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Pointwise Estimate of Cahn-Hilliard Equation with Inertial Term in $\mathbb{R}^3$

□ **XU Hongmei, ZHU Yue**

College of Science, Hohai University, Nanjing 211100, Jiangsu, China

Abstract: Cauchy problem of Cahn-Hilliard equation with inertial term in three-dimensional space is considered. Using delicate analysis of its Green function and its convolution with nonlinear term, pointwise decay rate is obtained.

Key words: Cahn-Hilliard equation with inertial term; Green function; pointwise decay

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Biography: XU Hongmei, female, Ph.D., Associate professor, research direction: partial differential equations. E-mail: xxu_hongmei@163.com

0 Introduction

This paper is devoted to pointwise estimate of following Cahn-Hilliard equation with inertial term:

$$\begin{cases} \eta u_{tt} + u_t + \Delta^2 u - \Delta f(u) = 0, & x \in \mathbb{R}^3,\ t>0, \\ u|_{t=0} = u_0(x), & u_t|_{t=0} = u_1(x). \end{cases} \tag{1}$$

Here Δ is the usual Laplace operator, $\eta>0$ is a given constant. The nonlinear term $f(u)$ has the form $|u|^{\theta+1}$ or $|u|^{\theta-q+1}\cdot u^q$, where θ, q are positive integers and $\theta-q+1\geqslant 0$, $\theta\geqslant 1$.

Equation (1) is closely related to the well-known Cahn-Hilliard system

$$u_t + \Delta^2 u - \Delta f(u) = 0 \tag{2}$$

System (2) is a hyperbolic equation with relaxation which describes phase separation of a binary mixture and u denotes the relative concentration of one phase. The fourth order differential operator of (2) makes its mathematical analysis more difficult than the corresponding second order equation[1]. Due to the physical background and mathematical difficulties, many mathematicians devoted their enthusiasm to the equation and got much qualitative behavior of the solution (see e. g. Refs.[2-8]). In order to model non-equilibrium decompositions caused by deep supercooling in certain glasses, Galenko *et al*[9] advised to append inertial term ηu_{tt} to (2). The unknown u reflects the relative concentration of one phase. The modified system (1) shows a good agreement with experiments performed on glasses[9,10]. For simplicity, we later suppose $\eta=1$.

The mathematical structure changed after the adjunction. Equation (1) is a hyperbolic equation with relaxation while (2) is a parabolic one, so they present different mathematical features. Eq. (1) has some mathematical difficulties because there is no regularization of the solution in finite time anymore. In order to get regularization, mathematicians often first study them with viscous term. Xu and Shi[11] got global existence with large initial data for any space dimension. Because of weak dissipation, previous work for (1) mainly focused on the so-called energy bounded solution and quasi-strong solution[12-14]. Wang and Wu[15] took advantage of frequency decomposition and energy method, and they got global existence and L^2 decay rate of classical solution of (1) for the case of $n\geqslant 3$ with small initial data. Based on their work, Li and Mi[16] got pointwise decay estimate of the solution for $n\geqslant 4$. Their decay rates are closely related to the space dimension n. The solution decays faster if n is larger which makes it much more difficult to deal with lower space dimension. Obviously, compared with $n\geqslant 4$, $n=3$ reflects the reality. We make much more delicate analysis of the nonlinear term with convolution of the Green function, and get the same decay rate as those of Refs.[15, 16].

