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Discrete Morse Theory on Join of Digraphs

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Abstract: For given two digraphs, we can construct a larger digraph through join. The two digraphs that make up the join are called the factors of the join. In this paper, we give a necessary and sufficient condition that the function on the join determined by the discrete Morse functions on factors is a discrete Morse function. Moreover, we further prove the discrete Morse theory on join when the factors satisfy certain conditions.

Key words: path homology; transitive closure; digraph; join

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0 Introduction

Digraphs are important topological models in complex networks and the homology groups of digraphs can reveal the topological and geometric characteristics of complex networks. Among many approaches for constructing (co) homology of digraphs[1, 2], a natural and important homology theory on digraphs is path homology introduced by Grigor'yan *et al*, which theory has been systematically developed with fruitful results[3-9]. In this paper, we consider the discrete Morse theory based on the path homology of digraphs.

Let $G=(V(G),E(G))$ be a digraph where $V(G)$ is the vertex set of G and $E(G)$ is the directed edge set of G. For any directed edge $(u,v)\in E(G)$, it can be denoted as $u\to v$. G is called transitive, if $u\to v$ and $v\to w$ are two directed edges of G, then $u\to w$ is a directed edge of G. The smallest transitive digraph containing G is called the transitive closure of G, denoted as $\bar{G}$.

Let R be a commutative ring with unit. An elementary n-path is a sequence $v_0v_1\cdots v_n$ of $n+1$ vertices in $V(G)$, $n\geqslant 0$. Let $\Lambda_n(V(G))$ be the R-module consisting of all the formal linear combinations of the n-paths on $V(G)$. The boundary map

$$\partial_n\colon \Lambda_n(V(G))\to\Lambda_{n-1}(V(G))$$

is defined as

$$\partial_n(v_0v_1\cdots v_n)=\sum_{i=0}^{n}(-1)^i d_i(v_0v_1\cdots v_n)$$

where d_i is the i-th face map

$$d_i\colon \Lambda_n(V(G))\to\Lambda_{n-1}(V(G))$$

such that

$$d_i(v_0v_1\cdots v_n)=v_0v_1\cdots\hat{v}_i\cdots v_n.$$

Here $\hat{v}_i$ means omission of the vertex v_i. Then

$\partial_n\partial_{n+1}=0$ for each $n\geqslant 0$. Hence, $\{\Lambda_n(V(G)),\partial_n\}_{n\geqslant 0}$ is a chain complex, simply denoted as $\Lambda(V(G))$ if there is no ambiguity.

An allowed elementary n-path is an elementary n-path $v_0\cdots v_i v_{i+1}\cdots v_n$ such that $v_i\neq v_{i+1}$ and $v_i\to v_{i+1}$ is a directed edge of G for each $0\leqslant i\leqslant n-1$. Let $P_n(G)$ be the free R-module consisting of all the formal linear combinations of allowed elementary n-paths on G. Then $P_n(G)$ is a submodule of $\Lambda_n(V(G))$. However, in general, $\{P_n(G),\partial_n\}_{n\geqslant 0}$ is not a subchain complex of $\Lambda(V(G))$. Consider the sub-R-module

$$\Omega_n(G)=\{x\in P_n(G)|\partial x\in P_{n-1}(G)\}$$

of $P_n(G)$. Then $\partial_n\Omega_n(G)\subseteq\Omega_{n-1}(G)$. The path homology groups of G are defined as the homology groups of chain complex $\{\Omega_n(G),\partial_n\}$, denoted as $H_*(G;R)$.

In topological data analysis, we are concerned with the calculation and simplification of homology groups. Morse theory is just an important tool to simplify the calculation of homology groups. In 1998, Forman[10] extended Morse theory on smooth manifolds to cell complexes and simplicial complexes. And based on this study, Forman studied discrete Morse theory, cohomology card product, Witten Morse theory and other related problems[11-13]. From 2007 to 2009, Ayala *et al*[14-17] studied the discrete Morse theory on graphs by using the discrete Morse theory of cell complexes and simplicial complexes given by Forman. Inspired by these, we studied the discrete Morse theory on digraphs[18-20].

In this paper, we study the discrete Morse theory on join of digraphs, hoping to give the discrete Morse theory of join by requiring the two factors constituting the connection to meet certain conditions, rather than directly limiting the join.

Let G_1 and G_2 be two digraphs where $G_1=(V(G_1),E(G_1))$ and $G_2=(V(G_2),E(G_2))$. Suppose $V(G_1)$ and $V(G_2)$ are disjoint. Then $E(G_1)$ and $E(G_2)$ are disjoint as well. The join of G_1 and G_2 is a digraph $G=G_1{*}G_2$ such that

• the set of vertices of the digraph G is $V(G_1)\cup V(G_2)$;

• the set of directed edges of the digraph G is $E(G_1)\cup E(G_2)\cup\{(u,v)|u\in V(G_1),v\in V(G_2)\}$.

The paper is organized as follows. In Section 1, we review the definition and some properties of discrete Morse functions on digraphs. In Section 2, we give some auxiliary results for main theorems in Lemma 5 and Lemma 8. Finally, in Section 3, we prove the main theorems of this paper.

Let G be a digraph and $f{:}V(G)\to[0,+\infty)$ a discrete Morse function on G as defined in Definition 1. Define an R-linear map grad$f$: $P_n(G)\to P_{n+1}(G)$ such that for any allowed elementary n-path α on G,

$$(\mathrm{grad}f)(\alpha)=-<\partial\gamma,\alpha>\gamma,$$

where $\gamma>\alpha$ and $f(\gamma)=f(\alpha)$. Otherwise, $(\mathrm{grad}f)(\alpha)=0$[10]. We call grad$f$ the (algebraic) discrete gradient vector field of f on G, denoted as V_f, abbreviated as V. The discrete gradient flow is defined as

$$\Phi=\mathrm{Id}+\partial V+V\partial$$

Here Id is the identity map from $P_n(G)$ to $P_n(G)$. Correspondingly, the discrete Morse function, the (algebraic) discrete gradient vector field and the discrete gradient flow of a transitive digraph are denoted as $\bar{f}$, $\bar{V}$ and $\bar{\Phi}$, respectively.

The main theorems of this paper are as follows.

Theorem 1 Let $G=G_1{*}G_2$ and f_1, f_2 discrete Morse functions on G_1 and G_2, respectively. Let f be the discrete Morse function on G determined by f_1 and f_2. Suppose $\Omega(G_i)$ is $\bar{V}_i$-invariant, $i=1,2$. Then

$$H_m(G;R)\cong H_m(\Omega_*(G)\cap P_*^{\bar{\Phi}}(\bar{G})), m\geqslant 0$$

where $\{P_*^{\bar{\Phi}}(\bar{G}),\partial_*\}$ is the subchain complex of $\{P_*(\bar{G}),\partial_*\}$ consisting of all $\bar{\Phi}$-invariant chains.

Denote Crit$_n(G)$ as the free R-module consisting of all the formal linear combinations of critical n-paths on G. We have that

Theorem 2 Let $G=G_1{*}G_2$. Let f_1 be a function on G_1 such that $f_1(v)>0$ for each vertex $v\in V(G_1)$ and f_2 a discrete Morse function on G_2 with a unique zero-point. Let f be the discrete Morse function on G determined by f_1 and f_2. Suppose $\Omega(G_i)$ is $\bar{V}_i$-invariant, $\bar{\Phi}_i(\alpha_i)\in\Omega(G_i)$ for any $\alpha_i\in\mathrm{Crit}(\bar{G}_i)\cap P_n(G)$ and $\mathrm{Crit}(\bar{G}_2)\cap P(G_2)=\mathrm{Crit}(\bar{G}_2)\cap\Omega(G_2)$, $i=1,2$. Then

$$H_m(\{\mathrm{Crit}_n(\bar{G})\cap P_n(G),\tilde{\partial}_n\}_{n\geqslant 0})\cong H_m(G;R)$$

where $\tilde{\partial}=(\bar{\Phi}^\infty)^{-1}\circ\partial\circ\bar{\Phi}^\infty$ and $\bar{\Phi}^\infty$ is stabilization map of $\bar{\Phi}$.

