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Complete *q*th-Moment Convergence of Moving Average Process for *m*-WOD Random Variable

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Abstract: In this paper, we obtained complete qth-moment convergence of the moving average processes, which is generated by m-WOD moving random variables. The results in this article improve and extend the results of the moving average process. m-WOD random variables include WOD, m-NA, m-NOD and m-END random variables, so the results in the paper also promote the corresponding ones in WOD, m-NA, m-NOD, m-END random variables .

Key words: m-WOD random variable; moving average processes; complete convergence; complete qth-moment convergence

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0 Introduction

In many statistical models, it is not reasonable to assume that random variables are independent, so it is very meaningful to extend the concept of independence to dependence cases. Scholars have given many types of dependent random variables, such as negatively associated (NA) random variables, negatively orthant dependent (NOD) random variables, and extend negatively dependent (END) random variables. One important dependence sequence of these dependence is widely orthant dependent (WOD) random variables, which was introduced by Wang *et al*[1] , as follows.

Definition 1 The random variables $\{X_n, n \geqslant 1\}$ are said to be widely upper orthant dependent (WUOD) random variables, if there exists a finite sequence of real numbers $\{g_U(n), n \geqslant 1\}$ such that for each $n \geqslant 1$, $x_1, x_2, \cdots, x_n \in \mathbb{R}$,

$$P(X_1 > x_1, X_2 > x_2, \cdots, X_n > x_n) \leqslant g_U(n) \prod_{i=1}^{n} P(X_i > x_i) \quad (1)$$

The random variables $\{X_n, n \geqslant 1\}$ are said to be widely lower orthant dependent (WLOD, for short) random variables, if there exists a finite sequence of real numbers $\{g_L(n), n \geqslant 1\}$ such that for each $x_1, x_2, \cdots, x_n \in \mathbb{R}$,

$$P(X_1 \leqslant x_1, X_2 \leqslant x_2, \cdots, X_n \leqslant x_n) \leqslant g_L(n) \prod_{i=1}^{n} P(X_i \leqslant x_i) \quad (2)$$

The random variables $\{X_n, n \geqslant 1\}$ are said to be WOD random variables, if the random variables $\{X_n, n \geqslant 1\}$ are both WUOD and WLOD, $g(n) = \max\{g_U(n), g_L(n)\}$

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are called dominated coefficients.

Inspired by m-NA and WOD, the concept of m-WOD random variables was introduced by Fang *et al*[2], as follows:

Definition 2 For fix integer $m\geqslant 1$, the random variables $\{X_n, n\geqslant 1\}$ is called to be m-WOD if for any $n\geqslant 2$, $i_1, i_2, \cdots, i_n\in N^+$, such that $|i_k-i_j|\geqslant m$ for all $1\leqslant k\neq j\leqslant n$, we get the $X_{i_1}, X_{i_2}, \cdots, X_{i_n}$ are WOD random variables.

By (1) and (2), we have $g_U(n)\geqslant 1, g_L(n)\geqslant 1$. If $g_U(n)=g_L(n)=1$, then WOD random variables are NOD random variables, which were introduced by Ebrahimi and Ghosh[3]. When $g_U(n)=g_L(n)=M\geqslant 1$, then WOD random variables are END random variables, which were introduced by Liu[4]. Liu[4] pointed out that the END random variables are more comprehensive, which imply NA and positive random variables, so m-WOD random variables includes independent random variables, WOD, m-NA, m-NOD, m-END random variables and so on. Therefore, it is interesting to investigate the probability limit theory and its applications for m-WOD random variables.

Let $\{Y_n, -\infty<n<\infty\}$ be a random variable sequence and the real number sequence $\{a_n, -\infty<n<\infty\}$ be absolute summable, i.e. $\sum_{i=-\infty}^{\infty}|a_i|<\infty$. Then $\{X_n, n\geqslant 1\}$ is said moving average process under the sequence $\{Y_n, -\infty<n<\infty\}$, if

$$X_n=\sum_{i=-\infty}^{\infty}a_iY_{n+i},\quad n\geqslant 1.$$

After the appearance of moving average process, a lot of conclusions on convergence properties have been obtained. When $\{Y_n, -\infty<n<\infty\}$ is identically distributed, many results about moving average process have been gained[5-8]. Recently, some results have been obtained under the assumption that the sequence $\{Y_n, -\infty<n<\infty\}$ is dependent. For example, Li[9] investigated the convergence properties under ρ-mixing assumptions; Zhang[10] and Chen *et al*[11] established complete convergence under φ-mixing assumptions; Song and Zhu[12] got the complete convergence of moving average process based on ρ^--mixing assumptions; Tao[13] discussed the complete convergence under WOD random variables; Guan[14] obtained the complete moment convergence under m-WOD random variables.

There are few results on the complete qth-moment convergence of moving average process for m-WOD random variables. Therefore, in this paper, based on Guan's research[14], we study the complete qth-moment convergence of moving average processes based on m-WOD random variables, the results extend and improve the corresponding ones under WOD, m-NA, m-NOD, m-END random variables.

Definition 3 The random variables $\{Y_n, -\infty<n<\infty\}$ are called be stochastically dominated by a random variable Y, if for any $x>0, P(|Y_n|>x)\leqslant CP(|Y|>x), -\infty<n<\infty$, where the constant $C>0$, and denote $\{Y_n, -\infty<n<\infty\}\prec Y$.