We introduce some notations in this paper . We denote C or $C(x)$ a constant or constant depending on variable x. L^p, $W^{m,p}$ denote usual Lebesgue and Sobolev spaces on $\mathbb{R}^n$ and $H^m=W^{m,2}$, with norms $\|\cdot\|_{L^p}$, $\|\cdot\|_{W^{m,p}}$, $\|\cdot\|_{H^m}$, respectively. $\alpha=(\alpha_1,\alpha_2,\alpha_3)$ is a multi-index with $|\alpha|=\alpha_1+\alpha_2+\alpha_3$. Fourier transform to the variable x of function $f(x,t)$ is $\hat{f}(\xi,t)$, that is $\hat{f}(\xi,t)=\int f(x,t)\mathrm{e}^{-\mathrm{i}x\cdot\xi}\mathrm{d}x$, where i is the imaginary unit. Thus the inverse Fourier transform to the variable ξ of $\hat{f}(\xi,t)$ is defined as

$$f(x,t)=F^{-1}(\hat{f})(x,t)=(2\pi)^{-\frac{n}{2}}\int\hat{f}(\xi,t)\mathrm{e}^{\mathrm{i}x\cdot\xi}\mathrm{d}\xi.$$

The rest of this paper is arranged as follows. In Section 1, we give some preparing work. The estimate of the solution will be given in Section 2.

1 Preliminary Work

Our work is a follow-through of the global existence of (1), that is the Theorem 1.1 of Ref.[15]. We list it here.

Theorem 1[15] If initial data u_0, u_1 satisfy

$$\|u_0\|_{H^{l+1}\cap W^{1,1}}+\|u_1\|_{H^l\cap L^1}\leqslant\varepsilon$$

for some small ε, $l\geqslant 6$, the Cauchy problem (1) admits a unique, global, classical solution $u(x,t)$ satisfying:

$$\|\partial_x^\alpha u(\cdot,t)\|_{L^2}\leqslant C\varepsilon(1+t)^{-\frac{3}{8}-\frac{|\alpha|}{4}},\quad\text{for }|\alpha|\leqslant l,$$

$$\|\partial_x^\alpha u(\cdot,t)\|_{L^\infty}\leqslant C\varepsilon(1+t)^{-\frac{3}{4}-\frac{|\alpha|}{4}},\quad\text{for }|\alpha|\leqslant l-2.$$

The Green function of (1) is defined as

$$\begin{cases}(\partial_{tt}+\partial_t+\Delta^2)G(x,t)=0, & x\in\mathbb{R}^3,\ t>0,\\ G|_{t=0}=0,\\ G_t|_{t=0}=\delta(x).\end{cases}$$

Then $\hat{G}(\xi,t)=\dfrac{1}{\lambda_+-\lambda_-}(\mathrm{e}^{\lambda_+t}-\mathrm{e}^{\lambda_-t})$, where

$$\lambda_+=\frac{1}{2}\left(-1+\sqrt{1-4|\xi|^4}\right),\lambda_-=\frac{1}{2}\left(-1-\sqrt{1-4|\xi|^4}\right).$$

Using Duhamel principle, the solution of (1) can be represented as

$$u(x,t)=G(\cdot,t)*(u_0+u_1)+\partial_tG*u_0+\int_0^tG(\cdot,t-\tau)*\Delta f(u)(\cdot,\tau)\mathrm{d}\tau \tag{3}$$

Operator $*$ denotes the convolution of space variable x in this paper.

We will use frequency decomposition to estimate $G(x,t)$. Set

$$\chi_1(\xi)=\begin{cases}1, & |\xi|<\varepsilon_1\\ 0, & |\xi|>2\varepsilon_1\end{cases},\chi_3(\xi)=\begin{cases}1, & |\xi|>R+1\\ 0, & |\xi|<R\end{cases}$$

are smooth cut-off functions. Here $2\varepsilon_1<R$. Set $\chi_2(\xi)=1-\chi_1(\xi)-\chi_3(\xi)$.

Denote $B_N(x,t)=(1+\dfrac{|x|^4}{1+t})^{-N}$, $\hat{G}^+(\xi,t)=\dfrac{\mathrm{e}^{\lambda_+t}}{\lambda_+-\lambda_-}$, $\hat{G}^-(\xi,t)=\dfrac{-\mathrm{e}^{\lambda_-t}}{\lambda_+-\lambda_-}$,

$$\hat{G}^{\pm}{}_i(\xi,t)=\chi_i(\xi)\hat{G}^{\pm}(\xi,t),\quad \hat{G}_i(\xi,t)=\chi_i(\xi)\hat{G}(\xi,t) \text{ for } i=1,2,3.$$

For $G_1(x,t), G_2(x,t)$, we can use the results of Ref.[16] which are Proposition 3.1 and Proposition 3.2. We list them here.