1 Preliminaries

In this section, we mainly review the definition and properties of discrete Morse functions on digraphs.

For any allowed elementary paths α and β, if β can be obtained from α by removing some vertices, then we write $\alpha>\beta$ or $\beta<\alpha$.

Definition 1[20] A map f: $V(G)\to[0,+\infty)$ is called a discrete Morse function on G, if for any allowed elementary path $\alpha = v_0 v_1 \cdots v_n$ on G, both of the followings hold:

(i) $\#\{\gamma^{(n+1)} > \alpha^{(n)} | f(\gamma) = f(\alpha)\} \leqslant 1$;

(ii) $\#\{\beta^{(n-1)} < \alpha^{(n)} | f(\beta) = f(\alpha)\} \leqslant 1$

where

$$f(\alpha) = f(v_0 v_1 \cdots v_n) = \sum_{i=0}^{n} f(v_i).$$

For an allowed elementary path α, if in both (i) and (ii), the inequalities hold strictly, then α is called critical. Precisely,

Definition 2 An allowed elementary n-path $\alpha^{(n)}$ is called critical, if both of the followings hold:

(i)′ $\#\{\gamma^{(n+1)} > \alpha^{(n)} | f(\gamma) = f(\alpha)\} = 0$;

(ii)′ $\#\{\beta^{(n-1)} < \alpha^{(n)} | f(\beta) = f(\alpha)\} = 0$.

It follows from Definition 2 that an allowed elementary p-path is not critical if and only if either of the following conditions holds

(i)″ there exists $\beta^{(n-1)} < \alpha^{(n)}$ such that $f(\beta) = f(\alpha)$;

(ii)″ there exists $\gamma^{(n+1)} > \alpha^{(n)}$ such that $f(\gamma) = f(\alpha)$.

A directed loop on G is an allowed elementary path $v_0 v_1 \cdots v_n v_0$, $n \geqslant 1$.

Lemma 1[18, Lemma 2.4] Let G be a digraph and f a discrete Morse function on G. Let $\alpha = v_0 v_1 \cdots v_n v_0$ be a directed loop. Then for each $0 \leqslant i \leqslant n, f(v_i) > 0$.

Lemma 2[18, Lemma 2.5] Let G be a digraph and f a discrete Morse function on G as defined in Definition 1. Then for any allowed elementary path on G, there exists at most one index such that the value of the corresponding vertex is zero.

Lemma 3[19, Lemma 2.5] Let f be a discrete Morse function on digraph G. Then for any allowed elementary path $\alpha = v_0 v_1 \cdots v_n$ in G, (i)″ and (ii)″ can not both be true.

2 Auxiliary Results for Main Theorems

Let G_1 and G_2 be two digraphs. Suppose $V(G_1)$ and $V(G_2)$ are disjoint. Then $E(G_1)$ and $E(G_2)$ are disjoint as well. The join of G_1 and G_2 is a digraph $G = G_1 * G_2$ such that

• the set of vertices of the digraph G is $V(G_1) \cup V(G_2)$;

• the set of directed edges of the digraph G is $E(G_1) \cup E(G_2) \cup \{(u, v) | u \in V(G_1), v \in V(G_2)\}$.

2.1 Discrete Morse Functions on the Join of Digraphs

In this subsection, we will give a necessary and sufficient condition that the function on the join determined by the discrete Morse functions on factors is a discrete Morse function.

Firstly, we prove that discrete Morse functions on the join of digraphs have the following important property.

Lemma 4 Let f be a discrete Morse function on $G = G_1 * G_2$. Then there exists at most one zero-point of f on G.

Proof Suppose to the contrary, there are two distinct vertices $v', v'' \in V(G)$ such that $f(v') = f(v'') = 0$. Then by the definition of join of digraphs, there are three cases to be considered.

Case 1 Both of v', v'' are in $V(G_1)$. Then for any allowed elementary path $\alpha = v_0 \cdots v_t$ on G_2, we have that

$$\alpha' = v' v_0 \cdots v_t$$

and

$$\alpha'' = v'' v_0 \cdots v_t$$

are two distinct allowed elementary $(p+1)$-paths on G such that $f(\alpha') = f(\alpha'') = f(\alpha)$. This contradicts that f is a discrete Morse function on G.

Case 2 Both of v', v'' are in $V(G_2)$. Similar to Case 1, for any allowed elementary path $\alpha = v_0 \cdots v_s$ on G_1,

$$\alpha' = v_0 \cdots v_t v'$$

and

$$\alpha'' = v_0 \cdots v_s v''$$

are two distinct allowed elementary $(p+1)$-paths on G such that $f(\alpha') = f(\alpha'') = f(\alpha)$ which contradicts that f is a discrete Morse function on G.

Case 3 $v' \in V(G_1)$ and $v'' \in V(G_2)$. Then $f(v'v'') = f(v') = f(v'')$. This also contradicts that f is a discrete Morse function on G.

Combining Case 1, Case 2 and Case 3, the lemma is proved.

Secondly, define a function f

$$f(v) = \begin{cases} f_1(v), & v \in V_1 \\ f_2(v), & v \in V_2 \end{cases} \tag{1}$$

on $G = G_1 * G_2$, where f_1 and f_2 are functions on G_1 and G_2, respectively. Then by Lemma 4 and (1), we have that

Lemma 5 f is a discrete Morse function on $G = G_1 * G_2$ if and only if there exist discrete Morse functions f_1 on G_1 and f_2 on G_2 respectively such that $f = f_1 * f_2$ and one of f_1 and f_2 is positive while the other one has at most one zero-point.

Proof ($\Rightarrow$) Let $f_1 = f\big|_{G_1}$ and $f_2 = f\big|_{G_2}$. Then by (1), $f = f_1 * f_2$ and by Definition 1, f_1, f_2 are discrete Morse functions on G_1 and G_2, respectively. Moreover, by Lemma 4, one of f_1 and f_2 is positive and the other one has at most one zero-point.

($\Leftarrow$) Without loss of generality, f_1 is a positive function on G_1 and f_2 is nonnegative on G_2. Let α be an arbitrary allowed elementary n-path on G. Then by Ref.[3, Proposition 6.4],

$$\alpha = \alpha_1 * \alpha_2$$

where $\alpha_1 = v_0 \cdots v_s \in P(G_1)$, $\alpha_2 = w_0 \cdots w_t \in P(G_2)$, $s+t+1=n$. Consider the following cases:

