Wuhan University Journal of Natural Sciences
2023, Vol.28 No.1, 011-014

Article ID 1007-1202(2023)01-0011-04
DOI https://doi.org/10.1051/wujns/2023281011

The Generalization of Ciric and Caristi Type Fixed Point Theorem

CHENG Zejia
School of Mathematics and Statistics, Wuhan University, Wuhan 430072, Hubei, China

Abstract: In this paper, we generalize the renowned Ciric and Caristi type fixed point theorem and some corollaries. Then we give an example to illustrate our result is really better than the theorem.
Key words: Ciric and Caristi type fixed point theorem; completed metric space; generalization
CLC number: O 189.3

0 Introduction and Preliminaries

Since Banach[1] established the famous Banach Contraction Principle (Theorem 1) in 1922, fixed point theory has become a hot reasearch field in functional analysis. Then, Ciric[2,3], Rhoades[4,5] and many other authors extended the Banach Contraction Principle into various forms. And the most remarkable form is Caristi[6,7] type fixed point theorem based on Banach Contraction Principle. Since Caristi type fixed point theorem equals to Ekeland's variational principle[8,9], it has various applications in nonlinear analysis and variational inequalities[10-13].

In recent years, the combination of two contraction type has become more and more popular. In 2013, Du and Karapinar[14] firstly merged Banach Contraction into Caristi Theorem to get Theorem 2. Then in 2019, Erdal Karapiner[15] merged Ciric-Type contraction into Caristi Theorem to get Theorem 3.

We give some important definitions and theorems mentioned above.

Definition 1 Given $T: X \to X$, we will say that a point $x \in X$ is a fixed point of T if $Tx = x$.

Theorem 1 (Banach[1]) Let (X, d) be a complete metric space, and $T: X \to X$ be a self-mapping. Suppose that there exists $q \in (0, 1)$ such that $d(Tx, Ty) \leqslant qd(x, y)$ for every $x, y \in X$. Then T has a unique fixed point in X.

Theorem 2 (Theorem 1 in Ref.[14]) Let (X, d) be a complete metric space, and $T: X \to X$ be a map. Suppose that there exists a function $\varphi: X \to R$ with φ is bounded from below ($\inf \varphi(x) > -\infty$)

$d(x, Tx) > 0$ implies $d(Tx, Ty) \leqslant (\varphi(x) - \varphi(Tx))d(x, y)$, for each $x, y \in X$. Then T has a fixed point in X.

Theorem 3 (Theorem 4 in Ref. [15]) Suppose that T is a self-mapping on complete metric space (X, d). If there is a $\varphi: X \to [0, \infty)$ with $d(x, Tx) > 0$, we get

$$d(Tx, Ty) \leqslant (\varphi(x) - \varphi(Tx))N(x, y)$$

where

$$N(x, y) = \max\{d(x, y), d(x, Tx), d(y, Ty), d(y, Tx), d(x, Ty)\}$$

for all $x, y \in X$. Then T has a fixed point.

We can easily see that Theorem 3 is a generaliza-

Received date: 2022-06-18
Biography: CHENG Zejia , female, Master candidate, research direction: fixed point theorem. E-mail: 2021202010064@whu.edu.cn

tion of Theorem 2.

And our main work of this paper is to generalize Theorem 3.

In this paper, we firstly generalize the renowned Theorem 3 into Theorem 4. Then we show Example 1 which satisfies Theorem 4 while not satisfies Theorem 3.

1 Main Result

Theorem 4 Suppose that T is a self-mapping on complete metric space (X, d). If there is a $\varphi : X \to [0, \infty)$ with $d(x, Tx) > 0$, then we get:

$$d(Tx, Ty) \leqslant (\varphi(x) - \varphi(Tx))K(x, y) \quad (1)$$

where

$K(x, y) = \max\{d(x, y), [d(x, Tx) + d(y, Ty)], [d(y, Tx) + d(x, Ty)]\}$

for all $x, y \in X$. Then T has a fixed point.

Proof If there exists $x_0 \in X$ such that $d(x_0, Tx_0) = 0$, the proof is completed.

Now we assume $d(x_0, Tx_0) > 0$ for $x_0 \in X$, and let $x_{n+1} = Tx_n$. If there exists a k such that $d(x_k, Tx_k) = 0$, then x_k is a fixed point of T and we proved the result. So we suppose for $\forall n$, $d(x_n, Tx_n) > 0$, then we get:

$$d(x_n, x_{n+1}) = d(x_n, Tx_n) > 0.$$

Suppose that $c_n = d(x_{n-1}, x_n) > 0$, from equation (1), we derive that

$$\begin{aligned} c_{n+1} &= d(x_n, x_{n+1}) = d(Tx_{n-1}, Tx_n) \\ &\leqslant (\varphi(x_{n-1}) - \varphi(Tx_{n-1}))K(x_{n-1}, x_n) \\ &= (\varphi(x_{n-1}) - \varphi(x_n)) \max\{d(x_{n-1}, x_n), [d(x_{n-1}, x_n) + d(x_n, x_{n+1})], d(x_{n-1}, x_{n+1})\} \\ &= (\varphi(x_{n-1}) - \varphi(x_n)) \max\{[d(x_{n-1}, x_n) + d(x_n, x_{n+1})], d(x_{n-1}, x_{n+1})\} \end{aligned} \quad (2)$$

Step 1 We prove $\{d(x_n, x_{n+1})\}$ is non-increasing and bounded below.