In this paper, $I(A)$ denotes the indicator function of an event A, the symbol C represents a positive constant, which can take different values in different places, even in the same formula. $\log n=\ln\max\{x, e\}, X^+=XI(X>0), g(n)=\max\{g_U(n), g_L(n)\}$.

1 Some Lemmas and Main Results

Lemma 1[2] The sequence $\{Y_n, -\infty<n<\infty\}$ are m-WOD random variables, if the function sequences $\{f_n, -\infty<n<\infty\}$ are non-decreasing(non-increasing), then random variables $\{f_n(Y_n), -\infty<n<\infty\}$ are also m-WOD random variables with same dominating coefficients.

Lemma 2[2] $p\geqslant 2$, the sequence $\{Y_n, -\infty<n<\infty\}$ are m-WOD random variables with dominating coefficients $g(n)$. For every $j\geqslant 1$, the $EY_j=0$ and $E|Y_j|^p<\infty$.

Then, there exist positive constants $C_1=C_1(p,m), C_2=C_2(p,m)$, depending only on p and m, such that

$$E(|\sum_{i=1}^{n}Y_i|^p)\leqslant C_1(p,m)\sum_{i=1}^{n}E|Y_i|^p+C_2(p,m)g(n)(\sum_{i=1}^{n}EY_i^2)^{p/2}.$$

Lemma 3[15] Let $\{Y_n, -\infty<n<\infty\}\prec Y, a>0, b>0$ are constant, then there exists positive constant C_1, C_2 such that following inequalities are established:

$$E|Y_n|^aI(|Y_n|\leqslant b)\leqslant C[E|Y|^aI(|Y|\leqslant b)+b^aI(|Y|>b)],$$
$$E|Y_n|^aI(|Y_n|>b)\leqslant CE|Y|^aI(|Y|>b)$$

Lemma 4 [14] Let $r>1, 1\leqslant p<2, \{X_n, n\geqslant 1\}$ be a moving average process under the sequence $\{Y_n, -\infty<n<\infty\}$ of m-WOD random variables with dominating coefficients $g(n)$, $\{Y_n, -\infty<n<\infty\}\prec Y$ and $EY_j=0$ for every j, the real number sequence $\{a_n, -\infty<n<\infty\}$ is absolute summable. If $E|Y|^{rp}<\infty, g(n)=o(n^\delta), \delta\geqslant 0$, then

$$\sum_{n=1}^{\infty} n^{r-2}P(|\sum_{j=1}^{n}X_j|\geqslant \varepsilon n^{1/p})<\infty, \forall\varepsilon>0.$$

Proof If $r>1, 1\leqslant p<2,$ obviously $r=rp(1/p)$, and $1/p>1/2, rp>1,$ so Lemma 4 satisfies the conditions of the Theorem 3.1 in Ref. [14].

If r=1, we have the following results:

Lemma 5 [14] Let $1\leqslant p<2, \{X_n, n\geqslant 1\}$ be a moving average process under the sequence $\{Y_n, -\infty<n<\infty\}$ of m-WOD random variables with dominating coefficients $g(n)$, $\{Y_n, -\infty<n<\infty\}\prec Y$ and $EY_j=0$ for every j. Suppose $\sum_{i=-\infty}^{\infty}|a_i|^\theta<\infty$, where $\theta\in(0, 1)$ if p=1, and $\theta=1$ if $1<p<2$. If $E|Y|^{p(1+\delta)}<\infty, g(n)=o(n^\delta), 0\leqslant\delta<(2-p)/p$, then

$$\sum_{n=1}^{\infty} n^{-1}P(|\sum_{j=1}^{n}X_j|\geqslant \varepsilon n^{1/p})<\infty, \forall\varepsilon>0.$$

Now, we present the main results, the proofs for them will be postponed in next section.

Theorem 1 Let $q>0, r>1, 1\leqslant p<2, \{X_n, n\geqslant 1\}$ be a moving average process under the sequence $\{Y_n, -\infty<n<\infty\}$ of m-WOD random variables with dominating coefficients $g(n)$,$\{Y_n, -\infty<n<\infty\}\prec Y$ and $EY_j=0$ for every j, the real number sequence $\{a_n, -\infty<n<\infty\}$ is absolute summable. Let $g(n)=o(n^\delta), \delta\geqslant 0$, if

$$\begin{cases} E|Y|^{rp}<\infty, & \text{if } q<rp,\\ E|Y|^{rp}\log(1+|Y|), & \text{if } q=rp,\\ E|Y|^{q}<\infty, & \text{if } q>rp,\end{cases}$$

then

$$\sum_{n=1}^{\infty} n^{r-2-q/p}E\{|\sum_{j=1}^{n}X_j|-\varepsilon n^{1/p}\}_+^q<\infty, \forall\varepsilon>0 \tag{3}$$

If r=1, we have the following result:

Theorem 2 Let $0<q<2, 1\leqslant p<2, \{X_n, n\geqslant 1\}$ be a moving average process under the sequence $\{Y_n, -\infty<n<\infty\}$ of m-WOD random variables with dominating coefficients $g(n)$, $\{Y_n, -\infty<n<\infty\}\prec Y$ and $EY_j=0$ for every j. Suppose $\sum_{i=-\infty}^{\infty}|a_i|^\theta<\infty$, where $\theta\in(0, 1)$ if $p=1$, and $\theta=1$ if $1<p<2$. Let $g(n)=o(n^\delta), 0\leqslant\delta<\min\{2/p-1, (2-q)/p\}$, if

$$\begin{cases} E|Y|^{p(1+\delta)}<\infty, & \text{if } q<p,\\ E|Y|^{p(1+\delta)}\log(1+|Y|), & \text{if } q=p,\\ E|Y|^{q(1+\delta)}<\infty, & \text{if } q>p,\end{cases}$$

then

$$\sum_{n=1}^{\infty} n^{-1-q/p}E\{|\sum_{j=1}^{n}X_j|-\varepsilon n^{1/p}\}_+^q<\infty, \forall\varepsilon>0 \tag{4}$$

Remark 1 Theorems 1 and 2 obtain the complete qth-moment convergence of moving average processes $\{X_n, n\geqslant 1\}$ under m-WOD random variables. We get the results without slowly varying function, so our results in the paper extend and improve the results in Ref.[14].

Remark 2 Noting, m-WOD random variables include WOD, m-NA, m-NOD, m-END random variables and so on, so our results also hold for WOD, m-NA, m-NOD, m-END random variables, therefore our Theorems 1 and 2 improve the known results.

Remark 3 We know that the condition $\{Y_n, -\infty<n<\infty\}\prec Y$ is weaker than the condition of identical distribution for $\{Y_n, -\infty<n<\infty\}$. Thus, our results still hold for identically distributed random variables.

2 Proof of Theorems

Proof of Theorem 1 By $E|Y|<\infty, \sum_{i=-\infty}^{\infty}|a_i|<\infty$, we get

$$E|X_n|\leqslant \sum_{i=-\infty}^{\infty}E|a_iY_{n+i}|\leqslant CE|Y|\sum_{i=-\infty}^{\infty}|a_i|<\infty, \qquad \forall n\geqslant 1,$$

Therefore X_n exists. Let $x>n^{q/p}$, write

$$Y_j'=-x^{1/q}I(Y_j<-x^{1/q})+Y_jI(|Y_j|\leqslant x^{1/q})+x^{1/q}I(Y_j>x^{1/q}),$$
$$Y_j''=Y_j-Y_j'=(Y_j-x^{1/q})I(Y_j>x^{1/q})+(Y_j+x^{1/q})I(Y_j<-x^{1/q}).$$

Note that

$$\sum_{j=1}^{n}X_j=\sum_{j=1}^{n}\sum_{i=-\infty}^{\infty}a_iY_{j+i}=\sum_{i=-\infty}^{\infty}a_i\sum_{j=i+1}^{i+n}Y_j.$$

For $r>1, rp>1, \sum_{i=-\infty}^{\infty}|a_i|<\infty, EY_j=0, |Y_j''|\leqslant|Y_j|I(|Y_j|>x^{1/q})$, by Lemma 3, we have

$$\begin{aligned}
&x^{-1/q}|E\sum_{i=-\infty}^{\infty}a_i\sum_{j=i+1}^{i+n}Y_j'|\\
&=x^{-1/q}|E\sum_{i=-\infty}^{\infty}a_i\sum_{j=i+1}^{i+n}Y_j''|\\
&\leqslant x^{-1/q}\sum_{i=-\infty}^{\infty}|a_i|\sum_{j=i+1}^{i+n}E|Y_j|I(|Y_j|>x^{1/q})\leqslant Cnx^{-1/q}E|Y|I(|Y|>x^{1/q})\\
&\leqslant Cnx^{-rp/q}E|Y|^{rp}I(|Y|>x^{1/q})\\
&\leqslant Cn^{1-r}E|Y|^{rp}I(|Y|>x^{1/q})\to 0, \text{as } n\to\infty
\end{aligned}$$

So, we have

$$x^{-1/q}|E\sum_{i=-\infty}^{\infty}a_i\sum_{j=i+1}^{i+n}Y_j'|<1/4 \tag{5}$$

For

$$\begin{aligned}
&\sum_{n=1}^{\infty}n^{r-2-q/p}E\{|\sum_{j=1}^{n}X_j|-\varepsilon n^{1/p}\}_+^q\\
=&\sum_{n=1}^{\infty}n^{r-2-q/p}\int_0^{\infty}P(|\sum_{j=1}^{n}X_j|>\varepsilon n^{1/p}+x^{1/q})\mathrm{d}x\\
\leqslant&\sum_{n=1}^{\infty}n^{r-2-q/p}\int_0^{n^{q/p}}P(|\sum_{j=1}^{n}X_j|>\varepsilon n^{1/p})\mathrm{d}x+\sum_{n=1}^{\infty}n^{r-2-q/p}\int_{n^{q/p}}^{\infty}P(|\sum_{j=1}^{n}X_j|>x^{1/q})\mathrm{d}x\\
=&\sum_{n=1}^{\infty}n^{r-2}P(|\sum_{j=1}^{n}X_j|>\varepsilon n^{1/p})+\sum_{n=1}^{\infty}n^{r-2-q/p}\int_{n^{q/p}}^{\infty}P(|\sum_{j=1}^{n}X_j|>x^{1/q})\mathrm{d}x\\
=&:I_1+I_2
\end{aligned}$$

by Lemma 4, we obtain

$$I_1=\sum_{n=1}^{\infty}n^{r-2}P(|\sum_{j=1}^{n}X_j|>\varepsilon n^{1/p})<\infty.$$

