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Dimension Estimate of the Global Attractor for a 3D Brinkman-Forchheimer Equation

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Abstract: In this paper, we study the dimension estimate of global attractor for a 3D Brinkman-Forchheimer equation. Based on the differentiability of the semigroup with respect to the initial data, we show that the global attractor of strong solution of the 3D Brinkman-Forchheimer equation has finite Hausdorff and fractal dimensions.

Key words: Brinkman-Forchheimer equation; global attractor; Hausdorff dimension; fractal dimension

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0 Introduction

In this paper, we study the following three-dimensional Brinkman-Forchheimer equation:

$$\begin{cases} u_t - \gamma\Delta u + au + b|u|u + c|u|^{\beta}u + \nabla p = f(x), & (x,t)\in\Omega\times\mathbb{R}^+ \\ \mathrm{div}u = 0, & (x,t)\in\Omega\times\mathbb{R}^+ \\ u(x,t)\big|_{\partial\Omega} = 0, & (x,t)\in\partial\Omega\times\mathbb{R}^+ \\ u(x,0) = u_0(x), & x\in\Omega \end{cases} \tag{1}$$

where $a>0$ is the Darcy coefficient, $b>0, c>0$ are the Forchheimer coefficients, $\beta>0$ is a constant, and $\Omega\subseteq\mathbb{R}^3$ is an open and bounded set, which is sufficiently regular. Here $u=(u_1(x,t), u_2(x,t), u_3(x,t))$ is the fluid velocity vector, $p(x,t)$ denotes the pressure field, $f(x)$ is the external force, γ is the Brinkman coefficient and $u_0=u_0(x)$ is the initial velocity.

As a mathematical model, Brinkman-Forchheimer equation describes the motion of a fluid flowing in saturated porous media [1-3], which has received much attention on several issues over the last decades. From a mathematical point of view, the research on the three-dimensional Brinkman-Forchheimer equation is mainly divided into two categories. One is the structural stability of the equation with respect to the coefficients γ, b and c[4-9], and the other is the long-term behavior of the solution of the equation [10-18].

If a system has a global attractor, then the attractor will contain all possible limit states of the solutions of the sys-

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tem. Therefore, studying the dynamic system restricted to the global attractor will be able to reveal a lot of information about the original system. So proving the existence of global attractors is a basic and important problem in infinite-dimensional dynamical systems. For the 3D Brinkman-Forchheimer equation, Ugurlu [10], Ouyang and Yang [11] showed the existence of global attractor in $\left(H_0^1(\Omega)\right)^3$ when $\beta=2$ and $1<\beta<4/3$ by condition-(C) method, respectively. Wang and Lin[12] showed the existence of global attractor in $(H^2(\Omega))^3$ when $\beta=2$. In Ref.[13], the existence of D-pullback attractors for three-dimensional non-autonomous Brinkman-Forchheimer equation is deduced by establishing the D-pullback asymptotical compactness of θ-cocycle. In Ref.[14], Song *et al* discussed the L^2 decay of the weak solution of the Brinkman-Forchheimer equation in three-dimensional full space. In Ref.[15], Song and Wu investigated the uniform boundedness of uniform attractor A^ε of equation (1) with singularly oscillating external force. They established the convergence of the attractor A^ε to the attractor A^0 of the averaged equation as $\varepsilon\to 0^+$. In Ref.[16], the pullback dynamics and asymptotic stability for a 3D Brinkman-Forchheimer equation with finite delay was concerned. In Ref.[17], by some estimates and the variable index to deal with the delay term, Yang *et al* got the sufficient conditions for asymptotic stability of trajectories inside the pullback attractors for a fluid flow model in porous medium by generalized Grashof numbers. In Ref.[18], Qiao *et al* proved the existence of a global attractor for the strong solution of the Brinkman-Forchheimer equation in a three-dimensional bounded domain.

In the geometric structure of the global attractor, the dimension is a very important property. This is because if the fractal dimension of the global attractor is finite, the original infinite-dimensional dynamical system can be reduced to a finite-dimensional ordinary differential equation system, so that the relatively complete theory of the finite-dimensional dynamical system can be used to study infinite dimensional dynamical system. As far as we know, there is no results on dimension estimate of global attractor of 3D Brinkman-Forchheimer. In this paper, inspired by Refs.[19-21], based on Ref.[18], we will discuss the Hausdorff dimension and fractal dimension of global attractors for strong solutions of the equation.

The structure of this paper is arranged as follows: In Section 1, we give some function space symbols and some inequalities that will be used later. In Section 2, we discuss the Hausdorff dimension and fractal dimension of global attractors for strong solution of the equation.

1 Preliminaries

In this section, we introduce some notations and preliminaries, which will be used throughout this paper.

First, let us introduce the following function spaces:

$$E=\{u\in(C_0^\infty(\Omega))^3\colon \operatorname{div}u=0\},\ H=\mathrm{cl}_{(L^2(\Omega))^3}E,\ V=\mathrm{cl}_{(H_0^1(\Omega))^3}E,$$

where cl_X denotes the closure in space X. H is the closure of the set E in $(L^2(\Omega))^3$ topology, and V is the closure of the set E in $(H_0^1(\Omega))^3$ topology. H' and V' are the dual spaces of H and V, respectively. H and V are equipped with the following inner products:

$$(u,v)=\int_\Omega u\cdot v\mathrm{d}x,\ \forall u,v\in H,\ \ ((u,v))=\sum_{i=1}^3\int_\Omega \nabla u_i\cdot\nabla v_i\,\mathrm{d}x,\ \forall u,v\in V,$$

and norms $\|\cdot\|=(\cdot,\cdot)^{\frac12}$, $\|\cdot\|_V=((\cdot,\cdot))^{\frac12}$. Let $\boldsymbol{L}^p(\Omega)=(L^p(\Omega))^3$, $\boldsymbol{H}^2(\Omega)=(H^2(\Omega))^3$. Throughout this paper, we use $\|\cdot\|_p$ to denote the norm in $\boldsymbol{L}^p(\Omega)$. C or C_i will stand for some generic positive constants, depending on Ω and some constants, but independent of time t.