Theorem 2[16] There exists positive constant $C(N)$, such that

$$|\partial_x^\alpha G_i(x,t)|\leqslant C(N)(1+t)^{-\frac{3+|\alpha|}{4}}\cdot B_N(x,t),\ i=1,2.$$

Since $\partial_t\hat{G}=\dfrac{\lambda_+\mathrm{e}^{\lambda+t}-\lambda_-\mathrm{e}^{\lambda_-t}}{\lambda_+-\lambda_-}=\dfrac{1-\mathrm{e}^{(\lambda_--\lambda_+)t}}{\lambda_+-\lambda_-}\cdot\lambda_+\mathrm{e}^{\lambda_+t}+\mathrm{e}^{\lambda_-t}$, using the same method of Theorem 2, we can get the same conclusion, that is

Theorem 3[16] There exists positive constant $C(N)$, such that

$$|\partial_x^\alpha\partial_t G_i(x,t)|\leqslant C(N)(1+t)^{-\frac{3+|\alpha|}{4}}\cdot B_N(x,t),\ i=1,2.$$

When $|\xi|$ is large enough, using Taylor expansion, we get

$$\hat{G}_3^+(\xi,t)=(-\frac{\mathrm{i}}{2}|\xi|^{-2}+o(|\xi|^{-4}))\mathrm{e}^{(-\frac{1}{2}+\mathrm{i}|\xi|^2-\frac{\mathrm{i}}{8}|\xi|^{-2}+o(|\xi|^{-4}))t} \tag{4}$$

$$\hat{G}_3^-(\xi,t)=(\frac{\mathrm{i}}{2}|\xi|^{-2}+o(|\xi|^{-4}))\mathrm{e}^{(-\frac{1}{2}-\mathrm{i}|\xi|^2+\frac{\mathrm{i}}{8}|\xi|^{-2}+o(|\xi|^{-4}))t} \tag{5}$$

In order to get estimate of high frequency part, we need to understand the construction of $G_3(x,t)$, and we can use Lemma 2.5 in Ref.[17]. That is

Lemma 1[17] Assume that supp$\hat{f}\subset O(R)=\{\xi:|\xi|>R\}$ with

$$|\hat{f}(\xi)|\leqslant C,\ \ |\partial_\xi^\beta\hat{f}(\xi)|\leqslant C|\xi|^{-1-|\beta|},\ \ |\beta|\geqslant 1,$$

then there exist distributions $f_1(x),\ f_2(x)$, and a constant C such that

$$f(x)=f_1(x)+f_2(x)+C\delta(x)$$

where $\delta(x)$ is the Dirac function. Furthermore, for a positive integer $2N_1>n+|\alpha|$,

$$|\partial_x^\alpha f_1(x)|\leqslant C(1+|x|^2)^{-N_1},\ \|f_2\|_{L^1}\leqslant C,\ \mathrm{supp}f_2\subset\{x;|x|<2\varepsilon_0\}$$

with ε_0 being sufficiently small.

From (4), (5) and Lemma 1, we have the following construction

$$\Delta G_3(x,t)=(f_1(x)+C\delta(x)+f_2(x))\mathrm{e}^{\lambda t},\ \ \mathrm{Re}\,\lambda\leqslant-\frac{1}{4} \tag{6}$$

where $f_1(x), f_2(x)$ satisfy Lemma 1.

2 Decay Estimation

We use (3) and decay rate of G to estimate the solution $u(x,t)$ step by step.