To prove (3) of Theorem 1, we only need to prove

$$I_2=\sum_{n=1}^{\infty}n^{r-2-q/p}\int_{n^{q/p}}^{\infty}P(|\sum_{j=1}^{n}X_j|>x^{1/q})\mathrm{d}x<\infty.$$

By (5), we get

$$\begin{aligned}
I_2\leqslant& C\sum_{n=1}^{\infty}n^{r-2-q/p}\int_{n^{q/p}}^{\infty}P(|\sum_{i=-\infty}^{\infty}a_i\sum_{j=i+1}^{i+n}Y_j''|\geqslant x^{1/q}/2)\mathrm{d}x+C\sum_{n=1}^{\infty}n^{r-2-q/p}\int_{n^{q/p}}^{\infty}P(|\sum_{i=-\infty}^{\infty}a_i\sum_{j=i+1}^{i+n}(Y_j'-EY_j')|\geqslant x^{1/q}/4)\,\mathrm{d}x\\
=&:I_{21}+I_{22}.
\end{aligned}$$

For I_{21}, by Markov inequality and Lemma 3, we get

$$\begin{aligned}
I_{21} &\leqslant C\sum_{n=1}^{\infty} n^{r-2-q/p}\int_{n^{q/p}}^{\infty} x^{-1/q} E|\sum_{i=-\infty}^{\infty} a_i \sum_{j=i+1}^{i+n} Y_j''|\,\mathrm{d}x \leqslant C\sum_{n=1}^{\infty} n^{r-2-q/p}\int_{n^{q/p}}^{\infty} x^{-1/q}\sum_{i=-\infty}^{\infty}|a_i|\sum_{j=i+1}^{i+n} E|Y_j''|\,\mathrm{d}x \\
&\leqslant C\sum_{n=1}^{\infty} n^{r-2-q/p}\int_{n^{q/p}}^{\infty} x^{-1/q}\sum_{i=-\infty}^{\infty}|a_i|\sum_{j=i+1}^{i+n} E|Y_j|\, I(|Y_j|>x^{1/q})\mathrm{d}x \leqslant C\sum_{n=1}^{\infty} n^{r-1-q/p}\int_{n^{q/p}}^{\infty} x^{-1/q} E|Y| I(|Y|>x^{1/q})\mathrm{d}x \\
&= C\sum_{n=1}^{\infty} n^{r-1-q/p}\sum_{m=n}^{\infty}\int_{m^{q/p}}^{(m+1)^{q/p}} x^{-1/q} E|Y| I(|Y|>x^{1/q})\mathrm{d}x \\
&\leqslant C\sum_{n=1}^{\infty} n^{r-1-q/p}\sum_{m=n}^{\infty} m^{q/p-1/p-1} E|Y| I(|Y|>m^{1/p}) \\
&= C\sum_{m=1}^{\infty} m^{q/p-1/p-1} E|Y| I(|Y|>m^{1/p})\sum_{n=1}^{m} n^{r-1-q/p} \\
&\leqslant \begin{cases} C\sum_{m=1}^{\infty} m^{r-1/p-1} E|Y| I(|Y|>m^{1/p}), & \text{if } q<rp \\ C\sum_{m=1}^{\infty} m^{r-1/p-1}\log(m+1) E|Y| I(|Y|>m^{1/p}), & \text{if } q=rp \\ C\sum_{m=1}^{\infty} m^{q/p-1/p-1} E|Y| I(|Y|>m^{1/p}), & \text{if } q>rp \end{cases} \\
&= \begin{cases} C\sum_{m=1}^{\infty} m^{r-1/p-1}\sum_{k=m}^{\infty} E|Y|\, I(k^{1/p}<|Y|\leqslant(k+1)^{1/p}), & \text{if } q<rp \\ C\sum_{m=1}^{\infty} m^{r-1/p-1}\log(m+1)\sum_{k=m}^{\infty} E|Y|\, I(k^{1/p}<|Y|\leqslant(k+1)^{1/p}), & \text{if } q=rp \\ C\sum_{m=1}^{\infty} m^{q/p-1/p-1}\sum_{k=m}^{\infty} E|Y|\, I(k^{1/p}<|Y|\leqslant(k+1)^{1/p}), & \text{if } q>rp \end{cases} \\
&\leqslant \begin{cases} C\sum_{k=1}^{\infty} k^{r-1/p} E|Y| I(k^{1/p}<|Y|\leqslant(k+1)^{1/p}), & \text{if } q<rp \\ C\sum_{k=1}^{\infty} k^{r-1/p}\log(k+1) E|Y| I(k^{1/p}<|Y|\leqslant(k+1)^{1/p}), & \text{if } q=rp \\ C\sum_{k=1}^{\infty} k^{q/p-1/p} E|Y| I(k^{1/p}<|Y|\leqslant(k+1)^{1/p}), & \text{if } q>rp \end{cases} \\
&\leqslant \begin{cases} CE|Y|^{rp}<\infty, & \text{if } q<rp \\ CE|Y|^{rp}\log(|Y|+1)<\infty, & \text{if } q=rp \\ CE|Y|^{q}<\infty, & \text{if } q>rp \end{cases} \qquad (6)
\end{aligned}$$