We call $u\in L^\infty(0,T;H)\bigcap L^2(0,T;V)\bigcap L^{\beta+2}(0,T;\boldsymbol{L}^{\beta+2}(\Omega))$ is a weak solution of problem (1) on $[0,T]$, if

$$\begin{cases}\dfrac{\mathrm{d}}{\mathrm{d}t}(u,v)+\gamma((u,v))+a(u,v)+(b|u|u,v)+(c|u|^\beta u,v)=(f,v),\ \forall v\in V,t>0\\ u(0)=u_0\end{cases}\tag{2}$$

The weak form (2) is equivalent to the following functional equation:

$$\begin{cases}\dfrac{\mathrm{d}u}{\mathrm{d}t}+\gamma Au+au+B(u)=f,\ \forall t>0\\ u(0)=u_0\end{cases}\tag{3}$$

Here $A=-P_H\Delta$ is the Stokes operator subject to the no-slip homogeneous Dirichlet boundary condition with the domain $(H^2(\Omega))^3\cap V$ defined as $\langle Au,v\rangle=((u,v))$. P_H is the orthogonal projection from $\boldsymbol{L}^2(\Omega)$ onto H. $F(u)=b|u|u+c|u|^{\beta}u$, $B(u)=P_HF(u), f\in H$. Obviously, the operator A is a non-negative self-adjoint operator in H with $V=D(A^{1/2})$ and $\langle Au,u\rangle=\|u\|_V$, for all $u\in V$.

For a bounded domain Ω, the operator A is invertible, and its inverse operator A^{-1} is bounded, self-adjoint and compact in H. Thus, the spectrum of A consists of an infinite sequence $0<\lambda_1\leqslant\lambda_2\leqslant\cdots\leqslant\lambda_k\leqslant\cdots$, with $\lambda_k\to\infty$ as its eigenvalue $k\to\infty$ (Theorem 2.2, Corollary 2.2, Ref. [22]). For all $k\geqslant 1$ and $n\in\mathbb{N}$, we have $\lambda_k\geqslant\tilde{C}k^{2/n}$, where $\tilde{C}=\frac{n}{2+n}\left(\frac{(2\pi)^n}{\omega_n(n-1)|\Omega|}\right)^{2/n}$, $\omega_n=\pi^{n/2}\Gamma(1+\frac{n}{2})$, and $|\Omega|$ is the n-dimensional Lebesgue measure of Ω. For $n=3$, we find

$$\tilde{C}=\frac{3^{1/3}2^{8/3}\pi^{2/3}}{5|\Omega|^{2/3}} \tag{4}$$

Next we formulate some well-known inequalities and a Gronwall type lemma that we will use in what follows.

Poincaré's inequality [23]:

$$\|u\|^2\leqslant\frac{1}{\lambda_1}\|\nabla u\|^2,\forall u\in V \tag{5}$$

where λ_1 is the first eigenvalue of operator $A=-P_H\Delta$ under the homogeneous Dirichlet boundary condition.

Agmon's inequality [23]:

$$\|u\|_\infty\leqslant C_1\|\nabla u\|^{1/2}\|\Delta u\|^{1/2},\forall u\in D(A) \tag{6}$$

C_q-inequality [24]:

$$|x^q-y^q|\leqslant C_q(|x|^{q-1}+|y|^{q-1})|x-y|,\text{for the integer } q\geqslant 2 \tag{7}$$

Series inequlity[25]:

$$\sum_{k=m}^{n}k^p<\frac{1}{p+1}(n^{p+1}-(m-1)^{p+1}),\text{for} -1<p<0,n>m \tag{8}$$

Lemma 1 (Gagliardo-Nirenberg's inequality)[24] Assuming that $\Omega=\mathbb{R}^n$ or $\Omega\subset\mathbb{R}^n$ be a bounded domain, which has a sufficiently smooth boundary $\partial\Omega$, $u\in L^q(\Omega), D^mu\in L^r(\Omega)$, $1\leqslant q,r\leqslant\infty$. Then there is a constant $C>0$ such that $\|D^ju\|_p\leqslant C\|D^mu\|_r^a\|u\|_q^{1-a}$, where $\frac{1}{p}=\frac{j}{n}+a(\frac{1}{r}-\frac{m}{n})+(1-a)\frac{1}{q}$, $1\leqslant p\leqslant\infty, 0\leqslant j\leqslant m, \frac{j}{m}\leqslant a\leqslant 1$, C depends on n,m,j,a,q,r.

Lemma 2 (Gronwall's inequality)[23] Let $u(t)$, $k(t)$ be non-negative integrable functions on $[0,T]$. If there is $K>0$, such that $u(t)\leqslant K+\int_0^t k(s)u(s)\mathrm{d}s, \forall t\in[0,T]$, then $u(t)\leqslant K\exp(\int_0^t k(s)\mathrm{d}s), \forall t\in[0,T]$.

Now we recall the existence and uniqueness theorem of the strong solutions of equation (1).

Theorem 1[18] Suppose $\beta>0$, $u_0\in V\cap\boldsymbol{L}^{\beta+2}(\Omega)$ and $f\in H$. Then there exists a strong solution of equation (1) satisfying

$$u\in L^\infty(0,T;V)\cap L^\infty(0,T;\boldsymbol{L}^{\beta+2}(\Omega))\cap L^2(0,T;(H^2(\Omega))^3),\ \nabla u|u|^{\beta/2}\in L^2(0,T;H), u_t\in L^2(0,T;H).$$

Moreover when $5/2\leqslant\beta\leqslant 4$, the strong solution is unique.

Now we will review the uniform estimates of strong solution to the problem (1) when $t\to\infty$.

