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The Minkowski Measure of Asymmetry for Spherical Bodies of Constant Width

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Abstract: In this paper, we introduce the Minkowski measure of asymmetry for the spherical bodies of constant width. Then we prove that the spherical balls are the most symmetric bodies among all spherical bodies of constant width, and the completion of the spherical regular simplexes are the most asymmetric bodies.

Key words: spherical convex body; spherical body of constant width; Minkowski measure of asymmetry; simplex; Reuleaux triangle

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0 Introduction

The Minkowski measure of asymmetry of convex bodies in the Euclidean space $\mathbb{R}^n$ was introduced by Minkowski[1]. It is well-known that the n-dimensional simplex has the biggest Minkowski measure n of asymmetry. In other words, we always say that the simplex is the most asymmetric convex body. There are many other measures of asymmetry for convex bodies, such as the Winternitz measure of symmetry, the Kovner-Besicovitch measure of symmetry, and the mean Minkowski measure of asymmetry. For the new development of the research of the Minkowski measure of asymmetry, see Refs.[2-7].

The Convex bodies of constant width are important ones in $\mathbb{R}^n$. The symmetry of convex bodies of constant width in $\mathbb{R}^2$ was studied by Besicovitch[8], who showed that the Reuleaux triangles are the most asymmetric convex bodies of constant width in $\mathbb{R}^2$, and the circles are the most symmetric convex bodies of constant width in $\mathbb{R}^2$. A special measure of asymmetry for convex bodies of constant width in $\mathbb{R}^2$ was introduced by Groemer and Wallen[9]. The Minkowski measure of asymmetry for convex bodies of constant width was studied by Guo and Jin[3, 5, 6, 10, 11]. In the sense of Minkowski measure of asymmetry, the complete bodies of the regular simplex are the most asymmetric convex bodies of constant width, and the Euclidean balls are the most symmetric convex bodies of constant width.

The convex geometry in the n-dimensional spherical space $\mathbb{S}^n$ was investigated by many mathematician, such as Robinson, Santaló, Dekster, Leichtweiss[12-16], Lassak[17] and Guo[10]. In this paper, we introduce the Minkowski measure of asymmetry for spherical convex bodies of constant width in $\mathbb{S}^n$. Then we prove a spherical analogy of a result obtained by Jin and Guo[11, 18], i.e. that the spherical balls are the most symmetric bodies among all spherical bodies of constant width, and the completions of the spherical regular simplex are the most asymmetric bodies. Concretely, we give the following theorem.

Main Theorem Let $W \subset \mathbb{S}^n$ be a spherical body of constant width. Then the Minkowski measure $\mathrm{as}_s(W)$ of asymmetry of W satisfies the following inequality:

$$1 \leqslant \mathrm{as}_s(W) \leqslant \frac{n+\sqrt{2n(n+1)}}{n+2} \tag{1}$$

The equality holds on the left-hand side if and only if W is a spherical ball, and on the right-hand side if and only if W is a completion of a spherical regular simplex.

1 Preliminaries

Let K^n denote the class of convex bodies (compact convex sets with nonempty interiors) in $\mathbb{R}^n$, and let $\mathrm{K}^n_{\mathrm{o}}$ denote the class of sets in K^n which contain the origin in their interiors. The convex body $K \in \mathrm{K}^n$ is said to be of constant width $\omega > 0$ if its projection on any straight line is a segment of length ω, which is equivalent to the geometrical fact that any two parallel support hyperplanes of K are always at the distance ω. The convex bodies of constant width in $\mathbb{R}^2$ and $\mathbb{R}^3$ are also called orbiforms and spheroforms, respectively. Euclidean balls are obviously bodies of constant width, however, there are many others[19]. We denote by W^n the set of all n-dimensional convex bodies of constant width.

Convex bodies of constant width have many interesting properties and applications which have gained much attention in the history, e. g., orbiforms were intensely studied during the nineteenth century and later, particularly by Reuleaux, whose name is now attached to the orbiforms obtained by intersecting a finite number of disks of equal radii. In $\mathbb{R}^3$, Meissner tetrahedrons may be the most famous spheroforms. Mathematicians believe that Meissner tetrahedrons have the minimal volume among all spheroforms of the same width.

Given a convex body $C \in \mathrm{K}^n$ and $x \in \mathrm{int}(C)$. For a hyperplane H through x and the pair of support hyperplanes H_1, H_2 (of C) parallel to H, let $\gamma(H,x)$ be the ratio, not less than 1, in which H divides the distance between H_1 and H_2. Put

$$\gamma(C,x) = \max\{\gamma(H,x) | H \ni x\} \tag{2}$$

and define the Minkowski measure $\mathrm{as}(C)$ of asymmetry of C by

$$\mathrm{as}(C) = \min_{x \in \mathrm{int}(C)} \gamma(C,x). \tag{3}$$

A point $x \in \mathrm{int}(C)$ satisfying $\gamma(C,x) = \mathrm{as}(C)$ is called a critical point of C. The set of all critical points of C is denoted by $\mathbb{C}(C)$. It is known that $\mathbb{C}(C)$ is a non-empty convex set.