Case 1 $t \geqslant 0$. Suppose β_1 and β_2 are two allowed elementary paths on G such that $\beta_1 < \alpha$, $\beta_2 < \alpha$ and $f(\beta_1) = f(\beta_2) = f(\alpha)$. Since $f_1(v) > 0$ for each $v \in V(G_1)$, it follows that $f_1(v_i) > 0$ for $0 \leqslant i \leqslant s$. Then there are two indices $0 \leqslant j \neq k \leqslant t$ such that $f_2(w_j) = f_2(w_k) = 0$. This contradicts Lemma 2. Therefore,

$$\#\{\beta^{(n-1)} < \alpha^{(n)} | f(\beta) = f(\alpha)\} \leqslant 1.$$

Suppose γ_1 and γ_2 are two allowed elementary paths on G such that $\gamma_1 > \alpha$, $\gamma_2 > \alpha$ and $f(\gamma_1) = f(\gamma_2) = f(\alpha)$. Since $f_1(v) > 0$ for each $v \in V(G_1)$, it follows that

$$\gamma_1 = \alpha_1 * \alpha_2'$$

and

$$\gamma_2 = \alpha_1 * \alpha_2''$$

where $\alpha_2' = w_0 \cdots w_i u w_{i+1} \cdots w_t$, $\alpha_2'' = w_0 \cdots w_j w w_{j+1} \cdots w_t$ and $f_2(u) = f_2(w) = 0$. Since there exists at most one zero-point of f_2, it follows that $u = w$ and $i \neq j$. Without loss of generality, $i < j$. Thus, there exists a directed loop

$$w w_{i+1} \cdots w_j w$$

on G_2 with $f_2(w) = 0$. This contradicts Lemma 1. Therefore,

$$\#\{\gamma^{(n+1)} > \alpha^{(n)} | f(\gamma) = f(\alpha)\} \leqslant 1.$$

Case 2 $t = -1$. Then for any allowed elementary n-path α on G,

$$\alpha = \alpha_1$$

where $\alpha_1 = v_0 \cdots v_n \in P(G_1)$. Obviously, since $f_1(v) > 0$ for all vertices $v \in V(G_1)$, it follows that $f(\beta) < f(\alpha)$ for any allowed elementary path $\beta < \alpha$. Hence,

$$\#\{\beta^{(n-1)} < \alpha^{(n)} | f(\beta) = f(\alpha)\} = 0.$$

Moreover, since there exists at most one vertex $w \in P(G_2)$ such that $f_2(w) = 0$, it follows that there exists at most one allowed elementary path

$$\gamma = v_0 \cdots v_n w$$

such that $\gamma > \alpha$ and $f(\gamma) = f(\alpha)$. Hence,

$$\#\{\gamma^{(n+1)} > \alpha^{(n)} | f(\gamma) = f(\alpha)\} \leqslant 1.$$

Summarizing Case 1 and Case 2, due to the arbitrariness of α, f is a discrete Morse function on G.

Therefore, the lemma is proved.

Moreover, we have

Theorem 3[18, Theorem 2.12] Let G be a digraph and $f: V(G) \to [0, +\infty)$ be a discrete Morse function on G. Then f can be extended to be a Morse function $\bar{f}$ on $\bar{G}$ such that $\bar{f}(v) = f(v)$ for each vertex $v \in V(G)$ if and only if the following condition (*) is satisfied.

(*) for each vertex $v \in V(G)$, there exists at most one zero-point of f in all allowed elementary paths starting or ending at v.

Therefore, by Lemma 5 and Theorem 3, we have that

Corollary 1 Let $f_1: V(G_1) \to (0, +\infty)$ be a function on G_1 and $f_2: V(G_2) \to [0, +\infty)$ be a discrete Morse function on G_2 with at most one zero-point. Then the function f defined in (1) is extendable.

Proof By Theorem 3, f_1 can be extended to be a discrete Morse function $\bar{f}_1$ on $\bar{G}_1$ such that $\bar{f}_1(v) = f_1(v)$ for $v \in V(G_1)$, and f_2 can be extended to be a discrete Morse function $\bar{f}_2$ on $\bar{G}_2$ such that $\bar{f}_2(w) = f_2(w)$ for $w \in V(G_2)$.

By Lemma 5, f is a discrete Morse function on G and f is extendable.

Define a function $\bar{f}: V(\bar{G}) \to [0, +\infty)$ such that

$$\bar{f}(v) = \begin{cases} \bar{f}_1(v), & v \in V(G_1), \\ \bar{f}_2(v), & v \in V(G_2). \end{cases}$$

Then $\bar{f}$ is the extension of f on $\bar{G}$ such that $\bar{f}(v) = f(v)$ for $v \in V(G)$.

Remark 1 Let f_1, f_2 be discrete Morse functions on digraphs G_1 and G_2, respectively. By Lemma 5 and Corollary 1, unless otherwise specified, we always assume that f_1 is positive and f_2 has at most a unique zero-point in this paper. Denote the extended discrete Morse function of f on $\bar{G} = \overline{G_1 * G_2}$ as $\bar{f}$.

2.2 Transitive Closure of Join of Digraphs, Discrete Gradient Vector Field on the Transitive Closure

Firstly, it is easy to prove that the transitive closure of join of two digraphs are the same as the join of their transitive closures. That is,

Proposition 1 Let G_1 and G_2 be two digraphs. Then

$$\overline{G_1 * G_2} = \bar{G}_1 * \bar{G}_2.$$

Proof Firstly,

$$V(\overline{G_1 * G_2}) = V(G_1 * G_2) = V(G_1) \cup V(G_2) = V(\bar{G}_1) \cup V(\bar{G}_2).$$

Secondly, we will prove

$$E(\overline{G_1 * G_2}) = E(\bar{G}_1 * \bar{G}_2) \tag{2}$$

and divide the proof into the following two steps.

Step 1 Since $G_1*G_2 \subseteq \bar{G}_1*\bar{G}_2$, it is sufficient to prove that for each directed edge $(u,w)\in E(\overline{G_1*G_2})\backslash E(G_1*G_2)$, $(u,w)\in E(\bar{G}_1*\bar{G}_2)$. Since for any $(u,w)\in E(\overline{G_1*G_2})\backslash E(G_1*G_2)$, there exist a sequence of directed edges $\{v_iv_{i+1}\}_{i=0}^{n-1}$ such that $v_iv_{i+1}\in E(G_1*G_2)$ with $v_0=u$ and $v_n=w$. Since $V(G_1)\cap V(G_2)=\varnothing$, there are three cases to be considered.

Case 1 Each $v_iv_{i+1}\in E(G_1)$, $0\leqslant i\leqslant n-1$. Then $uw=v_0v_n\in E(\bar{G}_1)\subseteq E(\bar{G}_1*\bar{G}_2)$.

Case 2 Each $v_iv_{i+1}\in E(G_2), 0\leqslant i\leqslant n-1$. Then $uw=v_0v_n\in E(\bar{G}_2)\subseteq E(\bar{G}_1*\bar{G}_2)$.

Case 3 There exists a directed edge $v_kv_{k+1}\in E(G_1*G_2)\backslash E(G_1)\cup E(G_2)$. Then by the definition of join of digraphs, vertices $v_0, v_1, \cdots, v_k$ are all in G_1 and $v_{k+1},\cdots,v_n$ are all in G_2. Thus,

$$uw=v_0v_n\in E(G_1*G_2)\backslash E(G_1)\cup E(G_2)\subseteq E(G_1*G_2).$$

Combing Case 1-Case 3, $E(\overline{G_1*G_2}\backslash G_1*G_2)\subseteq E(\bar{G}_1*\bar{G}_2)$. Hence, $E(\overline{G_1*G_2})\subseteq E(\bar{G}_1*\bar{G}_2)$.

Step 2 By the definition of join of digraphs, we have that

$$E(\bar{G}_1*\bar{G}_2)=E(\bar{G}_1)\cup E(\bar{G}_2)\cup\{(u,v)|u\in V(\bar{G}_1), v\in V(\bar{G}_2)\}$$
$$=E(\bar{G}_1)\cup E(\bar{G}_2)\cup\{(u,v)|u\in V(G_1), v\in V(G_2)\}.$$

Moreover,

- $E(\bar{G}_1)\subseteq E(\overline{G_1*G_2})$;
- $E(\bar{G}_2)\subseteq E(\overline{G_1*G_2})$;
- for each $(u,v)\in E(\bar{G}_1*\bar{G}_2)$ such that $u\in V(G_1)$ and $v\in V(G_2)$,

$$(u,v)\in E(G_1*G_2)\subseteq E(\overline{G_1*G_2}).$$

Hence, $E(\overline{G_1*G_2})\supseteq E(\bar{G}_1*\bar{G}_2)$.