Lemma 3[18] Suppose $5/2\leqslant\beta\leqslant 4$, $u_0\in V, f\in H$. Then there exists a time t_0, constants ρ_1, I_1, such that when $t>t_0$, we have $\|u(t)\|\leqslant\rho_1$, $\int_t^{t+1}\|u(s)\|_V^2\mathrm{d}s+\int_t^{t+1}\|u(s)\|^2\mathrm{d}s+\int_t^{t+1}\|u(s)\|_3^3\mathrm{d}s+\int_t^{t+1}\|u(s)\|_{\beta+2}^{\beta+2}\mathrm{d}s\leqslant I_1$, for $t>t_0$.

Lemma 4[18] Suppose $5/2\leqslant\beta\leqslant 4$, $u_0\in V, f\in H$. Then there exists a time t_1, a constant ρ_2, such that $\|u(t)\|_V+\|u(t)\|+\|u(t)\|_3+\|u(t)\|_{\beta+2}\leqslant\rho_2, \forall t>t_1$.

Lemma 5[18] Suppose $5/2\leqslant\beta\leqslant 4$, $u_0\in V, f\in H$. Then there exists a time t_2, a constant ρ_3, such that $\|u_t(s)\|\leqslant\rho_3, \forall s\geqslant t_2$.

Lemma 6[18] Suppose $5/2\leqslant\beta\leqslant 4, u_0\in V, f\in H$. Then there exists a constant ρ_4, such that $\|Au(t)\|\leqslant\rho_4, \forall t\geqslant t_2$.

Finally, we give the result of existence of global attractor in V for the 3D Brinkman-Forchheimer equation.

Theorem 2[18] Suppose $5/2 \leqslant \beta \leqslant 4$, $u_0 \in V$, $f \in H$. Then the problem (1) has a global attractor A_V in V, which is invariant and compact in V and attracts every bounded subset of V with the norm in V.

2 Estimates of Dimensions of the Global Attractor

In this section, we will establish the differentiability of the semigroup with respect to the initial data. We show that the global attractor of the 3D Brinkman-Forchheimer system has finite Hausdorff and fractal dimensions. We will use the similar techniques as in Refs.[19-21], etc to obtain the desired results.

Let $u(\cdot)$ be the unique strong solution of the autonomous system (1) belonging to the global attractor A_V.

Let us take inner product with $-\Delta u$ in H to the first equation in (1) to obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\|u\|_V^2+2\gamma\|\Delta u\|^2+2a\|u\|_V^2+2b\int_\Omega|u||\nabla u|^2\mathrm{d}x+\frac{8b}{9}\int_\Omega\left|\nabla|u|^{3/2}\right|^2\mathrm{d}x+2c\int_\Omega|u|^\beta|\nabla u|^2\mathrm{d}x+\frac{8c\beta}{(\beta+2)^2}\int_\Omega\left|\nabla|u|^{\frac{\beta+2}{2}}\right|^2\mathrm{d}x$$
$$=-2(f,\Delta u)\leqslant 2\|f\|\|\Delta u\|\leqslant\gamma\|\Delta u\|^2+\frac{1}{\gamma}\|f\|^2 \tag{9}$$

So we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\|u\|_V^2+2a\|u\|_V^2\leqslant\frac{1}{\gamma}\|f\|^2 \tag{10}$$

Applying Gronwall's inequality, we find

$$\|u\|_V^2\leqslant\|u_0\|_V^2\mathrm{e}^{-2at}+\frac{\|f\|^2}{2a\gamma},\forall t>0 \tag{11}$$

So there is a time t' which we can take as $t'=\max\{-\frac{1}{2a}\ln\frac{\|f\|^2}{2a\gamma\|u_0\|_V^2},0\}$, such that for all $t\geqslant t'$, we have

$$\|u\|_V^2\leqslant\frac{\|f\|^2}{a\gamma}=M_1^2,M_1=\frac{\|f\|}{\sqrt{a\gamma}} \tag{12}$$

Integrating the inequality (10) from 0 to T, we obtain

$$\|u(T)\|_V^2+2a\int_0^T\|u(s)\|_V^2\mathrm{d}s\leqslant\|u_0\|_V^2+\frac{T}{\gamma}\|f\|^2 \tag{13}$$

so we get

$$\limsup_{T\to\infty}\frac{1}{T}\int_0^T\|u(t)\|_V^2\mathrm{d}t\leqslant K_1=\frac{\|f\|^2}{2a\gamma} \tag{14}$$

Theorem 3 Let u_0 and v_0 be two members of A_V. Then there exists a constant $K=K(\|u_0\|_V,\|v_0\|_V)$ such that

$$\|S(t)u_0-S(t)v_0-\Lambda(t)(u_0-v_0)\|_V\leqslant K\|u_0-v_0\|_V \tag{15}$$

where the linear operator $\Lambda(t):V\to V$, for $t>0$ is the solution operator of the problem:

$$\begin{cases}\frac{\mathrm{d}\xi}{\mathrm{d}t}+\gamma A\xi+a\xi+2bP_H(|u(t)|\xi(t))+c(\beta+1)P_H(|u(t)|^\beta\xi(t))=0,t\in(0,T)\\ \xi(0)=\xi_0\in V\end{cases} \tag{16}$$

$\xi_0=u_0-v_0$ and $u(t)=S(t)u_0$, $v(t)=S(t)v_0$. In other words, for every $t>0$, the solution $S(t)u_0$ as a map $S(t):V\to V$ is Fréchet differentiable for the initial data, and its Fréchet derivative $D_{u_0}(S(t)u_0)w_0=\Lambda(t)w_0$.