For I_{22}, by Lemmas 1-3, Markov and Hölder inequalities, we have that for any $v\geqslant 2$,

$$\begin{aligned}
I_{22} &\leqslant C\sum_{n=1}^{\infty} n^{r-2-q/p}\int_{n^{q/p}}^{\infty} x^{-v/q} E\left\{|\sum_{i=-\infty}^{\infty} a_i\sum_{j=i+1}^{i+n}(Y_j'-EY_j')|^v\right\}\mathrm{d}x \\
&\leqslant C\sum_{n=1}^{\infty} n^{r-2-q/p}\int_{n^{q/p}}^{\infty} x^{-v/q} E\left\{\sum_{i=-\infty}^{\infty}|a_i||\sum_{j=i+1}^{i+n}(Y_j'-EY_j')|\right\}^v\mathrm{d}x \\
&= C\sum_{n=1}^{\infty} n^{r-2-q/p}\int_{n^{q/p}}^{\infty} x^{-v/q} E\left\{\sum_{i=-\infty}^{\infty}|a_i|^{1-1/v}(|a_i|^{1/v}|\sum_{j=i+1}^{i+n}(Y_j'-EY_j')|)\right\}^v\mathrm{d}x \\
&\leqslant C\sum_{n=1}^{\infty} n^{r-2-q/p}\int_{n^{q/p}}^{\infty} x^{-v/q}(\sum_{i=-\infty}^{\infty}|a_i|)^{v-1}\left\{\sum_{i=-\infty}^{\infty}|a_i| E(|\sum_{j=i+1}^{i+n}(Y_j'-EY_j')|^v)\right\}\mathrm{d}x \\
&\leqslant C\sum_{n=1}^{\infty} n^{r-2-q/p}\int_{n^{q/p}}^{\infty} x^{-v/q}\sum_{i=-\infty}^{\infty}|a_i|\left\{\sum_{j=i+1}^{i+n} E|Y'|^v + g(n)(\sum_{j=i+1}^{i+n} E|Y'|^2)^{v/2}\right\}\mathrm{d}x \\
&\leqslant C\sum_{n=1}^{\infty} n^{r-2-q/p}\int_{n^{q/p}}^{\infty} x^{-v/q}\sum_{i=-\infty}^{\infty}|a_i|\left\{\sum_{j=i+1}^{i+n}[E|Y_j|^v I(|Y_j|\leqslant x^{1/q}) + x^{v/q}P(|Y_j|>x^{1/q})]\right\}\mathrm{d}x
\end{aligned}$$

$$+C\sum_{n=1}^{\infty}n^{r-2-q/p+\delta}\int_{n^{q/p}}^{\infty}x^{-v/q}\sum_{i=-\infty}^{\infty}|a_i|\left\{\sum_{j=i+1}^{i+n}[E|Y_j|^2 I(|Y_j|\leqslant x^{1/q})+x^{2/q}P(|Y_j|>x^{1/q})]\right\}^{v/2}\mathrm{d}x$$

$$\leqslant C\sum_{n=1}^{\infty}n^{r-1-q/p}\int_{n^{q/p}}^{\infty}x^{-v/q}[E|Y|^v I(|Y|\leqslant x^{1/q})+x^{v/q}P(|Y|>x^{1/q})]\mathrm{d}x$$

$$+C\sum_{n=1}^{\infty}n^{r-2-q/p+v/2+\delta}\int_{n^{q/p}}^{\infty}x^{-v/q}[E|Y|^2 I(|Y|\leqslant x^{1/q})+x^{2/q}P(|Y|>x^{1/q})]^{v/2}\mathrm{d}x$$

$$=:I_{221}+I_{222}.$$

For I_{221}, taking $v>\max\{2,q,rp\}$, we have

$$I_{221}=C\sum_{n=1}^{\infty}n^{r-1-q/p}\sum_{m=n}^{\infty}\int_{m^{q/p}}^{(m+1)^{q/p}}[x^{-v/q}E|Y|^v I(|Y|\leqslant x^{1/q})+P(|Y|>x^{1/q})]\mathrm{d}x$$

$$\leqslant C\sum_{n=1}^{\infty}n^{r-1-q/p}\sum_{m=n}^{\infty}[m^{q/p-v/p-1}E|Y|^v I(|Y|\leqslant(m+1)^{1/p})+m^{q/p-1}P(|Y|>m^{1/q})]$$

$$\leqslant C\sum_{m=1}^{\infty}[m^{q/p-v/p-1}E|Y|^v I(|Y|\leqslant(m+1)^{1/p})+m^{q/p-1}P(|Y|>m^{1/q})]\sum_{n=1}^{m}n^{r-1-q/p}$$

$$\leqslant\begin{cases}C\sum_{m=1}^{\infty}[m^{r-v/p-1}E|Y|^v I(|Y|\leqslant(m+1)^{1/p})+m^{r-1}P(|Y|>m^{1/q})], & \text{if } q<rp\\ C\sum_{m=1}^{\infty}[m^{r-v/p-1}E|Y|^v I(|Y|\leqslant(m+1)^{1/p})+m^{r-1}P(|Y|>m^{1/q})]\log(m+1), & \text{if } q=rp\\ C\sum_{m=1}^{\infty}[m^{q/p-v/p-1}E|Y|^v I(|Y|\leqslant(m+1)^{1/p})+m^{q/p-1}P(|Y|>m^{1/q})], & \text{if } q>rp\end{cases}$$