Therefore, (2) is proved and the proposition holds.

By Proposition 1 and Ref. [3, Proposition 6.4], we have

Corollary 2 Suppose α is an arbitrary allowed elementary n-path on $\overline{G_1*G_2}$. Then there exist $\alpha_1\in P_s(\bar{G}_1)$ and $\alpha_2\in P_t(\bar{G}_2)$ such that $\alpha=\alpha_1*\alpha_2$, $s+t+1=n$ and $s,t\geqslant -1$.

For each $n\geqslant 0$, denote $\mathrm{Crit}_n(G)$ as the free R-module consisting of all the formal linear combinations of critical n-paths on digraph G.

Lemma 6 Suppose $\alpha\in\mathrm{Crit}_n(\bar{G})$. Then there exist $\alpha_1\in\mathrm{Crit}_s(\bar{G}_1)$ and $\alpha_2\in\mathrm{Crit}_t(\bar{G}_2)$ such that $\alpha=\alpha_1*\alpha_2, s+t+1=n$.

Proof By Corollary 2, $\alpha=\alpha_1*\alpha_2$ where $\alpha_1\in P_s(\bar{G}_1)$ and $\alpha_2\in P_t(\bar{G}_2)$. By Lemma 5, since f is the discrete Morse function on G decided by f_1 and f_2, it follows that one of f_1 and f_2 is positive and the other one has at most one zero-point. Without loss of generality, f_1 is positive on $V(G_1)$ and f_2 has at most one zero-point. Then the extension $\bar{f}_1$ of f_1 is positive on $V(\bar{G}_1)$ and each allowed elementary path on $\bar{G}_1$ is critical on $\bar{G}_1$. Hence, α_1 is a critical path on $\bar{G}_1$ and the crucial part of the proof is to verify that α_2 is a critical path on $\bar{G}_2$. Consider the following two cases.

Case 1 There exists no zero-point of f_2. Obviously, $\alpha_1\in\mathrm{Crit}_s(\bar{G}_1)$ and $\alpha_2\in\mathrm{Crit}_t(\bar{G}_2)$.

Case 2 There exists one zero-point of f_2. Then each allowed elementary path on $\bar{G}_1$ is not critical on $\bar{G}$. Since α is critical on $\bar{G}$, it follows that $t\geqslant 0$. We assert that $\alpha_2\in\mathrm{Crit}_t(\bar{G}_2)$. Suppose to the contrary, α_2 is not critical on $\bar{G}_2$. By Lemma 3, there are two cases to be considered.

Subcase 2.1 There exists an allowed elementary path β_2 on $\bar{G}_2$ such that $\beta_2<\alpha_2$ and $\bar{f}_2(\beta_2)=\bar{f}_2(\alpha_2)$. Let $\alpha'=\alpha_1*\beta_2$. Then α' is an allowed elementary path on $\bar{G}$ such that $\alpha'<\alpha$ and $\bar{f}(\alpha')=\bar{f}(\alpha)$ which contradicts α is critical on $\bar{G}$.

Subcase 2.2 There exists an allowed elementary path γ_2 on $\bar{G}_2$ such that $\gamma_2>\alpha_2$ and $\bar{f}_2(\gamma_2)=\bar{f}_2(\alpha_2)$. Let $\alpha''=\alpha_1*\gamma_2$. Then α'' is an allowed elementary path on $\bar{G}$ such that $\alpha''>\alpha$ and $\bar{f}(\alpha'')=\bar{f}(\alpha)$ which contradicts α is critical on $\bar{G}$.

Combining Case 2.1 and Case 2.2, the assertion holds.

Summarizing Case 1 and Case 2, the lemma is proved.

Remark 2 The inverse of Lemma 6 may not hold. For example, suppose $f_1:V(G_1)\to(0,+\infty)$ is a function on G_1 and $f_2:V(G_2)\to[0,+\infty)$ is a discrete Morse function on G_2 with $f_2(w)=0$. Let $\alpha=v$ where v is an arbitrary vertex of G_1 and $\alpha'=w$. Then α and α' are critical paths on $\bar{G}_1$ and $\bar{G}_2$, respectively. Let $\gamma=vw$. Then since $\bar{f}(\gamma)=\bar{f}(\alpha)$, it follows that γ is not critical on $\bar{G}$.

Secondly, denote the discrete gradient vector field on $\bar{G}_i$ and the discrete gradient flow of $\bar{G}_i$ as $\bar{V}_i$ and $\bar{\Phi}_i$ respectively, $i=1,2$. Then we have that

Proposition 2 Let $\bar{V}=\mathrm{grad}\bar{f}$ be the discrete gradient vector filed on $\bar{G}$ and $\alpha=\alpha_1*\alpha_2$ be an allowed elementary path on $\bar{G}$ where $\alpha_2\neq 0$. Then $\bar{V}(\alpha)\neq 0$ if and only if one of $\bar{V}_1(\alpha_1)\neq 0$ and $\bar{V}_2(\alpha_2)\neq 0$ holds.

Proof ($\Rightarrow$) Suppose $\bar{V}(\alpha)\neq 0$. Then there exists a unique allowed elementary $(n+1)$-path $\gamma\in P(\bar{G})$ such that $\gamma>\alpha$ and $\bar{f}(\gamma)=\bar{f}(\alpha)$. By Corollary 2, $\gamma=\gamma_1*\gamma_2$ where

$\gamma_1 \in P_s(\bar{G}_1)$, $\gamma_2 \in P_t(\bar{G}_2)$ and $s+t+1=n+1$. Since $\alpha_2 \neq 0$, it follows that $t \geqslant 0$. Thus, either

$$\gamma_1 > \alpha_1 \text{ and } \bar{f}_1(\gamma_1) = \bar{f}_1(\alpha_1) \quad (3)$$

or

$$\gamma_2 > \alpha_2 \text{ and } \bar{f}_2(\gamma_2) = \bar{f}_2(\alpha_2) \quad (4)$$

Hence, either

$$\bar{V}_1(\alpha_1) \neq 0 \quad (5)$$

or

$$\bar{V}_2(\alpha_2) \neq 0 \quad (6)$$

We assert that only one of (3) and (4) holds. Otherwise, γ can be written as either $\gamma_1 * \alpha_2$ or $\alpha_1 * \gamma_2$. This contradicts the uniqueness of γ. Therefore, only one of (5) and (6) holds.

($\Leftarrow$) Without loss of generality, $\bar{V}_1(\alpha_1)=0$ and $\bar{V}_2(\alpha_2) \neq 0$. Then there exists a unique allowed elementary path γ_2 on $\bar{G}_2$ such that $\gamma_2 > \alpha_2$ and $\bar{f}_2(\gamma_2) = \bar{f}_2(\alpha_2)$. Let $\gamma = \alpha_1 * \gamma_2$. Then γ is an allowed elementary path on $\bar{G}$ such that $\gamma > \alpha$ and $\bar{f}(\gamma) = \bar{f}(\alpha)$. Hence, $\bar{V}(\alpha) \neq 0$.

The lemma is proved.

Remark 3 The condition "$\alpha_2 \neq 0$" in Proposition 2 can not be omitted. Consider the example in Remark 2. Let $\alpha = v$. Then $\alpha_2 = 0, \alpha = \alpha_1 = v$ and $\bar{V}_1(\alpha_1) = 0$. However, since $f_2(w)=0$, it follows that $\bar{V}(\alpha) = vw \neq 0$.