Proof Let $S(t)u_0=u(t)$, $S(t)v_0=v(t)$, for any $t\geqslant 0$. Then we have

$$\begin{cases}\frac{\mathrm{d}u}{\mathrm{d}t}+\gamma Au+au+P_H(b|u|u+c|u|^\beta u)=f\\ u(0)=u_0\end{cases} \tag{17}$$

and

$$\begin{cases}\dfrac{\mathrm{d}v}{\mathrm{d}t}+\gamma Av+av+P_H(b|v|v+c|v|^{\beta}v)=f\\ v(0)=v_0\end{cases} \tag{18}$$

Let $w(t)=u(t)-v(t)$. Combining (17) with (18), we obtain

$$\begin{cases}\dfrac{\mathrm{d}w}{\mathrm{d}t}+\gamma Aw+aw+P_H(b|u|u-b|v|v+c|u|^{\beta}u-c|v|^{\beta}v)=0\\ w(0)=u_0-v_0\end{cases} \tag{19}$$

Let us define $\eta(t)=u(t)-v(t)-\xi(t)=S(t)(u_0-v_0)-\xi(t)$. Then $\eta(t)$ satisfies:

$$\begin{cases}\dfrac{\mathrm{d}\eta(t)}{\mathrm{d}t}+\gamma A\eta(t)+a\eta(t)+P_H(b|u(t)|u(t)-b|v(t)|v(t)+c|u(t)|^{\beta}u(t)-c|v(t)|^{\beta}v(t))\\ \qquad -2bP_H(|u(t)|\xi(t))-c(\beta+1)P_H(|u(t)|^{\beta}\xi(t))=0\\ \eta(0)=0\end{cases} \tag{20}$$

Let us take inner product with $A\eta(t)$ in H to the first equation in (20) to obtain

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla\eta(t)\|^2+\gamma\|A\eta(t)\|^2+a\|\nabla\eta(t)\|^2=2b(P_H(|u(t)|w(t)),A\eta(t))-2b(P_H(|u(t)|\eta(t)),A\eta(t))\\ &+c(\beta+1)(P_H(|u(t)|^{\beta}w(t)),A\eta(t))-c(\beta+1)(P_H(|u(t)|^{\beta}\eta(t)),A\eta(t))\\ &-(P_H(b|u(t)|u(t)-b|v(t)|v(t)+c|u(t)|^{\beta}u(t)-c|v(t)|^{\beta}v(t)),A\eta(t))\end{aligned} \tag{21}$$

Let us consider the first term on the right-hand side of (21). We have

$$\begin{aligned}&2b(P_H(|u(t)|w(t)),A\eta(t))\leqslant\frac{\gamma}{5}\|A\eta(t)\|^2+\frac{5b^2}{\gamma}\int_\Omega|u(t)|^2|w(t)|^2\mathrm{d}x\\ &\leqslant\frac{\gamma}{5}\|A\eta(t)\|^2+\frac{5b^2}{\gamma}\|u(t)\|_\infty^2\|w(t)\|^2\leqslant\frac{\gamma}{5}\|A\eta(t)\|^2+\frac{5b^2C_1^2}{\gamma}\|\nabla u(t)\|\|\Delta u(t)\|\|w(t)\|^2\\ &\leqslant\frac{\gamma}{5}\|A\eta(t)\|^2+C\|w(t)\|^2\end{aligned} \tag{22}$$

In (22), Agmon's inequality is used. And in the last inequality of (22), because u_0, v_0 are members of A_V, so we used the uniform estimates of solutions in Lemma 4 and Lemma 6 to obtain the desired result.

Similar with (22), for the second term on the right-hand side of (21), we have

$$-2b(P_H(|u(t)|\eta(t)),A\eta(t))\leqslant\frac{\gamma}{5}\|A\eta(t)\|^2+\frac{5b^2C_1^2}{\gamma}\|\nabla u(t)\|\|\Delta u(t)\|\|\eta(t)\|^2\leqslant\frac{\gamma}{5}\|A\eta(t)\|^2+C\|\eta(t)\|^2 \tag{23}$$

For the third term on the right-hand side of (21), we get

$$\begin{aligned}c(\beta+1)(P_H(|u(t)|^{\beta}w(t)),A\eta(t))&\leqslant\frac{\gamma}{5}\|A\eta(t)\|^2+\frac{5c^2(\beta+1)^2}{4\gamma}\int_\Omega|u(t)|^{2\beta}|w(t)|^2\mathrm{d}x\leqslant\frac{\gamma}{5}\|A\eta(t)\|^2+\frac{5c^2(\beta+1)^2}{4\gamma}\|u(t)\|_\infty^{2\beta}\|w(t)\|^2\\ &\leqslant\frac{\gamma}{5}\|A\eta(t)\|^2+\frac{5c^2(\beta+1)^2C_1^{2\beta}}{4\gamma}\|\nabla u(t)\|^{\beta}\|\Delta u(t)\|^{\beta}\|w(t)\|^2\leqslant\frac{\gamma}{5}\|A\eta(t)\|^2+C\|w(t)\|^2\end{aligned} \tag{24}$$

Similar with (24), for the fourth term on the right-hand side of (21), we have

$$-c(\beta+1)(P_H(|u(t)|^{\beta}\eta(t)),A\eta(t))\leqslant\frac{\gamma}{5}\|A\eta(t)\|^2+\frac{5c^2(\beta+1)^2C_1^{2\beta}}{4\gamma}\|\nabla u(t)\|^{\beta}\cdot\|\Delta u(t)\|^{\beta}\|\eta(t)\|^2\leqslant\frac{\gamma}{5}\|A\eta(t)\|^2+C\|\eta(t)\|^2 \tag{25}$$

For the fifth term on the right-hand side of (21), we have

$$\begin{aligned}&-(P_H(b|u(t)|u(t)-b|v(t)|v(t)+c|u(t)|^{\beta}u(t)-c|v(t)|^{\beta}v(t)),A\eta(t))\\ &\leqslant\frac{\gamma}{5}\|A\eta(t)\|^2+\frac{5}{2\gamma}\int_\Omega\left|b|u(t)|u(t)-b|v(t)|v(t)\right|^2\mathrm{d}x+\frac{5}{2\gamma}\int_\Omega\left|c|u(t)|^{\beta}u(t)-c|v(t)|^{\beta}v(t)\right|^2\mathrm{d}x\\ &\leqslant\frac{\gamma}{5}\|A\eta(t)\|^2+C\int_\Omega(|u(t)||w(t)|+||u|-|v|||v|)^2\mathrm{d}x+C\int_\Omega\left(|u(t)|^{\beta}|w(t)|+\left||u(t)|^{\beta}-|v(t)|^{\beta}\right||v(t)|\right)^2\mathrm{d}x\\ &\leqslant\frac{\gamma}{5}\|A\eta(t)\|^2+C\int_\Omega|u(t)|^2|w(t)|^2\mathrm{d}x+C\int_\Omega|v(t)|^2|w(t)|^2\mathrm{d}x+C\int_\Omega|u(t)|^{2\beta}|w(t)|^2\mathrm{d}x\\ &+C\int_\Omega\left||u(t)|^{\beta-1}+|v(t)|^{\beta-1}\right|^2|v(t)|^2|w(t)|^2\mathrm{d}x\end{aligned}$$