Theorem 4 If $u_0(x), u_1(x)$ satisfy the condition of Theorem 1, and

$$|u_0(x)|+|u_1(x)|\leqslant C\varepsilon(1+|x|^4)^{-r},\ r>\frac{3}{4},$$

we have $|\partial_x^\alpha G*(u_0+u_1)|\leqslant C(N,r)\varepsilon(1+t)^{-\frac{3+|\alpha|}{4}}B_r(x,t),\ |\alpha|\leqslant l.$

Proof Here and afterwards we take $N>2r>\frac{3}{2}$. We divide the following integral into three parts.

$$\begin{aligned}\int_{\mathbb{R}^3}B_N(x-y,t)(1+|y|^4)^{-r}\mathrm{d}y&=\Big(\int_{|y|\leqslant\frac{|x|}{2}}+\int_{|y|\geqslant\frac{|x|}{2},|x|^4\geqslant t}+\int_{|y|\geqslant\frac{|x|}{2},|x|^4\leqslant t}\Big)B_N(x-y,t)(1+|y|^4)^{-r}\mathrm{d}y\\&:=I_1+I_2+I_3\end{aligned} \tag{7}$$

If $N_1>\frac{3}{4}$, we have

$$\int B_{N_1}(x,t)\mathrm{d}x=\int\frac{\mathrm{d}x(1+t)^{-\frac{1}{4}}}{\left(1+(|x|(1+t)^{-\frac{1}{4}})^4\right)^{N_1}}\cdot(1+t)^{\frac{3}{4}}\leqslant(1+t)^{\frac{3}{4}}\int\frac{\mathrm{d}x}{\left(1+x^4\right)^{N_1}}\leqslant C(N_1)(1+t)^{\frac{3}{4}} \tag{8}$$

Thus

$$I_1+I_2\leqslant C(N)B_N(x,t)\int_{|y|\leqslant \frac{|x|}{2}}(1+|y|^4)^{-r}\mathrm{d}y+(1+|x|^4)^{-r}\int_{|y|\geqslant \frac{|x|}{2},\,|x|^4\geqslant t}B_N(x-y,t)\mathrm{d}y$$
$$\leqslant C(N,r)B_N(x,t)+(1+|x|^4)^{-r}(1+t)^{\frac{3}{4}}$$
$$\leqslant C(N,r)B_r(x,t)+(1+|x|^4)^{-r}(1+t)^{r}\leqslant C(N,r)B_r(x,t) \tag{9}$$

When $|x|^4\leqslant t$, we have $B_r(x,t)\geqslant \dfrac{1}{2}$, then

$$I_3\leqslant \int\left(1+|y|^4\right)^{-r}\mathrm{d}y\leqslant C(r)\leqslant C(r)B_r(x,t) \tag{10}$$

From (7), (9), (10) and Theorem 2, we have

$$|\partial_x^\alpha G_i*(u_0+u_1)|\leqslant C(r,N)\varepsilon(1+t)^{-\frac{3+|\alpha|}{4}}B_r(x,t),\quad i=1,2 \tag{11}$$

Suppose function $v\in H^l$, $|\alpha|\leqslant l$, from (4), (5), we have

$$|x^\beta\partial_x^\alpha G_3^{\pm}*v|\leqslant \int_{|\xi|\geqslant R}|\partial_\xi^\beta\xi^\alpha\hat{G}_3^{\pm}\cdot\hat{v}|\mathrm{d}\xi\leqslant C\mathrm{e}^{-\frac{t}{2}}\int_{|\xi|\geqslant R}|\xi|^{|\alpha|-|\beta|-2}|\hat{v}|\mathrm{d}\xi\leqslant C\mathrm{e}^{-\frac{t}{2}}\||\xi|^{|\alpha|-|\beta|}|\hat{v}|\|_{L^2}\cdot\|\xi^{-2}\|_{L^2(|\xi|\geqslant R)}\leqslant C\mathrm{e}^{-\frac{t}{2}}\|v\|_{H^l} \tag{12}$$

Take $\beta=0$, we have

$$|\partial_x^\alpha G_3^{\pm}*v|\leqslant C\mathrm{e}^{-\frac{t}{2}}\|v\|_{H^l} \tag{13}$$