Remark 4 Let $\alpha = \alpha_1 * \alpha_2 = v_0 \cdots v_p w_0 \cdots w_q$ be an allowed elementary path on $\bar{G}$ where $\alpha_1 \in P(\bar{G}_1)$ and $\alpha_2 \in P(\bar{G}_2)$. Then under the assumption that " f_1 is positive and f_2 has at most a unique zero-point", we have that

$$\begin{aligned}\bar{V}(\alpha) &= -<\partial\gamma, \alpha>\gamma \\ &= -(-1)^{p+i+2}\gamma \\ &= (-1)^{p+1}\alpha_1 * (-(-1)^{i+1})\gamma_2 \\ &= (-1)^{p+1}\alpha_1 * \bar{V}_2(\alpha_2) \quad (7)\end{aligned}$$

where $\gamma = v_0 \cdots v_p w_0 \cdots w_i w w_{i+1} \cdots w_q$, $\gamma_2 = w_0 \cdots w_i w w_{i+1} \cdots w_q$ and $f_2(w) = 0$, $-1 \leqslant i \leqslant q$ (Particularly, $i=-1$, $\gamma = v_0 \cdots v_p w w_0 \cdots w_q$, $\gamma_2 = w w_0 \cdots w_q$; $i = q$, $\gamma = v_0 \cdots v_p w_0 \cdots w_q w$, $\gamma_2 = w_0 \cdots w_q w$). Therefore, if $\bar{V}(\alpha) \neq 0$, then there must exist a unique zero-point f_2 of on G_2 and for any allowed elementary path $\alpha \in P(\bar{G}_1) \subseteq P(\bar{G})$, $\bar{V}_1(\alpha) = 0$ and $\bar{V}(\alpha) = 0$.

Thirdly, consider a structural feature of elements in $\Omega_*(G)$.

Lemma 7 Let $G = G_1 * G_2$. Suppose $x = \sum_{i=1}^{m} a^{(i)}\alpha^{(i)} \in \Omega_n(G)$, where $\alpha^{(i)} = \alpha_1^{(i)} * \alpha_2^{(i)}$, $\alpha_1^{(i)} \in P_{s_i}(G_1)$, $\alpha_2^{(i)} \in P_{t_i}(G_2)$ and $s_i + t_i + 1 = n$. Then x can be written as a finite sum of $y*z$,where $y \in \Omega_*(G_1)$ and $z \in \Omega_*(G_2)$.

Proof For each $0 \leqslant i \leqslant m$,

$$\partial\alpha^{(i)} = (\partial\alpha_1^{(i)}) * \alpha_2^{(i)} + (-1)^{s_i+1}\alpha_1^{(i)} * (\partial\alpha_2^{(i)}).$$

Thus

$$d_k(\alpha^{(i)}) \notin P_{n-1}(G), 0<k<s_i \Leftrightarrow d_k(\alpha_1^{(i)}) \notin P_{s_i-1}(G_1),\ 0<k<s_i$$

and

$$d_k(\alpha^{(i)}) \notin P_{n-1}(G),$$
$$s_i+1<k<s_i+t_i+1 \Leftrightarrow d_r(\alpha_2^{(i)}) \notin P_{t_i-1}(G_2),\ 0<r<t_i$$

where $r = k-(s_i+1)$.

Since $x \in \Omega_n(G)$, it follows that $\partial x \in P_{n-1}(G)$. Hence, the coefficient for each fixed $d_k(\alpha^{(i)}) \notin P_{n-1}(G)$, $1<k<n$ must sum up to zero in ∂x. Specifically, there are two cases.

Case 1 There exists a certain index $0<k<s_i$ such that $d_k(\alpha^{(i)}) \notin P_{n-1}(G)$. Then the coefficient of $d_k(\alpha^{(i)})$ in ∂x is

$$\sum_{\{(l,j) | d_l(\alpha_1^j) = d_k(\alpha_1^i)\}} a_j(-1)^l = 0$$

where $\alpha_1^{(j)} \in P_{s_i}(G_1)$, $\alpha_2^{(j)} \in P_{t_i}(G_2)$ and $\alpha_2^{(j)} = \alpha_2^{(i)}$. Hence, by finite steps, we can obtain a formal linear combination y of allowed elementary s_i-paths on G_1 containing $\alpha_1^{(i)}$ such that $y \in \Omega_{s_i}(G_1)$.

Case 2 There exists a certain index $s_i+1<k<s_i+t_i+1$ such that $d_k(\alpha^{(i)}) \notin P_{n-1}(G)$. Then

$d_r(\alpha_2^{(i)}) \notin P_{t_i-1}(G_2)$, $r=k-(s_i+1)$, $0<r<t_i$.

Similar to Case 1 above, we can obtain a formal linear combination z of allowed elementary paths on G_2 which containing $\alpha_2^{(i)}$ such that $z \in \Omega_{t_i}(G_2)$.

Therefore, the lemma is proved.

Finally, by Lemma 7, we have that

Lemma 8 Let $G = G_1 * G_2$ and f_1, f_2 be discrete Morse functions on G_1 and G_2, respectively. Let f be the discrete Morse function on G decided by f_1 and f_2. Suppose $\Omega(G_i)$ is $\bar{V}_i$-invariant. Then $\Omega(G)$ is $\bar{V}$-invariant.

Proof By Lemma 5 and Corollary 1, f is extendable. Without loss of generality, f_1 is positive and f_2 has at most one zero-point. Let $x \in \Omega_n(G)$. By Lemma 7, it is sufficient to prove that for each $x = y*z \in \Omega_n(G)$ where $y \in \Omega_s(G_1)$, $z \in \Omega_t(G_2)$ and $s+t+1=n$, we have that $\bar{V}(x) \in \Omega(G)$. According to the number of zero-points of f_2, there are two cases.

Case 1 f_2 is positive on G_2. Then by Theorem 3, f_1, f_2 are both extendable and $\bar{V}(\alpha) = 0$ for any allowed elementary path α on $\bar{G}$. Hence, $\bar{V}(x) = 0$.

Case 2 There exists one vertex $w \in V(G_2)$ such that $f_2(w) = 0$. Since $f_1(v) = 0$ for any vertex $v \in V(G_1)$, it follows that f_1 is extendable by Theorem 3. Moreover,

$$\bar{V}_1(\alpha_1) = 0 \quad (8)$$

for any allowed elementary path $\alpha_1 \in P(\bar{G}_1)$. Consider the following two subcases.

Subcase 2.1 $t \geqslant 0$. Let $y = \sum_{i=1}^{m} a^{(i)}\alpha_1^{(i)}$ and $z = \sum_{j=1}^{l} b^{(j)}\alpha_2^{(j)}$ where $\alpha_1^{(i)} \in P_s(G_1)$ and $\alpha_2^{(j)} \in P_t(G_2)$. Then by (7) and (8),

$$\begin{aligned}\bar{V}(x) &= \bar{V}(y*z)\\ &= \bar{V}(a^{(1)}\alpha_1^{(1)} * \sum_{j=1}^{l} b^{(j)}\alpha_2^{(j)} + \cdots + a^{(m)}\alpha_1^{(m)} * \sum_{j=1}^{l} b^{(j)}\alpha_2^{(j)})\\ &= a^{(1)}\alpha_1^{(1)} * ((-1)^{p+1}\sum_{j=1}^{l} b^{(j)}\bar{V}_2(\alpha_2^{(j)})) + \cdots\\ &\quad + a^{(m)}\alpha_1^{(m)} * ((-1)^{p+1}\sum_{j=1}^{l} b^{(j)}\bar{V}_2(\alpha_2^{(j)}))\\ &= (-1)^{p+1}(\sum_{i=1}^{m} a^{(i)}\alpha_1^{(i)}) * \bar{V}_2(z)\\ &= (-1)^{p+1} y * \bar{V}_2(z)\end{aligned}$$

Since $\Omega(G_2)$ is $\bar{V}_2$-invariant, $\bar{V}_2(z) \in \Omega(G_2)$. Hence, by Ref. [3, Proposition 6.4], $\bar{V}(x) \in \Omega(G)$. That is, $\Omega(G)$ is $\bar{V}$-invariant.