$$\leqslant \frac{\gamma}{5}\|A\eta(t)\|^2+C\|u(t)\|_4^2\|w(t)\|_4^2+C\|v(t)\|_4^2\|w(t)\|_4^2+C\|u(t)\|_{3\beta}^{2\beta}\|w(t)\|_6^2+C(\|u(t)\|_{6(\beta-1)}^{2(\beta-1)}+\|v(t)\|_{6(\beta-1)}^{2(\beta-1)})\|v(t)\|_6^2\|w(t)\|_6^2$$

$$\leqslant \frac{\gamma}{5}\|A\eta(t)\|^2+C\|\nabla u(t)\|^2\|\nabla w(t)\|^2+C\|\nabla v(t)\|^2\|\nabla w(t)\|^2+C\|u(t)\|_{\beta+2}^{\frac{2(\beta+2)^2}{\beta+8}}\|\Delta u(t)\|^{\frac{8(\beta-1)}{\beta+8}}\|\nabla w(t)\|^2$$

$$+C(\|u(t)\|_{\beta+2}^{\frac{2(\beta^2+2\beta)}{\beta+8}}\|\Delta u(t)\|^{\frac{10\beta-16}{\beta+8}}+\|v(t)\|_{\beta+2}^{\frac{2(\beta^2+2\beta)}{\beta+8}}\|\Delta v(t)\|^{\frac{10\beta-16}{\beta+8}})\|\nabla v(t)\|^2\|\nabla w(t)\|^2$$

$$\leqslant \frac{\gamma}{5}\|A\eta(t)\|^2+C\|\nabla w(t)\|^2 \tag{26}$$

In inequality (26), we used C_q-inequality and the following Gagliardo-Nirenberg's inequality:

$$\|u(t)\|_{3\beta}^{2\beta}\leqslant C\|u(t)\|_{\beta+2}^{\frac{2(\beta+2)^2}{\beta+8}}\|\Delta u(t)\|^{\frac{8(\beta-1)}{\beta+8}},\ \|u(t)\|_{6(\beta-1)}^{2(\beta-1)}\leqslant C\|u(t)\|_{\beta+2}^{\frac{2(\beta^2+2\beta)}{\beta+8}}\|\Delta u(t)\|^{\frac{10\beta-16}{\beta+8}}.$$

Combining (21)-(26), we find

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla\eta(t)\|^2+2a\|\nabla\eta(t)\|^2\leqslant C\|w(t)\|^2+C\|\eta(t)\|^2+C\|\nabla w(t)\|^2 \tag{27}$$

Taking inner product with Aw in H to the first equation of (19), we have

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla w(t)\|^2+\gamma\|Aw(t)\|^2+a\|\nabla w(t)\|^2+b(P_H(|u(t)|u(t)-|v(t)|v(t)),Aw(t))+c(P_H(|u(t)|^\beta u(t)-|v(t)|^\beta v(t)),Aw(t))=0 \tag{28}$$

where

$$|b(P_H(|u(t)|u(t)-|v(t)|v(t)),Aw(t))|\leqslant\frac{\gamma}{2}\|Aw(t)\|^2+\frac{b^2}{2\gamma}\||u(t)|u(t)-|v(t)|v(t)\|^2 \tag{29}$$

and

$$|c(P_H(|u(t)|^\beta u(t)-|v(t)|^\beta v(t)),Aw(t))|\leqslant\frac{\gamma}{2}\|Aw(t)\|^2+\frac{c^2}{2\gamma}\||u(t)|^\beta u(t)-|v(t)|^\beta v(t)\|^2 \tag{30}$$

For the second term on the right-hand side of (29), from inequality (26), we have

$$\frac{b^2}{2\gamma}\int_\Omega||u(t)|u(t)-|v(t)|v(t)|^2\mathrm{d}x\leqslant C\|\nabla u(t)\|^2\|\nabla w(t)\|^2+C\|\nabla v(t)\|^2\|\nabla w(t)\|^2\leqslant C\|\nabla w(t)\|^2 \tag{31}$$

For the second term on the right-hand side of (30), we have

$$\frac{c^2}{2\gamma}\int_\Omega||u(t)|^\beta u(t)-|v(t)|^\beta v(t)|^2\mathrm{d}x\leqslant C\|u(t)\|_{\beta+2}^{\frac{2(\beta+2)^2}{\beta+8}}\|\Delta u(t)\|^{\frac{8(\beta-1)}{\beta+8}}\|\nabla w(t)\|^2$$

$$+C(\|u(t)\|_{\beta+2}^{\frac{2(\beta^2+2\beta)}{\beta+8}}\|\Delta u(t)\|^{\frac{10\beta-16}{\beta+8}}+\|v(t)\|_{\beta+2}^{\frac{2(\beta^2+2\beta)}{\beta+8}}\|\Delta v(t)\|^{\frac{10\beta-16}{\beta+8}})\cdot\|\nabla v(t)\|^2\|\nabla w(t)\|^2\leqslant C\|\nabla w(t)\|^2 \tag{32}$$

Combining (29)-(32) with (28), we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla w(t)\|^2\leqslant C\|\nabla w(t)\|^2 \tag{33}$$

Integrating (33) from 0 to t, we have

$$\|\nabla w(t)\|^2\leqslant\|\nabla w(0)\|^2+C\int_0^t\|\nabla w(s)\|^2\mathrm{d}s \tag{34}$$