$$\leqslant\begin{cases}C\sum_{k=1}^{\infty}k^{r-v/p}E|Y|^v I(k^{1/p}<|Y|\leqslant(k+1)^{1/p})+CE|Y|^{rp}, & \text{if } q<rp\\ C\sum_{k=1}^{\infty}k^{r-v/p}\log(k+1)E|Y|^v I(k^{1/p}<|Y|\leqslant(k+1)^{1/p})+CE|Y|^{rp}\log(|Y|+1), & \text{if } q=rp\\ C\sum_{k=1}^{\infty}k^{q/p-v/q}E|Y|^v I(k^{1/p}<|Y|\leqslant(k+1)^{1/p})+CE|Y|^q, & \text{if } q>rp\end{cases}$$

$$\leqslant\begin{cases}CE|Y|^{rp}<\infty, & \text{if } q<rp\\ CE|Y|^{rp}\log(|Y|+1)<\infty, & \text{if } q=rp\\ CE|Y|^q<\infty, & \text{if } q>rp\end{cases}\tag{7}$$

For I_{222}, taking $v>\max\{2,2q/p\}$, we obtain

$$I_{222}\leqslant C\sum_{n=1}^{\infty}n^{r-2-q/p+v/2+\delta}\int_{n^{q/p}}^{\infty}x^{-v/q}\left\{[E|Y|^2 I(|Y|\leqslant x^{1/q})]^{v/2}+x^{v/q}P^{v/2}(|Y|>x^{1/q})\right\}\mathrm{d}x$$

$$=C\sum_{n=1}^{\infty}n^{r-2-q/p+v/2+\delta}\sum_{m=n}^{\infty}\int_{m^{q/p}}^{(m+1)^{q/p}}\{x^{-v/q}[E|Y|^2 I(|Y|\leqslant x^{1/q})]^{v/2}+P^{v/2}(|Y|>x^{1/q})\}\mathrm{d}x$$

$$\leqslant C\sum_{n=1}^{\infty}n^{r-2-q/p+v/2+\delta}\sum_{m=n}^{\infty}[m^{q/p-v/p-1}(E|Y|^2 I(|Y|\leqslant(m+1)^{1/p})^{v/2}+m^{q/p-1}P^{v/2}(|Y|>m^{1/p})]$$

$$\leqslant C\sum_{m=1}^{\infty}[m^{q/p-v/p-1}(E|Y|^2 I(|Y|\leqslant(m+1)^{1/p})^{v/2}+m^{q/p-1}P^{v/2}(|Y|>m^{1/p})]\sum_{n=1}^{m}n^{r-2-q/p+v/2+\delta}$$

$$\leqslant C\sum_{m=1}^{\infty}[m^{r-2-v/p+v/2+\delta}(E|Y|^2 I(|Y|\leqslant(m+1)^{1/p})^{v/2}+m^{r-2+v/2+\delta}P^{v/2}(|Y|>m^{1/p})]$$

If $rp<2$, taking $v>\max\{2q/p,2+2\delta/(r-1)\}$, so $r-2+v/2-rv/2+\delta<-1$, then

$$I_{222}\leqslant C\sum_{m=1}^{\infty}m^{r-2+v/2-rv/2+\delta}[(E|Y|^{rp}I(|Y|\leqslant(m+1)^{1/p})^{v/2}+(E|Y|^{rp}(|Y|>m^{1/p}))^{v/2}]$$

$$\leqslant C\sum_{m=1}^{\infty}m^{r-2+v/2-rv/2+\delta}(E|Y|^{rp})^{v/2}<\infty\tag{8}$$

If $rp\geqslant 2$, we get $E|Y|^2<\infty$, taking $v>\max\{2,2q/p,(r-1+\delta)2p/(2-p)\}$, then $r-2+v/2-v/p+\delta<-1$, and

$$I_{222} \leqslant C\sum_{m=1}^{\infty}[m^{r-2-v/p+v/2+\delta}[(E|Y|^2 I(|Y| \leqslant (m+1)^{1/p})^{v/2} + (E|Y|^2(|Y| > m^{1/p}))^{v/2}]$$
$$\leqslant C\sum_{m=1}^{\infty} m^{r-2-v/p+v/2+\delta}(E|Y|^2)^{v/2} < \infty \tag{9}$$

By (5)-(9), the proof of Theorem 1 is completed.

Next, we prove Theorem 2.