Subcase 2.2 $t = -1$. Then $x = \sum_{i=1}^{m} a^{(i)}\alpha^{(i)} \in \Omega_n(G_1)$, where $\alpha^{(i)}$, $1 \leqslant i \leqslant m$ are allowed elementary n-paths on G_1.

Hence,

$$\begin{aligned}\bar{V}(x) &= -\sum_{i=1}^{m} a^{(i)} < \partial\gamma^{(i)}, \alpha^{(i)} > \gamma^{(i)}\\ &= (-1)^{n+2}\sum_{i=1}^{m} a^{(i)}\gamma^{(i)}\end{aligned}$$

where $\gamma^{(i)} = \alpha^{(i)} * \alpha_2 \in P_{n+1}(G)$ and $\alpha_2 = w$. Therefore,

$$\begin{aligned}\partial\bar{V}(x) &= (-1)^{n+2}\sum_{i=1}^{m} a^{(i)}\sum_{k=0}^{n+1}(-1)^k d_k(\gamma^{(i)})\\ &= (-1)^{n+2}(\sum_{i=1}^{m} a^{(i)}\sum_{k=0}^{n}(-1)^k d_k(\gamma^{(i)}) + \sum_{i=1}^{m} a^{(i)}(-1)^{n+1}d_{n+1}(\gamma^{(i)}))\\ &= (-1)^{n+2}\sum_{i=1}^{m} a^{(i)}\sum_{k=0}^{n}(-1)^k d_k(\gamma^{(i)}) - \sum_{i=1}^{m} a^{(i)}\alpha^{(i)}\\ &= ((-1)^{n+2}\sum_{i=1}^{m} a^{(i)}\sum_{k=0}^{n}(-1)^k d_k(\alpha^{(i)})) * \alpha_2 - x\\ &= (-1)^{n+2}(\partial x) * \alpha_2 - x\end{aligned}$$

By Ref. [3, Proposition 6.4], $(\partial x) * \alpha_2 \in \Omega_n(G)$. Thus, $\partial\bar{V}(x) \in \Omega_n(G) \subseteq P_n(G)$ which implies that $\bar{V}(x) \in \Omega(G)$. That is, $\Omega(G)$ is $\bar{V}$-invariant.

Summarizing Case 1 and Case 2, the lemma is proved.

3 Proof of Main Theorems

In this section, we will give the proof of Theorem 1 and the proof of Theorem 2.

Let $\{P_*^{\Phi}(\bar{G}), \partial_*\}$ be the subchain complex of $\{P_*(\bar{G}), \partial_*\}$ consisting of all $\bar{\Phi}$-invariant chains. By Ref. [18], we have the following theorem.

Theorem 4[18, Corollary 2.16] Let G be a digraph and f be a discrete Morse function on G satisfying condition (*). Let $\bar{f}$ be the extension of f on $\bar{G}$ and $\bar{V} = \mathrm{grad}\bar{f}$ be the discrete gradient vector field on $\bar{G}$. Suppose $\Omega_*(G)$ is $\bar{V}$-invariant. Then

$$H_m(G) \cong H_m(\Omega_*(G) \cap P_*^{\Phi}(\bar{G})), m \geqslant 0.$$

Then we can give the proof of Theorem 1.

Proof of Theorem 1 By Lemma 5 and Corollary 1, f is extendable. By Lemma 8, $\Omega_*(G)$ is $\bar{V}$-invariant. By Theorem 4, Theorem 1 is proved.

Furthermore, by Ref. [19], we have that

Theorem 5[19, Corollary 4.11] Let G be a digraph and $\bar{G}$ be the transitive closure of G. Suppose $\Omega_*(G)$ is $\bar{V}$-invariant and $\bar{\Phi}(\alpha) \in \Omega(G)$ for any $\alpha \in \mathrm{Crit}(G) \cap P(G)$ where $\bar{V}$ is the discrete gradient vector field on $\bar{G}$ and $\bar{\Phi}$ is the discrete gradient flow of $\bar{G}$, respectively. Then

$$H_m(\{\mathrm{Crit}_n(\bar{G}) \cap P_n(G), \tilde{\partial}_n\}_{n \geqslant 0}) \cong H_m(G; R)$$

where $\tilde{\partial} = (\bar{\Phi}^{\infty})^{-1} \circ \partial \circ \bar{\Phi}^{\infty}$ and $\bar{\Phi}^{\infty}$ is the stabilization map of $\bar{\Phi}$.

Then we can give the proof of Theorem 2.

Proof of Theorem 2 By Lemma 5, f is a discrete Morse function on G. By Corollary 1, f is extendable. Let $\alpha = \alpha_1 * \alpha_2 \in \mathrm{Crit}_n(\bar{G}) \cap P_n(G)$, where $\alpha_1 \in P_s(G_1)$, $\alpha_2 \in P_t(G_2)$, and $s + t + 1 = n$. Since $f_1(v) > 0$ for $v \in P(G_1)$ and f_2 has a unique zero-point, it follows that any allowed elementary path on $\bar{G}_1$ is not critical on $\bar{G}$. Hence, $t \geqslant 0$. Moreover, by Lemma 6, $\alpha_1 \in \mathrm{Crit}_s(\bar{G}_1)$ and $\alpha_2 \in \mathrm{Crit}_t(\bar{G}_2)$. Then $\alpha_1 \in \mathrm{Crit}_s(\bar{G}_1) \cap P_s(G_1)$ and $\alpha_2 \in \mathrm{Crit}_t(\bar{G}_2) \cap P_t(G_2)$.

Since

$$\mathrm{Crit}(\bar{G}_2) \cap P(G_2) = \mathrm{Crit}(\bar{G}_2) \cap \Omega(G_2),$$

it follows that

$$\alpha_2 \in \mathrm{Crit}_t(\bar{G}_2) \cap \Omega_t(G_2) \tag{9}$$

By (8), since $\bar{\Phi}_i(\alpha_i) \in \Omega(G_i)$, it follows that

$$\begin{aligned}\bar{\Phi}_1(\alpha_1) &= (\mathrm{Id} + \partial\bar{V}_1 + \bar{V}_1\partial)(\alpha_1)\\ &= \alpha_1 + \bar{V}_1\partial(\alpha_1)\\ &= \alpha_1 \in \Omega_s(G_1) \end{aligned}\tag{10}$$

and

$$\begin{aligned}\bar{\Phi}_2(\alpha_2) &= (\mathrm{Id} + \partial\bar{V}_2 + \bar{V}_2\partial)(\alpha_2)\\ &= \alpha_2 + \bar{V}_2\partial(\alpha_2) \in \Omega_t(G_2)\end{aligned}\tag{11}$$

Hence, by (8), (9), (10) and (11),

$$\begin{aligned}\Phi(\alpha) &= (\mathrm{Id}+\partial V+V\partial)(\alpha)\\ &= (\mathrm{Id}+\partial V+V\partial)(\alpha_1 * \alpha_2)\\ &= \alpha_1 * \alpha_2 + \bar{V}\partial(\alpha_1 * \alpha_2)\\ &= \alpha_1 * \alpha_2 + \bar{V}((\partial\alpha_1) * \alpha_2 + (-1)^{p+1}\alpha_1 * (\partial\alpha_2))\\ &= \alpha_1 * \alpha_2 + (-1)^p(\partial\alpha_1) * \bar{V}_2(\alpha_2) + (-1)^{p+1}(-1)^{p+1}\alpha_1 * \bar{V}_2(\partial\alpha_2)\\ &= \alpha_1 * (\alpha_2 + \bar{V}_2\partial(\alpha_2)) + (-1)^p(\partial\alpha_1) * (\alpha_2 + \bar{V}(\alpha_2))\\ &\quad -(-1)^p(\partial\alpha_1) * \alpha_2 \in \Omega_n(G).\end{aligned}$$

Since $\Omega(G_i)$ is $\bar{V}_i$-invariant, by Lemma 8, it follows that $\Omega(G)$ is $\bar{V}$-invariant. Therefore, by Theorem 5, the theorem is proved.