Applying Gronwall's inequality to (34), we get

$$\|\nabla w(t)\|^2\leqslant\|\nabla w(0)\|^2\mathrm{e}^{Ct}\leqslant\frac{\|\nabla w(0)\|^4+1}{2}\mathrm{e}^{Ct} \tag{35}$$

Integrating (27) from 0 to t, due to (35), we infer that

$$\|\eta(t)\|_V^2\leqslant C\int_0^t\|w(s)\|^2\mathrm{d}s+C\int_0^t\|\eta(s)\|^2\mathrm{d}s+C\int_0^t\|\nabla w(s)\|^2\mathrm{d}s$$

$$\leqslant C\int_0^t\|\nabla w(s)\|^2\mathrm{d}s+C\int_0^t\|\eta(s)\|_V^2\mathrm{d}s\leqslant C\|\nabla w(0)\|^4\mathrm{e}^{Ct}+C\int_0^t\|\eta(s)\|_V^2\mathrm{d}s \tag{36}$$

An application of Gronwall's inequality in (36) yields

$$\|\eta(t)\|_V^2\leqslant C\|\nabla w(0)\|^4\mathrm{e}^{2Ct} \tag{37}$$

Thus

$$\frac{\| u(t)-v(t)-\xi(t) \|_V}{\| u_0 - v_0 \|_V} \leqslant C \| u_0 - v_0 \|_V \mathrm{e}^{Ct} \tag{38}$$

which completes the proof.

Now we rewrite the system (3) as

$$\begin{cases} \dfrac{\mathrm{d}u(t)}{\mathrm{d}t} + \gamma A u(t) + a u(t) + b P_H(|u(t)|u(t)) + c P_H(|u(t)|^{\beta} u(t)) = f \\ u(0) = u_0 \in V \end{cases} \tag{39}$$

Let us now set $\tilde{u} = A^{1/2} u, \tilde{v} = A^{1/2} v,$ and using it in (39) to obtain

$$\begin{cases} \dfrac{\mathrm{d}\tilde{u}(t)}{\mathrm{d}t} = -\gamma A \tilde{u}(t) - a\tilde{u}(t) - bA^{1/2} P_H(|A^{-1/2}\tilde{u}(t)|A^{-1/2}\tilde{u}(t)) - cA^{1/2} P_H(|A^{-1/2}\tilde{u}(t)|^{\beta} A^{-1/2}\tilde{u}(t)) + A^{1/2} f \\ \tilde{u}(0) = A^{1/2} u_0 \end{cases} \tag{40}$$

where $A^{1/2} u_0 \in H$. Note that the systems (40) and (39) are equivalent. Remember that the systems (39) is well posed in V, while the system (40) is well posed in H. Therefore, there exists a unique weak solution $\tilde{u}(\cdot)$ of (40) in $C([0,T];H)$. Moreover, the system (40) generates one family of strongly continuous semigroup $\tilde{S}(t)$ of solution operators

$$\tilde{S}(t): H \to H, \tilde{u}_0 \mapsto \tilde{u}(t) = \tilde{S}(t)\tilde{u}_0.$$

Since $\tilde{u}(t) = A^{1/2} u(t)$ and $\tilde{u}_0 = A^{1/2} u_0$, the semigroup $\tilde{S}(t)$ is connected to the original semigroup $S(t)$ through the relation

$$\tilde{S}(t) = A^{1/2} S(t) A^{-1/2} \tag{41}$$

Thus, the semigroup $\tilde{S}(t)$ has the global attractor $\widetilde{A}_H$, where

$$\widetilde{A}_H = A^{1/2} A_V \tag{42}$$

and A_V is the global attractor for $S(t)$.

Now we will show a bound for the fractal dimension of $\widetilde{A}_H$ in H. Besides, using the following argument, the fractal dimension A_V in V can easily yield the same bound. From Proposition 3.1, Chapter VI of Ref.[23], we know that under the Lipschitz maps, the fractal dimension estimates can be obtained. Furthermore, we infer that

$$\dim_F^V(A_V) = \dim_F^V(A^{-1/2}\widetilde{A}_H) = \dim_F^H(\widetilde{A}_H) \tag{43}$$

Let us first consider the linear variations of the system (40). The linear variational equation corresponding to (40) has this form

$$\frac{\mathrm{d}w(t)}{\mathrm{d}t} = L(t,\tilde{u})w(t) \tag{44}$$

where

$$L(t,\tilde{u})w(t) = -\gamma A w(t) - a w(t) - 2bA^{1/2} P_H(|A^{-1/2}\tilde{u}(t)|A^{-1/2}w(t)) - (\beta+1)cA^{1/2} P_H(|A^{-1/2}\tilde{u}(t)|^{\beta} A^{-1/2}w(t)) \tag{45}$$

The adjoint $L^*(t,\tilde{u})$ of $L(t,\tilde{u})$ is given by

$$L^*(t,\tilde{u})w(t) = -\gamma A w(t) - a w(t) - 2bA^{1/2} P_H(|A^{-1/2}\tilde{u}(t)|A^{-1/2}w(t)) - (\beta+1)cA^{1/2} P_H(|A^{-1/2}\tilde{u}(t)|^{\beta} A^{-1/2}w(t)) \tag{46}$$

Hence, $\tilde{L}(t,\tilde{u})w(t) = L(t,\tilde{u})w(t) + L^*(t,\tilde{u})w(t)$ can be computed as

$$\tilde{L}(t,\tilde{u})w(t) = -2\gamma A w(t) - 2a w(t) - 4bA^{1/2} P_H(|A^{-1/2}\tilde{u}(t)|A^{-1/2}w(t)) - 2(\beta+1)cA^{1/2} P_H(|A^{-1/2}\tilde{u}(t)|^{\beta} A^{-1/2}w(t)) \tag{47}$$

Then, we derive the following results.