Proof of Theorem 2 From the proof of Theorem 1, we get

$$\sum_{n=1}^{\infty} n^{-1-q/p} E\{|\sum_{j=1}^{n} X_j| - \varepsilon n^{1/p}\}_+^q$$
$$\leqslant \sum_{n=1}^{\infty} n^{-1} P(|\sum_{j=1}^{n} X_j| > \varepsilon n^{1/p}) + \sum_{n=1}^{\infty} n^{-1-q/p}\int_{n^{q/p}}^{\infty} P(|\sum_{j=1}^{n} X_j| > x^{1/q})\mathrm{d}x$$
$$=: J_1 + J_2.$$

By Lemma 5, we get

$$J_1 = \sum_{n=1}^{\infty} n^{-1} P(|\sum_{j=1}^{n} X_j| > \varepsilon n^{1/p}) < \infty.$$

In order to prove (4), we need to prove

$$J_2 = \sum_{n=1}^{\infty} n^{-1-q/p}\int_{n^{q/p}}^{\infty} P(|\sum_{j=1}^{n} X_j| > x^{1/q})\mathrm{d}x < \infty.$$

Similar to (5), we have

$$x^{-1/q}|E\sum_{i=-\infty}^{\infty} a_i \sum_{j=i+1}^{i+n} Y_j'| \leqslant Cnx^{-1/q} E|Y|I(|Y| > x^{1/q})$$
$$\leqslant Cnx^{-p(1+\delta)/q} E|Y|^{p(1+\delta)} I(|Y| > x^{1/q}) \leqslant Cn^{-\delta} E|Y|^{p(1+\delta)} I(|Y| > x^{1/q}) \to 0, \text{ as } n \to \infty$$

Therefore

$$x^{-1/q}|E\sum_{i=-\infty}^{\infty} a_i \sum_{j=i+1}^{i+n} Y_j'| < 1/4.$$

We have

$$J_2 \leqslant C\sum_{n=1}^{\infty} n^{-1-q/p}\int_{n^{q/p}}^{\infty} P(|\sum_{i=-\infty}^{\infty} a_i \sum_{j=i+1}^{i+n} Y_j''| \geqslant x^{1/q}/2)\mathrm{d}x + C\sum_{n=1}^{\infty} n^{-1-q/p}\int_{n^{q/p}}^{\infty} P(|\sum_{i=-\infty}^{\infty} a_i \sum_{j=i+1}^{i+n}(Y_j' - EY_j')| \geqslant x^{1/q}/4)\mathrm{d}x$$
$$=: J_{21} + J_{22}.$$

For J_{21}, from Lemma 3, Markov and C_r-inequalities, similar to the proof of I_{21}, we have

$$J_{21} \leqslant C\sum_{n=1}^{\infty} n^{-1-q/p}\int_{n^{q/p}}^{\infty} x^{-\theta/q} E|\sum_{i=-\infty}^{\infty} a_i \sum_{j=i+1}^{i+n} Y_j''|^{\theta}\mathrm{d}x$$
$$\leqslant C\sum_{n=1}^{\infty} n^{-q/p}\int_{n^{q/p}}^{\infty} x^{-\theta/q} E|Y|^{\theta} I(|Y| > x^{1/q})\mathrm{d}x$$
$$= C\sum_{n=1}^{\infty} n^{-q/p}\sum_{m=n}^{\infty}\int_{m^{q/p}}^{(m+1)^{q/p}} x^{-\theta/q} E|Y|^{\theta} I(|Y| > x^{1/q})\mathrm{d}x$$
$$\leqslant C\sum_{n=1}^{\infty} n^{-q/p}\sum_{m=n}^{\infty} m^{q/p-\theta/p-1} E|Y|^{\theta} I(|Y| > m^{1/q})$$
$$= C\sum_{m=1}^{\infty} m^{q/p-\theta/p-1} E|Y|^{\theta} I(|Y| > m^{1/q})\sum_{n=1}^{m} n^{-q/p}$$
$$\leqslant \begin{cases} C\sum_{m=1}^{\infty} m^{-\theta/p} E|Y|^{\theta} I(|Y| > m^{1/p}), & \text{if } q < p \\ C\sum_{m=1}^{\infty} m^{-\theta/p}\log(m+1) E|Y|^{\theta} I(|Y| > m^{1/p}), & \text{if } q = p \\ C\sum_{m=1}^{\infty} m^{q/p-\theta/p-1} E|Y|^{\theta} I(|Y| > m^{1/p}), & \text{if } q > p \end{cases}$$

$$\leqslant\begin{cases} C\sum_{k=1}^{\infty}k^{1-\theta/p}E|Y|^{\theta}I(k^{1/p}<|Y|\leqslant(k+1)^{1/p}), & \text{if } q<p \\ C\sum_{k=1}^{\infty}k^{1-\theta/p}\log(k+1)E|Y|^{\theta}I(k^{1/p}<|Y|\leqslant(k+1)^{1/p}), & \text{if } q=p \\ C\sum_{k=1}^{\infty}k^{q/p-\theta/p}E|Y|^{\theta}I(k^{1/p}<|Y|\leqslant(k+1)^{1/p}), & \text{if } q>p \end{cases}$$

$$\leqslant\begin{cases} CE|Y|^{p}<\infty, & \text{if } q<p \\ CE|Y|^{p}\log(|Y|+1)<\infty, & \text{if } q=p \\ CE|Y|^{q}<\infty, & \text{if } q>p \end{cases} \tag{10}$$