Next, let G_1, G_2 be transitive digraphs. Then by Proposition 1, we have that

$$\overline{G_1 * G_2} = \bar{G}_1 * \bar{G}_2 = G_1 * G_2.$$

Thus, $G_1 * G_2$ is transitive. Therefore, by Lemma 5 and Corollary 1, Theorem 5 can be restated as follows.

Corollary 3 Let $G = G_1 * G_2$ where G_1 and G_2 are transitive digraphs. Let f_1, f_2 be discrete Morse functions on G_1 and G_2 respectively and f the discrete Morse function on G decided by f_1 and f_2. Then

$$H_m(\{\mathrm{Crit}_n(G), \tilde{\partial}_n\}_{n \geqslant 0}) \cong H_m(G; R).$$

Finally, we give some examples. The following example illustrates Theorem 1 and Theorem 2.

Example 1 Let $G_1 = \{V(G_1), E(G_1)\}$ be a digraph where $V(G_1) = \{v_0, v_1, v_2\}$ and $E(G_1) = \{v_0v_1, v_1v_2\}$. Let f_1: $V(G_1) \to [0, +\infty)$ be a function on G_1 such that $f_1(v_1) = 0$, $f_1(v_0) > 0$ and $f_1(v_2) > 0$.

Let $G_2 = \{V(G_2), E(G_2)\}$ be a digraph where $V(G_2) = \{v_3, v_4\}$ and $E(G_2) = \varnothing$. Let f_2 be a function on G_2 such that $f_2(v_3) > 0$ and $f_2(v_4) > 0$. Then f_1 and f_2 are discrete Morse functions on G_1 and G_2 respectively and both f_1 and f_2 are extendable.

Let $G = G_1 * G_2$. In fact, G is a suspension on G_1(Ref. [3, Definition 6.13]). By Corollary 1, f_1 and f_2 can define a discrete Morse function f as in (1) which is extendable (see Fig.1). Let $\bar{f}$ be the extension of f on $\bar{G}$. Then

$$\Omega(G_1) = P(G_1) = \{v_0, v_1, v_2, v_0v_1, v_1v_2\},$$

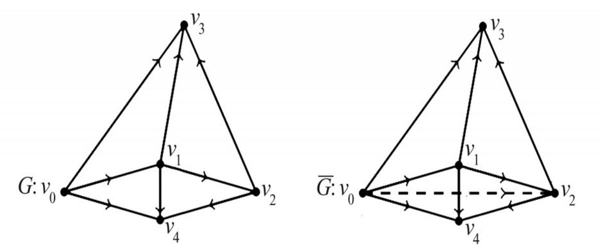

Fig.1 Example 1

$$\begin{aligned}&P(\bar{G}_1) = \{v_0, v_1, v_2, v_0v_1, v_1v_2, v_0v_2, v_0v_1v_2\},\\ &\mathrm{Crit}(\bar{G}_1) = \{v_1\},\\ &\mathrm{Crit}(\bar{G}_1) \cap P(G_1) = \{v_1\} = \mathrm{Crit}(\bar{G}_1) \cap \Omega(G_1)\\ &\Omega(G_2) = \{v_3, v_4\},\ P(\bar{G}_2) = \{v_3, v_4\}\\ &\mathrm{Crit}(\bar{G}_2) = \{v_3, v_4\},\ \mathrm{Crit}(\bar{G}_2) \cap P(G_2) = \{v_3, v_4\}\end{aligned}$$

and

$$\bar{V}_1(v_0) = v_0v_1,\ \bar{V}_1(v_2) = -v_1v_2,\ \bar{V}_1(v_0v_2) = v_0v_1v_2$$

$\bar{V}_1(\alpha_1) = 0$ for any other allowed elementary path α_1 on $\bar{G}_1$,

$\bar{V}_2(\alpha_2) = 0$ for any allowed elementary path α_2 on $\bar{G}_2$.

Hence, $\Omega(G_i)$ is $\bar{V}_i$-invariant. Moreover,

$$\begin{aligned}\bar{\Phi}_1(v_1) &= (\mathrm{Id}+\partial\bar{V}_1+\bar{V}_1\partial)(v_1)\\ &= v_1 \in \Omega(G_1),\\ \bar{\Phi}_2(v_3) &= (\mathrm{Id}+\partial\bar{V}_2+\bar{V}_2\partial)(v_3)\\ &= v_3 \in \Omega(G_2)\end{aligned}$$

and

$$\begin{aligned}\bar{\Phi}_2(v_4) &= (\mathrm{Id}+\partial\bar{V}_2+\bar{V}_2\partial)(v_4)\\ &= v_4 \in \Omega(G_2)\end{aligned}$$

Hence $\bar{\Phi}_i(\alpha_i) \in \Omega(G_i)$ for any $\alpha_i \in \mathrm{Crit}(\bar{G}_i) \cap P_n(G_i)$, $i = 1, 2$.

On the other hand,

$$\Omega(G) = \{v_0, v_1, v_2, v_3, v_4, v_0v_1, v_0v_3, v_0v_4, v_1v_2, v_1v_3, v_1v_4, v_2v_3, v_2v_4, v_0v_1v_3, v_0v_1v_4, v_1v_2v_3, v_1v_2v_4\}$$

and

$\bar{V}(v_0) = v_0v_1,\ \bar{V}(v_2) = -v_1v_2,\ \bar{V}(v_3) = v_1v_3,\ \bar{V}(v_4) = -v_1v_4,$
$\bar{V}(v_0v_2) = v_0v_1v_2,\ \bar{V}(v_0v_3) = v_0v_1v_3,\ \bar{V}(v_0v_4) = v_0v_1v_4,$
$\bar{V}(v_2v_3) = -v_1v_2v_3,\ \bar{V}(v_2v_4) = -v_1v_2v_4,\ \bar{V}(v_0v_2v_3) = v_0v_1v_2v_3,$
$\bar{V}(v_0v_2v_4) = v_0v_1v_2v_4,$
$\bar{V}(\alpha) = 0$ for any other allowed elementary path α on $\bar{G}$.
$\bar{\Phi}(v_0) = v_1,\ \bar{\Phi}(v_1) = v_1, \bar{\Phi}(v_2) = v_1, \bar{\Phi}(v_3) = v_1,\ \bar{\Phi}(v_4) = v_1,$
$\bar{\Phi}(\alpha) = 0$ for any other allowed elementary path α on $\bar{G}$.

By Theorem 1, since

$$\Omega_*(G) \cap P_*^{\bar{\Phi}}(\bar{G}) = \{v_1\},$$

it follows that

$$H_0(\Omega_*(G) \cap P_*^{\bar{\Phi}}(\bar{G})) = R$$

$$H_m(\Omega_*(G) \cap P_*^{\bar{\Phi}}(\bar{G})) = 0 \text{ for } m \geqslant 1$$

which is consistent with Ref.[3, Proposition 6.14].

