpha_2}\cdots p_k^{\alpha_k}},\Phi)$ and $|\Phi|\geqslant 3$. Now there are the following three cases.

Case 1 Suppose $p_1=2$. Clearly, Σ is disconnected and has two connected components $\Sigma_1=(G_1,\sigma)$ having vertex set $V(\Sigma_1)=\{(u,v)$: $u=1$ and v is odd$\}\cup\{(u,v)$: $u=0$ and v is even$\}$ and $\Sigma_2=(G_2,\sigma)$ having vertex set $V(\Sigma_2)=\{(u,v)$: $u=0$ and v is odd$\}\cup\{(u,v)$: $u=1$ and v is even$\}$.

According to the proof of Theorem 1, Σ_1 is balanced. By Lemma 7, Σ_1 can be considered all-positive, thus $C_E(\Sigma_1)$ is balanced. For Σ_2, the degrees of all members in $V(\Sigma_2)$ are negative and greater than 3 since Σ_2 is all-negative. Thus the condition (i) and (ii) of Lemma 9 does not hold for Σ_2, which implies that $C_E(\Sigma)$ is unbalanced. This is a contradiction.

Case 2 Suppose $p_1\geqslant 3$ and $\exists\, p_i=2$, for $i=2, 3, \cdots, k$. In this case, $|\Phi|>3$, Σ is a connected and $|\Phi|$-regular graph. Let $V(\Sigma)=V_1\cup V_2$, where

$V_1=\{(u,v)$: $u=0$ and v is even$\}\cup\{(u,v)$: $u\in\{1,2,\cdots,p_1-1\}$ and v is not odd-multiple of prime factors$\}$;

$V_2=\{(u,v)$: $u=0$ and v is odd$\}\cup\{(u,v)$: $u\in\{1,2,\cdots,p_1-1\}$ and v is odd-multiple of prime factors$\}$.

It follows from the proof of Lemma 4 that any two vertices in V_2 are not adjacent. Then there exists an edge $\{(u,v), (u',v')\}$ whose one vertex in V_1 and the other in V_2 since Σ is connected. Note that $d((u',v'))=|\Phi|>3$, and by the proof of Lemma 4, $\{(u,v), (u',v')\}$ is negative. Therefore, it follows from Lemma 9 (i) that $C_E(\Sigma)$ is unbalanced. This a contradiction.

Case 3 Suppose that $p_i\geqslant 3$ for all $i=1,2,\cdots,k$. In this case, $2\in\varphi(p_1)$ and $\varphi(p_1^{\alpha_1}p_2^{\alpha_2}\cdots p_k^{\alpha_k})$ and $|\Phi|\geqslant 4$. Then $d(v)\geqslant 3$ for any $v\in V(\Sigma)$. Note that the vertex (0, 1) is adjacent to (1, 0), (2, 0), $\cdots$, $(p_1-1, 0)$ and all p_1-1 edges are negative. Also, $d((0,1))=|\Phi|\geqslant 4$ since $(0,1)\notin\Phi$. Thus the condition (i) of Lemma 9 does not hold for (0, 1), which implies that $C_E(\Sigma)$ is unbalanced. This is a contradiction.

Sufficiency: It can be concluded that $C_E(\Sigma)$ is balanced in the following three cases.

Case 1 Let $G=\mathrm{Cay}^+(Z_2\times Z_2,\Phi)$. In this case, $|\Phi|=1$. According to Remark 2, $C_E(\Sigma)$ is balanced.

Case 2 Let $G=\mathrm{Cay}^+(Z_2\times Z_{2\times 3},\Phi)$ (see Fig. 3). In this case, $|\Phi|=2$. Obviously, $C_E(\Sigma)$ is balanced (see Fig. 8).

Case 3 Let $G=\mathrm{Cay}^+(Z_2\times Z_{2^2},\Phi)$. In this case, $|\Phi|=2$. From Theorem 1, Σ is balanced, and Σ can be considered all-positive according to Lemma 7. Therefore $C_E(\Sigma)$ is balanced.

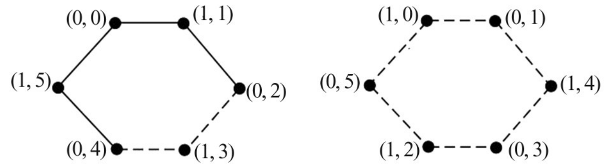

Fig. 8 The derived graph $C_E(\Sigma)$ of $\Sigma=(\mathrm{Cay}^+(Z_2\times Z_{2\times 3},\Phi,\sigma)$

3 Clusterability and Sign-Compatibility of Σ

In this section, we give a necessary and sufficient condition for Σ being clusterable and a sufficient condition for Σ being sign-compatible.

Theorem 5 Let $\Sigma=(G,\sigma)$ be an addition Cayley signed graph, where $G=\mathrm{Cay}^+(Z_{p_1}\times Z_{p_1^{\alpha_1}p_2^{\alpha_2}\cdots p_k^{\alpha_k}},\Phi)$ and all $p_1,p_2,\cdots,p_k$ are distinct prime factors. Then Σ is clusterable if and only if Σ is balanced.

Proof

Necessity: Contrarily, assume that Σ is unbalanced. From Theorem 1, we have $p_i\geqslant 3$ for any $i=1,2,\cdots,k$. Let us consider a cycle $C=((0,1), (1,0), (1,1), (0,1))$. It follows from Lemma 1 that Σ is not clust- erable since the cycle C contains only one negative edge $\{(0,1), (1,0)\}$. This is a contradiction.

Sufficiency: Let Σ be balanced. Then every cycle in Σ is positive. This implies every cycle of Σ has either all positive edges or an even number of negative edges. Thus the required result can be concluded from Lemma 1.

Theorem 6 Let $\Sigma=(G,\sigma)$ be an addition Cayley signed graph, where $G=\mathrm{Cay}^+(Z_{p_1}\times Z_{p_1^{\alpha_1}p_2^{\alpha_2}\cdots p_k^{\alpha_k}},\Phi)$ and all $p_1,p_2,\cdots,p_k$ are distinct prime factors. Then Σ is always sign-compatible.

Proof Contrarily, assume that Σ is not signcompatible. According to Lemma 2, Σ contains a sub-sigraph isomorphic to either S_1 or S_2 (see Fig. 5).

Case 1 Suppose that Σ contains a sub-sigraph isomophic to S_1. Denote by $P_4=((x_1,y_1), (x_2,y_2), (x_3,y_3), (x_4,y_4))$ the path with $\sigma(\{(x_1,y_1), (x_2,y_2)\})=-1$, $\sigma(\{(x_2,y_2), (x_3,y_3)\})=+1$ and $\sigma(\{(x_3,y_3), (x_4,y_4)\})=-1$. Notice that the vertices (x_2,y_2) and (x_3,y_3) can not be in Φ at the same time, otherwise P_4 is all-positive. Without loss of generality, let $(x_2,y_2)\in\Phi$ and $(x_3,y_3)\notin\Phi$. Then $\sigma(\{(x_1,y_1), (x_2,y_2)\})=+1$ and $\sigma(\{(x_2,y_2), (x_3,y_3)\})=+1$ which is contradictary to the choice of P_4. Thus Σ can not contain a sub-sigraph isomorphic to S_1.

Case 2 Assume that Σ contains a sub-sigraph isomorphic to S_2. In this case, the required result can be concluded completely similar to Case 1 since we need only consider $(x_1, y_1)=(x_4, y_4)$.

Therefore Σ is always sign-compatible.

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