For J_{22}, from Lemma 1, Lemma 3, Markov and Hölder inequalities, taking $v=2$, since $0<\delta<\min\left\{2/p-1,\left(2-q\right)/p\right\}$, then

$$J_{22}\leqslant C\sum_{n=1}^{\infty}n^{-1-q/p}\int_{n^{q/p}}^{\infty}x^{-2/q}E\left\{|\sum_{i=-\infty}^{\infty}a_i\sum_{j=i+1}^{i+n}(Y_j'-EY_j')|^2\right\}\mathrm{d}x$$

$$\leqslant C\sum_{n=1}^{\infty}n^{-1-q/p}\int_{n^{q/p}}^{\infty}x^{-2/q}E\left\{\sum_{i=-\infty}^{\infty}|a_i|^{1/2}(|a_i|^{1/2}|\sum_{j=i+1}^{i+n}(Y_j'-EY_j')|)\right\}^2\mathrm{d}x$$

$$\leqslant C\sum_{n=1}^{\infty}n^{-1-q/p}\int_{n^{q/p}}^{\infty}x^{-2/q}(\sum_{i=-\infty}^{\infty}|a_i|)\left\{\sum_{i=-\infty}^{\infty}|a_i|E(|\sum_{j=i+1}^{i+n}(Y_j'-EY_j')|^2)\right\}\mathrm{d}x$$

$$\leqslant C\sum_{n=1}^{\infty}n^{-1-q/p}\int_{n^{q/p}}^{\infty}x^{-2/q}\sum_{i=-\infty}^{\infty}|a_i|\sum_{j=i+1}^{i+n}(1+g(n))E|Y'|^2\,\mathrm{d}x$$

$$\leqslant C\sum_{n=1}^{\infty}n^{-1-q/p+\delta}\int_{n^{q/p}}^{\infty}x^{-2/q}\sum_{i=-\infty}^{\infty}|a_i|\left\{\sum_{j=i+1}^{i+n}[E|Y_j|^2I(|Y_j|\leqslant x^{1/q})+x^{2/q}P(|Y_j|>x^{1/q})]\right\}\mathrm{d}x$$

$$\leqslant C\sum_{n=1}^{\infty}n^{-q/p+\delta}\int_{n^{q/p}}^{\infty}x^{-2/q}[E|Y|^2I(|Y|\leqslant x^{1/q})+x^{2/q}P(|Y|>x^{1/q})]\mathrm{d}x$$

$$=C\sum_{n=1}^{\infty}n^{-q/p+\delta}\sum_{m=n}^{\infty}\int_{m^{q/p}}^{(m+1)^{q/p}}[x^{-2/q}E|Y|^2I(|Y|\leqslant x^{1/q})+P(|Y|>x^{1/q})]\mathrm{d}x$$

$$\leqslant C\sum_{n=1}^{\infty}n^{-q/p+\delta}\sum_{m=n}^{\infty}[m^{q/p-2/p-1}E|Y|^2I(|Y|\leqslant(m+1)^{1/q})+m^{q/p-1}P(|Y|>x^{1/q})]$$

$$\leqslant C\sum_{m=1}^{\infty}[m^{q/p-2/p-1}E|Y|^2I(|Y|\leqslant(m+1)^{1/q})+m^{q/p-1}P(|Y|>x^{1/q})]\sum_{n=1}^{m}n^{-q/p+\delta}$$

$$\leqslant C\sum_{m=1}^{\infty}[m^{q/p-2/p-1}E|Y|^2I(|Y|\leqslant(m+1)^{1/q})+m^{q/p-1}P(|Y|>x^{1/q})]m^{\delta}\sum_{n=1}^{m}n^{-q/p}$$

$$\leqslant\begin{cases} C\sum_{m=1}^{\infty}[m^{-2/p+\delta}E|Y|^2I(|Y|\leqslant(m+1)^{1/p})+m^{\delta}P(|Y|>m^{1/q})], & \text{if } q<p \\ C\sum_{m=1}^{\infty}[m^{-2/p+\delta}E|Y|^2I(|Y|\leqslant(m+1)^{1/p})+m^{\delta}P(|Y|>m^{1/q})]\log(m+1), & \text{if } q=p \\ C\sum_{m=1}^{\infty}[m^{q/p-2/p+\delta-1}E|Y|^2I(|Y|\leqslant(m+1)^{1/p})+m^{q/p+\delta-1}P(|Y|>m^{1/q})], & \text{if } q>rp \end{cases}$$

$$\leqslant\begin{cases} C\sum_{k=1}^{\infty}k^{1-2/p+\delta}E|Y|^2I(k^{1/p}<|Y|\leqslant(k+1)^{1/p})+CE|Y|^{p\delta}, & \text{if } q<p \\ C\sum_{k=1}^{\infty}k^{1-2/p+\delta}\log(k+1)E|Y|^2I(k^{1/p}<|Y|\leqslant(k+1)^{1/p})+CE|Y|^{p\delta}\log(|Y|+1), & \text{if } q=p \\ C\sum_{k=1}^{\infty}k^{q/p-2/p+\delta}E|Y|^2I(k^{1/p}<|Y|\leqslant(k+1)^{1/p})+CE|Y|^{q(1+\delta)}, & \text{if } q>p \end{cases}$$

$$\leqslant \begin{cases} CE|Y|^{p(1+\delta)} < \infty, & \text{if } q < p \\ CE|Y|^{r^{p(1+\delta)}} \log(|Y|+1) < \infty, & \text{if } q = p \\ CE|Y|^{q^{q(1+\delta)}} < \infty, & \text{if } q > p \end{cases} \tag{11}$$

By (10) and (11), the proof of Theorem 2 is completed.

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