If $x\neq 0$, take $|\beta|=N$, we have

$$|\partial_x^\alpha G_3^{\pm}*v|\leqslant C\mathrm{e}^{-\frac{t}{2}}|x|^{-|\beta|}\cdot\|v\|_{H^l} \tag{14}$$

From (13), (14), we have

$$|\partial_x^\alpha G*v|\leqslant C(N)\mathrm{e}^{-\frac{t}{4}}\cdot\|v\|_{H^l}\cdot B_N(x,t) \tag{15}$$

We know $u_0\in H^{l+1}$, $u_1\in H^l$, from (15), we get

$$|\partial_x^\alpha G_3*(u_0+u_1)|\leqslant C(N)\mathrm{e}^{-bt}\varepsilon B_N(x,t) \tag{16}$$

From (11), (16), the theorem is proved.

Using the same method as that of Theorem 4, we also get the following theorem.

Theorem 5 If $u_0\in H^{l+1}$, $|u_0(x)|\leqslant C\varepsilon(1+|x|^4)^{-4}$, $r>\dfrac{3}{4}$, we have

$$|\partial_x^\alpha\partial_t G*u_0|\leqslant C(N,r)(1+t)^{-\frac{3+|\alpha|}{4}}\varepsilon B_r(x,t),\quad |\alpha|\leqslant l.$$

Next, we begin to estimate the nonlinear term $\int_0^t G(\cdot,t-\tau)*\Delta f(u)(\cdot,\tau)\mathrm{d}\tau$.

Set $\varphi_\alpha(x,t)=(1+t)^{-\frac{3+|\alpha|}{4}}B_r(x,t)$, $M(t)=\sup\limits_{(x,\tau)\in\mathbb{R}^n\times[0,t],\,|\alpha|\leqslant l}|\partial_x^\alpha u(x,\tau)|\cdot\varphi_\alpha(x,\tau)$. Then

$$|\partial_x^\alpha u(x,\tau)|\leqslant M(t)(1+\tau)^{-\frac{3+|\alpha|}{4}}B_r(x,\tau),\quad |\alpha|\leqslant l \tag{17}$$

From (17), for $|\alpha|\leqslant l$, noticing $\theta\geqslant 1$, we have

$$|\partial_x^\alpha f(u)(y,\tau)|=|\sum_{|\alpha_1|+|\alpha_2|+|\alpha_3|=|\alpha|}\partial_{x_1}^{\alpha_1}u\cdot\partial_{x_2}^{\alpha_2}u\cdot\partial_{x_3}^{\alpha_3}u^{\theta-1}|\leqslant CM^{\theta+1}(t)(1+\tau)^{-\frac{3}{2}-\frac{|\alpha|}{4}}B_{2r}(y,\tau) \tag{18}$$

Theorem 6 For $|\alpha|\leqslant l$, we have

$$|\partial_x^\alpha\int_0^t G(\cdot,t-\tau)*\Delta f(u)(\cdot,\tau)\mathrm{d}\tau|\leqslant C(1+t)^{-\frac{3+|\alpha|}{4}}B_r(x,t)M^{\theta+1}(t).$$

Proof We first divide the following integral into four items.

$$\int_0^t\int_{\mathbb{R}^n}(1+t-\tau)^{-\frac{3+2}{4}}B_N(x-y,t-\tau)(1+\tau)^{-\frac{3}{2}}B_{2r}(y,\tau)\mathrm{d}y\mathrm{d}\tau$$
$$=\left(\int_0^{\frac{t}{2}}\int_{|y|\leqslant\frac{|x|}{2}}+\int_0^{\frac{t}{2}}\int_{|y|\geqslant\frac{|x|}{2}}+\int_{\frac{t}{2}}^t\int_{|y|\leqslant\frac{|x|}{2}}+\int_{\frac{t}{2}}^t\int_{|y|\geqslant\frac{|x|}{2}}\right)(1+t-\tau)^{-\frac{3+2}{4}}(1+\tau)^{-\frac{3}{2}}B_N(x-y,t-\tau)B_{2r}(y,\tau)\mathrm{d}y\mathrm{d}\tau$$
$$:=I_1+I_2+I_3+I_4 \tag{19}$$