Proposition 1 Let $w \in H$. Then, we have

$$(\tilde{L}(t,\tilde{u})w(t), w(t)) \leqslant -2a\| w(t) \|^2 + \left(\frac{4b^2 C_1^2}{\gamma\sqrt{\lambda_1}} \| \Delta u \|^2 + \frac{4(\beta+1)^2 c^2 C_1^{2\beta}}{\gamma \lambda_1^{\beta/2}} \| \Delta u \|^{2\beta}\right) \cdot \| A^{-1/2} w(t) \|^2 \tag{48}$$

where $\gamma > 0, b > 0, c > 0, C_1 > 0$ is given in (6) which only depends on Ω.

Proof Let us take the inner product with $w(t)$ in H to equation (47), we obtain

$$\begin{aligned} (\tilde{L}(t)w(t), w(t)) = & -2\gamma\| A^{1/2} w(t) \|^2 - 2a\| w(t) \|^2 - 4b(P_H(|A^{-1/2}\tilde{u}(t)|A^{-1/2}w(t)), A^{1/2}w(t)) \\ & -2(\beta+1)c(P_H(|A^{-1/2}\tilde{u}(t)|^{\beta} A^{-1/2}w(t)), A^{1/2}w(t)) \end{aligned} \tag{49}$$

And because

$$\left|4b(P_H(\left|A^{-1/2}\tilde{u}(t)\right|A^{-1/2}w(t)),A^{1/2}w(t))\right|\leqslant 4b\int_\Omega\left|A^{-1/2}\tilde{u}(t)\right|\left|A^{-1/2}w(t)\right|\left|A^{1/2}w(t)\right|\mathrm{d}x\leqslant 4b\left\|A^{-1/2}\tilde{u}(t)\right\|_\infty\left\|A^{-1/2}w(t)\right\|\left\|A^{1/2}w(t)\right\|$$

$$\leqslant\gamma\left\|A^{1/2}w(t)\right\|^2+\frac{4b^2}{\gamma}\|u\|_\infty^2\left\|A^{-1/2}w(t)\right\|^2\leqslant\gamma\left\|A^{1/2}w(t)\right\|^2+\frac{4b^2C_1^2}{\gamma}\|\nabla u\|\|\Delta u\|\left\|A^{-1/2}w(t)\right\|^2$$

$$\leqslant\gamma\left\|A^{1/2}w(t)\right\|^2+\frac{4b^2C_1^2}{\gamma\sqrt{\lambda_1}}\|\Delta u\|^2\left\|A^{-1/2}w(t)\right\|^2 \tag{50}$$

and

$$\left|2(\beta+1)c(P_H(\left|A^{-1/2}\tilde{u}(t)\right|^\beta A^{-1/2}w(t)),A^{1/2}w(t))\right|\leqslant 2(\beta+1)c\int_\Omega\left|A^{-1/2}\tilde{u}(t)\right|^\beta\left|A^{-1/2}w(t)\right|\left|A^{1/2}w(t)\right|\mathrm{d}x$$

$$\leqslant 2(\beta+1)c\left\|A^{-1/2}\tilde{u}(t)\right\|_\infty^\beta\left\|A^{-1/2}w(t)\right\|\left\|A^{1/2}w(t)\right\|\leqslant\gamma\left\|A^{1/2}w(t)\right\|^2+\frac{4(\beta+1)^2c^2}{\gamma}\|u\|_\infty^{2\beta}\left\|A^{-1/2}w(t)\right\|^2$$

$$\leqslant\gamma\left\|A^{1/2}w(t)\right\|^2+\frac{4(\beta+1)^2c^2C_1^{2\beta}}{\gamma}\|\nabla u\|^\beta\|\Delta u\|^\beta\left\|A^{-1/2}w(t)\right\|^2\leqslant\gamma\left\|A^{1/2}w(t)\right\|^2+\frac{4(\beta+1)^2c^2C_1^{2\beta}}{\gamma\lambda_1^{\beta/2}}\|\Delta u\|^{2\beta}\left\|A^{-1/2}w(t)\right\|^2 \tag{51}$$

Combining (50) and (51) with (49), we deduce that

$$(\widetilde{L}(t)w(t),w(t))\leqslant-2a\left\|w(t)\right\|^2+\left(\frac{4b^2C_1^2}{\gamma\sqrt{\lambda_1}}\|\Delta u\|^2+\frac{4(\beta+1)^2c^2C_1^{2\beta}}{\gamma\lambda_1^{\beta/2}}\|\Delta u\|^{2\beta}\right)\cdot\left\|A^{-1/2}w(t)\right\|^2 \tag{52}$$

Proposition 2 Suppose $5/2\leqslant\beta\leqslant 4, u_0\in V, f\in H$. Then the global attractor $\widetilde{A}_H$ has the finite fractal dimension in H, with

$$\dim_H(\widetilde{A}_H)\leqslant\dim_F(\widetilde{A}_H)\leqslant\left(\frac{6b^2C_1^2\rho_4^2}{a\widetilde{C}\gamma\sqrt{\lambda_1}}+\frac{6(\beta+1)^2c^2C_1^{2\beta}\rho_4^{2\beta}}{a\widetilde{C}\gamma\lambda_1^{\beta/2}}\right)^{\frac{3}{2}} \tag{53}$$

where $\widetilde{C}$ is defined in (4), $b>0, c>0, \rho_4>0$ is given in Lemma 6 and $C_1>0$ is given in (6) which only dependents on Ω.