If $C \in \mathrm{K}^n$, then

$$1 \leqslant \mathrm{as}(C) \leqslant n \tag{4}$$

Equality holds on the left-hand side if and only if C is centrally symmetric, and on the right-hand side if and only if C is a simplex.

The Minkowski measure of asymmetry for convex bodies of constant width was studied by Jin and Guo[11, 18].

The critical set $\mathbb{C}(K)$ of $K \in W^n$ is a singleton, and the unique critical point of K is the center of circumscribed sphere of K, also the center of inscribed sphere of K. Denoted by $r(K)$ and $R(K)$ be the radii of insphere and circumsphere of K, respectively. For $K \in W^n$, we have $\mathrm{as}(K) = \dfrac{R(K)}{r(K)}$. By Jung's theorem, we have

$$1 \leqslant \mathrm{as}(K) \leqslant \frac{n+\sqrt{2n(n+1)}}{n+2} \tag{5}$$

The equality holds on the left-hand side if and only if K is an Euclidean ball, and on the right-hand side if and only if K is a completion of a regular simplex.

In $\mathbb{R}^n$, a completion of a convex body C is a convex body of constant width, which has the same diameter as C.

2 Spherical Bodies of Constant Width

Let $\mathbb{S}^n$ be the unit sphere of the $(n+1)$-dimensional Euclidean space $\mathbb{R}^{n+1}$, where $n \geqslant 2$. The intersection of $\mathbb{S}^n$ with an $(m+1)$-dimensional subspace of $\mathbb{R}^{n+1}$, where $0 \leqslant m \leqslant n$, is called an m-dimensional subsphere of $\mathbb{S}^d$. In particular, if $m=0$, we get the 0-dimensional subsphere consisting of a pair of antipodal points, and if $m=1$ we obtain the so-called great circle.

If $a, b \in \mathbb{S}^n$ are not antipodes, by the arc ab connecting them we mean the shorter part of the great circle containing a and b. By the spherical distance $|ab|$, or distance in short, of these points we know the length of the arc connecting them. Clearly $|ab| = \angle aob$, the angel between vectors a, b, where o is the center of $\mathbb{R}^{n+1}$. The diameter diam(A) of a set $A\subset\mathbb{S}^n$ is the supremum of the spherical distances between pairs of points in A.

By a spherical ball of radius $\rho\in(0,\frac{\pi}{2}]$, or an s-ball in short, we understand the set of points of $\mathbb{S}^n$ having distance at most ρ from a fixed point, called the center of this ball. Spherical balls of radius $\frac{\pi}{2}$ are called hemispheres. In other words, by a hemisphere of $\mathbb{S}^n$ we mean the common part of $\mathbb{S}^n$ with any closed half-space of $\mathbb{R}^{n+1}$. We denote by $H(p)$ the hemisphere whose center is p. Two hemispheres whose centers are antipodes are called opposite hemispheres.

By a spherical $(n-1)$-dimensional ball of radius $\rho\in(0,\frac{\pi}{2})$ we mean the set of points of a $(n-1)$-dimensional sphere of $\mathbb{S}^n$ which are at distance at most ρ from a fixed point, called the center of this ball. The $(n-1)$-dimensional balls of radius $\frac{\pi}{2}$ are called $(n-1)$-dimensional hemisphere. If $n=2$, we call them semi-circles.

We say that a set $C\subset\mathbb{S}^n$ is s-convex if it does not contain any pair of antipodes, and if together with for every two points of C, the whole arc connecting them is a subset of C. By an s-convex body on $\mathbb{S}^n$ we mean a closed convex set with non-empty interior.

If an $(n-1)$-dimensional sphere G of $\mathbb{S}^n$ has a common point t with a convex body $C\subset\mathbb{S}^n$ and its intersection with the interior of C is empty, we say that G is a supporting hypersphere of C passing through t. We also say that G supports C at t. If H is the hemisphere bounded by G and containing C, we say that H is a supporting hemisphere of C and that H supports C at t.

For any distinct non-opposite hemispheres G and H the set $L=G\cap H$ is called a lune of $\mathbb{S}^n$. The $(n-1)$-dimensional hemispheres bounding the lune which are contained in G and H, respectively, are denoted by G/H and H/G. By the thickness $\Delta(L)$ of a lune $L=G\cap H\subset\mathbb{S}^n$ we mean the spherical distance of the centers of the $(n-1)$-dimensional hemispheres G/H and H/G bounding L[20].