From (8), we have

$$I_1 \leqslant C(N)(1+t)^{-\frac{3+2}{4}} B_N(x,t)\int_0^{\frac{t}{2}}(1+\tau)^{-\frac{3}{2}}(1+\tau)^{\frac{3}{4}}\mathrm{d}\tau \leqslant C(N)(1+t)^{-\frac{3+2}{4}+\frac{1}{4}} B_N(x,t),$$

$$I_4 \leqslant C(1+t)^{-\frac{3}{2}} B_r(x,t)\int_{\frac{t}{2}}^{t}(1+t-\tau)^{-\frac{3+2}{4}}(1+t-\tau)^{\frac{3}{4}}\mathrm{d}\tau \leqslant C(1+t)^{-\frac{3}{2}+\frac{1}{2}} B_r(x,t) \leqslant C(1+t)^{-\frac{3}{4}} B_r(x,t).$$

Noticing $N > 2r > \frac{3}{2}$, from (8), we have

$$I_2 \leqslant C(1+t)^{-\frac{3+2}{4}} B_r(x,t)\int_0^{\frac{t}{2}}(1+\tau)^{-\frac{n}{2}}(1+\tau)^{\frac{3}{4}}\mathrm{d}\tau \leqslant C(1+t)^{-\frac{n+2}{4}} \cdot (1+t)^{\frac{1}{4}} B_r(x,t) \leqslant C(1+t)^{-\frac{3}{4}} B_r(x,t),$$

$$I_3 \leqslant C(1+t)^{-\frac{3}{2}} B_r(x,t)\int_{\frac{t}{2}}^{t}(1+t-\tau)^{-\frac{3+2}{4}}(1+t-\tau)^{\frac{3}{4}}\mathrm{d}\tau \leqslant C(1+t)^{-\frac{3}{2}+\frac{1}{2}} B_r(x,t) \leqslant C(1+t)^{-\frac{3}{4}} B_r(x,t).$$

From (19) and above four inequalities, we get

$$\int_0^t\int_{\mathbb{R}^n}(1+t-\tau)^{-\frac{3+2}{4}} B_N(x-y,t-\tau)(1+\tau)^{-\frac{3}{2}} B_{2r}(y,\tau)\mathrm{d}y\mathrm{d}\tau \leqslant C(N,r)(1+t)^{-\frac{3}{4}} B_r(x,t) \tag{20}$$

From (20), (18), Theorem 2, we have

$$|\partial_x^\alpha\int_0^t G_i(\cdot,t-\tau)*\Delta f(u)(\cdot,t)\mathrm{d}\tau| = |\partial_x^\alpha\int_0^t \Delta G_i(\cdot,t-\tau)*f(u)(\cdot,t)\mathrm{d}\tau| \leqslant C(N,r)M(t)^{\theta+1}(1+t)^{-\frac{3+|\alpha|}{4}} B_r(x,t),\quad i=1,2 \tag{21}$$

If $N_1 > 4r$, from (20), we get

$$\begin{aligned}&\int_0^t\int_{\mathbb{R}^3}\mathrm{e}^{-b(t-\tau)}(1+|x-y|^2)^{-N_1}(1+\tau)^{-\frac{3}{2}} B_{2r}(y,\tau)\mathrm{d}y\mathrm{d}\tau\\ &\leqslant \int_0^t\int_{\mathbb{R}^3}(1+t-\tau)^{-\frac{3+2}{4}} B_{2r}(x-y,t-\tau)(1+\tau)^{-\frac{3}{2}} B_{2r}(y,\tau)\mathrm{d}y\mathrm{d}\tau \leqslant C(r)(1+t)^{-\frac{3}{4}} B_r(x,t)\end{aligned} \tag{22}$$