By Theorem 2, since

$$\mathrm{Crit}(\bar{G}) = \{v_0\},\ \mathrm{Crit}(\bar{G}) \cap P(G) = \{v_0\},$$

it follows that

$$H_0(\{\mathrm{Crit}_n(\bar{G}) \cap P(G), \tilde{\partial}_n\}_{n \geqslant 0}) = R$$

$$H_m(\{\mathrm{Crit}_n(\bar{G}) \cap P(G), \tilde{\partial}_n\}_{n \geqslant 0}) = 0 \text{ for } m \geqslant 1$$

which is consistent with Ref.[3, Proposition 6.14].

And the next is an example illustrating Corollary 3.

Example 2 Let G_1 be a digraph with vertex set $V(G_1)=\{v_0, v_1, v_2, v_3\}$ and directed edge set $E(G_1)=\{v_0v_2, v_0v_3, v_1v_2, v_1v_3\}$. Let $f_1:V(G_1)\to[0,+\infty)$ be a function on G_1 such that $f_1(v_i)>0,\ 0\leqslant i\leqslant 3$.

Let $G_2=\{V(G_2), E(G_2)\}$ be a digraph where $V(G_2)=\{v_4, v_5\}$ and $E(G_2)=\varnothing$. Let f_2 be a function on G_2 such that $f_2(v_4)=0$ and $f_2(v_5)>0$. Then f_1 and f_2 are discrete Morse functions on G_1 and G_2 respectively and both f_1 and f_2 are extendable.

Let $G=G_1*G_2$. By Corollary 1, f_1 and f_2 can define a discrete Morse function f as (1) which is extendable (see Fig.2). Let $\bar{f}$ be the extension of f on $\bar{G}$. Then

$$P(G_1)=\Omega(G_1)=\Omega(\bar{G}_1)$$
$$=\{v_0, v_1, v_2, v_3, v_0v_2, v_0v_3, v_1v_2, v_1v_3\}$$
$$P(G_2)=\Omega(G_2)=\Omega(\bar{G}_2)=\{v_4, v_5\}$$
$$P(G)=\Omega(G)=\Omega(\bar{G})$$
$$=\{v_0, v_1, v_2, v_3, v_4, v_5, v_0v_2, v_0v_3, v_0v_4, v_0v_5, v_1v_2, v_1v_3, v_1v_4, v_1v_5, v_2v_4, v_2v_5, v_3v_4, v_3v_5, v_0v_2v_4, v_0v_2v_5, v_0v_3v_4, v_0v_3v_5, v_1v_2v_4, v_1v_2v_5, v_1v_3v_4, v_1v_3v_5\}.$$

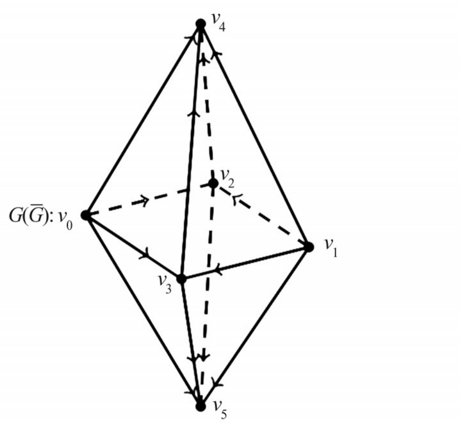

Fig. 2 Example 2

Hence

$$\mathrm{Crit}(\bar{G})=\mathrm{Crit}(G)=\{v_4, v_5, v_0v_5, v_1v_5, v_2v_5, v_3v_5, v_0v_2v_5, v_0v_3v_5, v_1v_2v_5, v_1v_3v_5\}$$

$$\bar{V}(v_0)=v_0v_4, \bar{V}(v_1)=v_1v_4, \bar{V}(v_2)=v_2v_4, \bar{V}(v_3)=v_3v_4,$$
$$\bar{V}(v_0v_2)=-v_0v_2v_4, \bar{V}(v_0v_3)=-v_0v_3v_4,$$
$$\bar{V}(v_1v_3)=-v_1v_3v_4, \bar{V}(v_1v_2)=-v_1v_2v_4,$$

$\bar{V}(\alpha)=0$ for any other allowed elementary path α on $\bar{G}$

and

$$\bar{\Phi}(v_0)=v_4, \bar{\Phi}(v_1)=v_4, \bar{\Phi}(v_2)=v_4,$$
$$\bar{\Phi}(v_3)=v_4, \bar{\Phi}(v_4)=v_4, \bar{\Phi}(v_5)=v_5,$$
$$\bar{\Phi}(v_0v_2)=0, \bar{\Phi}(v_0v_3)=0, \bar{\Phi}(v_1v_3)=0,$$
$$\bar{\Phi}(v_1v_2)=0, \bar{\Phi}(v_0v_4)=0,\ \bar{\Phi}(v_0v_5)=v_0v_5-v_0v_4,$$
$$\bar{\Phi}(v_1v_4)=0, \bar{\Phi}(v_1v_5)=v_1v_5-v_1v_4, \bar{\Phi}(v_2v_4)=0,$$
$$\bar{\Phi}(v_2v_5)=v_2v_5-v_2v_4,\ \bar{\Phi}(v_3v_4)=0,\ \bar{\Phi}(v_3v_5)=v_3v_5-v_3v_4,$$
$$\bar{\Phi}(v_0v_2v_4)=0,\ \bar{\Phi}(v_0v_2v_5)=v_0v_2v_5-v_0v_2v_4,\ \bar{\Phi}(v_0v_3v_4)=0,$$
$$\bar{\Phi}(v_0v_3v_5)=v_0v_3v_5-v_0v_3v_4,\ \bar{\Phi}(v_1v_2v_4)=0,$$
$$\bar{\Phi}(v_1v_2v_5)=v_1v_2v_5-v_1v_2v_4,\ \bar{\Phi}(v_1v_3v_4)=0,$$
$$\bar{\Phi}(v_1v_3v_5)=v_1v_3v_5-v_1v_3v_4.$$

Therefore,

$$\tilde{\partial}(v_0v_5)=v_5-v_4,\ \tilde{\partial}(v_1v_5)=v_5-v_4,\ \tilde{\partial}(v_2v_5)=v_5-v_4,$$
$$\tilde{\partial}(v_3v_5)=v_5-v_4,\ \tilde{\partial}(v_0v_2v_5)=v_2v_5-v_0v_5,$$
$$\tilde{\partial}(v_0v_3v_5)=v_3v_5-v_0v_5,\ \tilde{\partial}(v_1v_2v_5)=v_2v_5-v_1v_5,$$
$$\tilde{\partial}(v_1v_3v_5)=v_3v_5-v_1v_5,$$
$$\tilde{\partial}(v_1v_2v_5-v_0v_2v_5+v_0v_3v_5-v_1v_3v_5)=0$$

By Corollary 3,

$$H_0(\{\mathrm{Crit}_n(G), \tilde{\partial}_n\}_{n\geqslant 0})=R$$
$$H_1(\{\mathrm{Crit}_n(G), \tilde{\partial}_n\}_{n\geqslant 0})=R$$
$$H_2(\{\mathrm{Crit}_n(G), \tilde{\partial}_n\}_{n\geqslant 0})=R$$
$$H_m(\{\mathrm{Crit}_n(G), \tilde{\partial}_n\}_{n\geqslant 0})=0 \text{ for } m>2$$

which is consistent with Ref.[3, Example 6.17].

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