We say that a lune passes through a boundary point p of a convex body $C\subset\mathbb{S}^n$ if the lune contains C and if the boundary of the lune contains p. If the centers of both $(n-1)$-dimensional hemispheres bounding a lune belong to C, then we call such a lune an orthogonally supporting lune of C.

For an s-convex body $C\subset\mathbb{S}^n$ and any hemisphere K supporting C we define the width of C determined by K, denoted by $\mathrm{width}_K(C)$, as the minimum thickness of a lune $K\cap K^*$ over all hemispheres $K^*\neq K$ supporting C[20]. By a compactness argument we see that at least one such a hemisphere K^* exists, and thus at least one corresponding lune $K\cap K^*$ exists. We say that a convex body $W\subset\mathbb{S}^n$ is of constant width ω provided for every supporting hemisphere K of W we have $\mathrm{width}_K(C)=\omega$[20].

Lemma 1[17, 20] Every two convex sets on $\mathbb{S}^n$ with disjoint interiors are subsets of two opposite hemispheres.

Lemma 2[17] If $W\subset\mathbb{S}^n$ is a body of constant width ω , then $\mathrm{diam}(W)=\omega$.

We say that a convex body $D\subset\mathbb{S}^n$ with diameter δ is of constant diameter δ provided that for arbitrary $p\in bd(D)$ there exists $p'\in bd(D)$ such that $|pp'|=\delta$[20, 21]. We say that any subset of a hemisphere of $\mathbb{S}^n$ which is the largest (in the sense of inclusion) set of a given diameter $\delta\in(0,\pi)$ is a complete set of diameter δ, or a complete set for brevity.

Lemma 3[17]

(i) Bodies of constant diameter on $\mathbb{S}^n$ coincide with complete bodies;

(ii) Bodies of constant diameter on $\mathbb{S}^n$ coincide with bodies of constant width.

3 The Circumscribed Ball of Spherical Body of Constant Width

In the n-dimensional Euclidean space $\mathbb{R}^n$, if W is a convex bodies of constant width ω, then the insphere and circumsphere are concentric and their radii, $r(W)$ and $R(W)$, respectively, satisfy

$$r(W)+R(W)=\omega$$

and

$$\omega(1-\sqrt{\frac{n}{n+2}})\leqslant r(W)\leqslant R(W)\leqslant\omega\sqrt{\frac{n}{2n+2}} \tag{6}$$

In this section, we give a similar result of spherical bodies of constant width to the one for convex bodies in Euclidean spaces. The following results and approach of their proofs are implied vaguely in Ref.[22].

For $x\in\mathbb{S}^n$ and $0\leqslant r\leqslant\frac{\pi}{2}$, denote $B(x,r):=\{y\in\mathbb{S}^n\,|\,|xy|\leqslant r\}$, the spherical ball with radius r and centered at x.

Lemma 4 Let $C\subset\mathbb{S}^n$ be an s-convex body with diameter δ. If $B(x,r)\subset C$, then $C\subset B(x,\delta-r)$.

Proof Suppose $B(\delta-r)$ does not contain C. Let $y\in C\backslash B(\delta-r)$ and let l be the great circle through x and

y. Take the point p being the intersection of $bd(B(x,\delta-r))$ with l such that p is in-between x and y, i.e. p locates on the short arc connecting x and y, and take $q=bd(B(x,r))\cap l$ such that x is in-between p and q. Then $|pq|=\delta$ and hence $|yq|>\delta$, contradicting the fact that the diameter of C is δ.

Lemma 5 Let $W\subset\mathbb{S}^n$ be an s-convex body of constant width δ. If $B(x,r)\supset W$, then $W\supset B(x,r-\delta)$.

Proof Suppose there is a point p of $B(x,\delta-r)$ that is not in W. Let l be the great circle through x and p. By Lemma 1, there exist two opposite hemispheres $H(q)$ and $H(q')$ such that $p\in H(q)$, $W\subset H(q')$ and l is perpendicular to $H(q')$. Let s, t be the two intersecting points of l with the boundary of W, such that s is in-between p and x. Let p' be a point of the intersection of l with $bd(B(x,r))$ such that x is in-between p and p'. Let H' be the unique hemisphere supports $B(x,r)$ at p'. Then the lune $H(q')\cap H'$ contains W. Notice that the thickness of the lune $H(q')\cap H'$ is less than δ, which implies that W is not a spherical body of constant width.

Theorem 1 Let $W\subset\mathbb{S}^n$ be an s-convex body of constant width δ. Then the insphere and circumsphere of W are concentric, and the sum of their radii is δ.