If $|x-y| < 2\varepsilon_0$ with ε_0 small enough, we have

$$1+\frac{|y|^4}{1+\tau} \geqslant \frac{1}{2}+\frac{1}{2}+\frac{|x|^4}{1+\tau}-\frac{|x-y|^4}{1+\tau} \geqslant \frac{1}{2}+\frac{1}{2}+\frac{|x|^4}{1+t}-\frac{|2\varepsilon_0|^4}{1+\tau} \geqslant \frac{1}{2}(1+\frac{|x|^4}{1+t}).$$

Thus

$$\begin{aligned}&\int_0^t\int_{\mathbb{R}^n}\mathrm{e}^{-b(t-\tau)} f_2(x-y)(1+\tau)^{-\frac{3}{2}} B_{2r}(y,\tau)\mathrm{d}y\mathrm{d}\tau\\ &\leqslant C(r)\int_0^t \mathrm{e}^{-b(t-\tau)}(1+\tau)^{-\frac{3}{2}} B_r(x,\tau)\mathrm{d}\tau \leqslant C(r)(1+t)^{-\frac{3}{4}} B_r(x,t)\end{aligned} \tag{23}$$

$$\int_0^t \mathrm{e}^{-b(t-\tau)}(1+\tau)^{-\frac{3}{2}}\delta(x-y)\cdot B_{2r}(y,\tau)\mathrm{d}\tau = \int_0^t \mathrm{e}^{-b(t-\tau)}(1+\tau)^{-\frac{3}{2}} B_{2r}(x,\tau)\mathrm{d}\tau \leqslant C(1+t)^{-\frac{3}{4}} B_r(x,t) \tag{24}$$

From (22), (23), (24), (18), (6), we get

$$|\partial_x^\alpha\int_0^t G_3(\cdot,t-\tau)*\Delta f(u)(\cdot,t)\mathrm{d}\tau| = |\partial_x^\alpha\int_0^t \Delta G_3(\cdot,t-\tau)*f(u)(\cdot,t)\mathrm{d}\tau| \leqslant C(N,r)(1+t)^{-\frac{3+|\alpha|}{4}} M(t)^{\theta+1} B_r(x,t) \tag{25}$$

From (21), (25), the theorem is proved.

From (3), Theorem 4, Theorem 5, and Theorem 6, we get

$$|\partial_x^\alpha u(x,t)| \leqslant C(r)\varepsilon(1+t)^{-\frac{3+|\alpha|}{4}} B_r(x,t) + C(r)M^{\theta+1}(t)(1+t)^{-\frac{3+|\alpha|}{4}} B_r(x,t).$$

From the definition of M, we get

$$M(t) \leqslant C(r)(\varepsilon + M^{\theta+1}).$$

Because ε, $M(0)$ are small enough, thus $M(t)$ is bounded. Then we have

$$|\partial_x^\alpha u(x,t)| \leqslant C(r)(1+t)^{-\frac{3+|\alpha|}{4}} B_r(x,t).$$

Theorem 7 is the main conclusion of this paper.

Theorem 7 If u_0, u_1 satisfy $\|u_0\|_{H^{l+1}\cap W^{1,1}} + \|u_1\|_{H^l\cap L^1} \leqslant \varepsilon$ with ε small enough, $l \geqslant 6$, $|u_0(x)| + |u_1(x)| \leqslant C(r)\varepsilon(1+|x|^4)^{-r}$ with $r > \frac{3}{4}$, then the Cauchy problem of (1) exists a global classical solution $u(x,t)$, furthermore $|\partial_x^\alpha u(x,t)| \leqslant C(r)(1+$

$t)^{-\frac{3+|\alpha|}{4}} B_r(x,t), \quad |\alpha| \leqslant l.$

Remark 1 From (8), our result coincides with L^2 and L^∞ decay rate of Refs.[14,15]. Furthermore, we move forward the solution's derivative order which can be estimated by L^∞ module.

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