Proof Let $w_{1,0},\cdots,w_{n,0}$, for some $n\geqslant 1$, be an initial orthogonal set of infinitesimal displacements. The volume of the parallelopiped spanned by $w_{1,0},\cdots,w_{n,0}$, is given by $V_n(0)=\left|w_{1,0}\wedge\cdots\wedge w_{n,0}\right|$, where $\wedge$ denotes the exterior product. The evolution of such displacements satisfies the following evolution equation:

$$\begin{cases}\dfrac{\mathrm{d}}{\mathrm{d}t}w_i(t)=\widetilde{L}(t,\tilde{u})w_i(t)\\ w_i(0)=w_{i,0}\end{cases} \tag{54}$$

for all $i=1,\cdots,n$. Using Lemma 3.5 in Ref.[26], we know that the volume elements $V_n(t)=\left|w_1(t)\wedge\cdots\wedge w_n(t)\right|$ satisfy

$$V_n(t)=V_n(0)\exp\left[\int_0^t\mathrm{Tr}(P_n(s)\widetilde{L}(s,\tilde{u}))\mathrm{d}s\right] \tag{55}$$

where $P_n(s)$ is the orthogonal projection onto the linear span of $\{w_1(t),\cdots,w_n(t)\}$ in H. And we also know that $\mathrm{Tr}(P_n(s)\widetilde{L}(s,\tilde{u}))=\sum_{k=1}^n(\widetilde{L}(s,\tilde{u})\varphi_k(s),\varphi_k(s))$, with $n\geqslant 1$ and $\{\varphi_1(s),\cdots,\varphi_n(s)\}$ an orthonormal set spanning $P_n(s)H$. Then, we define

$$[P_n\widetilde{L}(\tilde{u})]=\limsup_{T\to\infty}\frac{1}{T}\int_0^T\mathrm{Tr}(P_n(t)\widetilde{L}(t,\tilde{u}))\mathrm{d}t \tag{56}$$

From (56), we have

$$V_n(t)=V_n(0)\exp\{t\sup_{\tilde{u}\in\tilde{A}_H}\sup_{P_n(0)}[P_n\widetilde{L}(\tilde{u})]\} \tag{57}$$

for all $t\geqslant 0$, where the supremum over $P_n(0)$ is a supremum over all choices of initial n orthogonal set of infinitesimal displacements that have taken around $\tilde{u}$. Now let us prove that the volume element $V_n(t)$ decays exponentially with time, whenever $n\geqslant N$, with $N>0$ to be determined later.

Let us use Proposition 1 to estimate $\frac{1}{T}\int_0^T\mathrm{Tr}(P_n(t)\widetilde{L}(t,\tilde{u}))\mathrm{d}t$.

$$
\begin{aligned}
&\frac{1}{T}\int_0^T \mathrm{Tr}(P_n(t)\widetilde{L}(t,\tilde{u}))\mathrm{d}t=\frac{1}{T}\int_0^T\sum_{k=1}^n(\widetilde{L}(t,\tilde{u})\varphi_k(t),\varphi_k(t))\,\mathrm{d}t\\
&\leqslant \frac{1}{T}\int_0^T\sum_{k=1}^n -2a\left\|\varphi_k(t)\right\|^2\mathrm{d}t+\frac{1}{T}\int_0^T\Big(\frac{4b^2C_1^2\rho_4^2}{\gamma\sqrt{\lambda_1}}+\frac{4(\beta+1)^2c^2C_1^{2\beta}\rho_4^{2\beta}}{\gamma\lambda_1^{\beta/2}}\Big)\cdot\sum_{k=1}^n\left\|A^{-1/2}\varphi_k(t)\right\|^2\mathrm{d}t\\
&\leqslant -2an+\frac{h}{T}\int_0^T\sum_{k=1}^n\frac{\left\|\varphi_k(t)\right\|^2}{\lambda_k}\mathrm{d}t\\
&\leqslant -2an+h\sum_{k=1}^n\frac{1}{\tilde{C}k^{\frac{2}{3}}}\qquad\text{(the fact } \lambda_k\geqslant\tilde{C}k^{\frac{2}{3}}\text{ is used)}\\
&\leqslant -2an+\frac{3n^{\frac{1}{3}}h}{\tilde{C}}\qquad\text{(series inequality (8) is used)}
\end{aligned}\tag{58}
$$

where $h=\dfrac{4b^2C_1^2\rho_4^2}{\gamma\sqrt{\lambda_1}}+\dfrac{4(\beta+1)^2c^2C_1^{2\beta}\rho_4^{2\beta}}{\gamma\lambda_1^{\beta/2}}$. So we obtain $[P_n\widetilde{L}(\tilde{u})]\leqslant -2an+\dfrac{3n^{\frac{1}{3}}h}{\tilde{C}}$.

We need the right hand side of the above inequality must be negative, therefore we require $n\geqslant\left(\dfrac{3h}{2a\tilde{C}}\right)^{\frac{3}{2}}$, where $h=\dfrac{4b^2C_1^2\rho_4^2}{\gamma\sqrt{\lambda_1}}+\dfrac{4(\beta+1)^2c^2C_1^{2\beta}\rho_4^{2\beta}}{\gamma\lambda_1^{\beta/2}}$ and $\tilde{C}$ is defined in (4), which completes the proof.

Since $\tilde{A}_H$ has finite fractal dimension in H with the bound (53), we can easily prove the following Theorem by using (43).

Theorem 4 Suppose $5/2\leqslant\beta\leqslant 4, u_0\in V, f\in H$**.** Then the global attractor A_V obtained in Theorem 2 has finite Hausdorff and fractal dimensions, which can be estimated by

$$
\dim_H(A_V)\leqslant\dim_F(A_V)\leqslant\left(\frac{6b^2C_1^2\rho_4^2}{a\tilde{C}\gamma\sqrt{\lambda_1}}+\frac{6(\beta+1)^2c^2C_1^{2\beta}\rho_4^{2\beta}}{a\tilde{C}\gamma\lambda_1^{\beta/2}}\right)^{\frac{3}{2}},
$$

where $\tilde{C}$ is defined in (4), $b>0, c>0, \rho_4>0$ are given in Lemma 6 and $C_1>0$ is given in (6) which only depends on Ω.

3 Conclusion

In this paper, we investigate the dimension of global attractor in $\left(H_0^1(\Omega)\right)^3$ of strong solution for a 3D Brinkman-Forchheimer equation. By setting $\tilde{u}=A^{1/2}u$, we rewrite system (3) as (40). And by proving (40) has a bound for the fractal dimension and Hausdorff dimension of $\tilde{A}_H$ in H, we obtain the system (3) has a bound for the fractal dimension and Hausdorff dimension of $\tilde{A}_V$ in V.

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