Proof Let W be a spherical body of constant width, and $B(x,r)$ be its circumsphere, where $x\in W$. Since the diameter of W is δ, we have $R(W)\leqslant\delta$. Let $B(\delta-R(W))$ be the ball concentric with $B_o(W)$ having radius $\delta-R(W)$. By Lemma 5, $B(\delta-R(W))$ is contained in W. We will prove that $B(\delta-R(W))$ is an insphere of W and is unique. Suppose that it is not; then there is a ball $B(r')$ different from $B(\delta-R(W))$, with radius $r'\geqslant\delta-R(W)$. Then, by Lemma 4, there exists a sphere $B_o(W)'$ concentric with $B(r')$ and having radius $\delta-r'$, which is a contradiction, since $\delta-r'\leqslant R(W)$ and $B_o(W)$ is different from $B_o(W)'$.

Remark 1 Bodies of constant width ω in an n-dimensional Riemannian manifold M^n were introduced and studied by Dekster[12]. Then Dekster studied the incenter and circumcenter of bodies K of constant width in M^n. He proved that each circumcenter of K is an incenter and vice versa, and the inradius r_i and circumradius r_c fulfill $r_i+r_c=\omega$.

The following is the theorem of Jung's type for spherical space.

Theorem 2 [13] Let C be a compact set in $\mathbb{S}^n$ of diameter δ and circumradius R. Let B be a ball of radius R containing C. Then

(i) $\delta\geqslant 2\arcsin(\sqrt{\dfrac{n+1}{2n}}\sin R)$, where $R\in[0,\pi]$ (7)

(ii) $\delta=2\arcsin(\sqrt{\dfrac{n+1}{2n}}\sin R)$ if and only if there exist $n+1$ points on the boundary of B such that these points are of equidistant δ. In other words, C contains an n-dimensional spherical regular simplex with diameter δ.

Remark 2 The definition of body of constant width in Theorem 2 is similar to the definition of body of constant diameter in Section 2. By Lemma 3, we know that the two notions coincide.

4 The Minkowski Measure of Spherical Body of Constant Width

In this section, we firstly give a definition of the Minkowski measure of asymmetry of spherical body of constant width, then we prove the main Theorem given in Section 0, that is Theorem 3 in this section.

Definition 1 Denote by $W\subset\mathbb{S}^n$ be an s-convex body of constant width ω. Let $R(W)$ be the radius of the circumsphere of W. The Minkowski measure of asymmetry of W is defined by

$$\mathrm{as}_s(W)=\frac{\sin R(W)}{2\sin\dfrac{\omega}{2}-\sin(R(W))}\tag{8}$$

Remark 3

(i) We have not found a suitable definition of the Minkowski measure of asymmetry for general spherical convex bodies;

(ii) The Definition 1 is motivated by the work of Brandenberg and Merino[2].

In the following, we prove our main result.

Theorem 3 Let $W\subset\mathbb{S}^n$ be an s-convex body of constant width. Then,

$$1\leqslant\mathrm{as}_s(W)\leqslant\frac{n+\sqrt{2n(n+1)}}{n+2}\tag{9}$$

The equality holds on the left-hand side if and only if K is a spherical ball, and on the right-hand side if and only if K is a completion of a spherical regular simplex.

Proof Let ω be the constant width of W, and $r(W)$, $R(W)$ the radii of the insphere, circumsphere respectively. Then $R(W)\geqslant\dfrac{\omega}{2}$, which implies that $\mathrm{as}_s(W)=\dfrac{\sin R(W)}{2\sin\dfrac{\omega}{2}-\sin(R(W))}\geqslant 1$. We have $\mathrm{as}_s(W)=1$ if and only if $R(W)=\dfrac{\omega}{2}=r(W)$. Hence, $\mathrm{as}_s(W)=1$ if and only if W

is a spherical ball.

By Theorem 2, we have $\sin\frac{\omega}{2} \geqslant \sqrt{\frac{n+1}{2n}}\sin R(W)$, then

$$\begin{aligned}\mathrm{as}_s(W) &= \frac{\sin R(W)}{2\sin\frac{\omega}{2} - \sin(R(W))} \\ &= \frac{1}{\frac{2\sin\frac{\omega}{2}}{\sin R(K)} - 1} \\ &\leqslant \frac{1}{2\sqrt{\frac{n+1}{2n}} - 1} \\ &= \frac{n+\sqrt{2n(n+1)}}{n+2} \qquad (10)\end{aligned}$$

By Theorem 2, the equality holds in the above inequality if and only if W contains a spherical regular simplex of diameter ω, which implies that W is a completion of a spherical regular simplex.

Corollary 1 Let $W \subset \mathbb{S}^2$ be an s-convex body of constant width. Then,

$$1 \leqslant \mathrm{as}_s(W) \leqslant \frac{1+\sqrt{3}}{2} \qquad (11)$$

The equality holds on the left-hand side if and only if W is a spherical disc, and on the right-hand side if and only if W is a spherical Reuleaux triangle.

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