To prove $\{d(x_n, x_{n+1})\}$ is non-increasing, we only need to find that there exists $\varepsilon \in (0, 1)$ such that

$$d(x_n, x_{n+1}) \leqslant \varepsilon d(x_{n-1}, x_n), \forall n \in \mathbb{N}^*.$$

We confirm it into two cases:

Case 1:

If

$$\max\{[d(x_{n-1}, x_n) + d(x_n, x_{n+1})], d(x_{n-1}, x_{n+1})\} = d(x_{n-1}, x_n) + d(x_n, x_{n+1}).$$

Then equation (2) implies that

$$c_{n+1} = d(x_n, x_{n+1}) \leqslant (\varphi(x_{n-1}) - \varphi(x_n)) \cdot [d(x_{n-1}, x_n) + d(x_n, x_{n+1})] \quad (3)$$

And we have $c_n = d(x_{n-1}, x_n)$, so the above equation (3) can be also written as

$$c_{n+1} \leqslant (\varphi(x_{n-1}) - \varphi(x_n))(c_n + c_{n+1}).$$

Therefore, we have

$$0 < \frac{c_n}{c_n + c_{n+1}} \leqslant \varphi(x_{n-1}) - \varphi(x_n) \quad (4)$$

From the above equation (4), we can get $\{\varphi(x_n)\}$ is positive and non-increasing.

So we can assume that it converges to some $r \geqslant 0$. For each $n \in \mathbb{N}$, we have

$$\begin{aligned} \sum_{k=1}^{n} \frac{c_{k+1}}{c_k + c_{k+1}} &\leqslant \sum_{k=1}^{n} (\varphi(x_{k-1}) - \varphi(x_k)) \\ &= (\varphi(x_0) - \varphi(x_1)) + (\varphi(x_1) - \varphi(x_2)) + \cdots + (\varphi(x_{n-1}) - \varphi(x_n)) \\ &= \varphi(x_0) - \varphi(x_n) \to \varphi(x_0) - r < \infty, \text{ as } n \to \infty. \end{aligned}$$

It means that

$$\sum_{n=1}^{\infty} \frac{c_{n+1}}{c_n + c_{n+1}} < \infty.$$

So we can get:

$$\lim_{n \to \infty} \frac{c_{n+1}}{c_n + c_{n+1}} = 0.$$

For $\forall \delta \in \left(0, \frac{1}{2}\right)$, there exists $n_0 \in \mathbb{N}$, for all $n \geqslant n_0$, we have

$$\frac{c_{n+1}}{c_n + c_{n+1}} \leqslant \delta.$$

It equals to

$$c_{n+1} \leqslant \frac{\delta}{1 - \delta} c_n, \ \delta \in \left(0, \frac{1}{2}\right).$$

Since that $\delta \in \left(0, \frac{1}{2}\right)$, we have $\frac{\delta}{1 - \delta} < 1$. By taking $\varepsilon = \frac{\delta}{1 - \delta}$, then we have:

$$c_{n+1} \leqslant \varepsilon c_n, \quad \varepsilon \in (0, 1), \quad \forall n \geqslant n_0.$$

Case 2

If

$$\max\{[d(x_{n-1}, x_n) + d(x_n, x_{n+1})], d(x_{n-1}, x_{n+1})\} = d(x_{n-1}, x_{n+1}).$$

Revisiting equation (2), we can have:

$$\begin{aligned} c_{n+1} &= d(x_n, x_{n+1}) \\ &\leqslant (\varphi(x_{n-1}) - \varphi(x_n))\, d(x_{n-1}, x_{n+1}) \\ &\leqslant (\varphi(x_{n-1}) - \varphi(x_n))\, (d(x_{n-1}, x_n) + d(x_n, x_{n+1})) \\ &= (\varphi(x_{n-1}) - \varphi(x_n))\, (c_n + c_{n+1}). \end{aligned}$$

Then we transpose it into:

$$\frac{c_n}{c_n + c_{n+1}} \leqslant \varphi(x_{n-1}) - \varphi(x_n).$$

Then it is similar to the process in Case 1.

Step 2 We prove $\{x_n\}$ converges to some $u \in X$.

Note that Step 1 shows that $\{d(x_n, x_{n+1})\}$ is nonincreasing and has inferior. So we can assume $d(x_n, x_{n+1})$ converges to some $p \geqslant 0$. From

$$d(x_n, x_{n+1}) \leqslant \varepsilon d(x_{n-1}, x_n), \ \varepsilon \in (0, 1), \ \forall n \in \mathbb{N}^*,$$

we can easily get $p = 0$. For each $m, n \in \mathbb{N}$ with $m > n$, we can get:

$$d\left(x_n, x_m\right) \leqslant \sum_{k=n}^{m-1} d\left(x_k, x_{k+1}\right) \leqslant \frac{\varepsilon^n}{1-\epsilon} d\left(x_0, x_1\right).$$

That means that $\lim_{m,n\to\infty} d\left(x_n, x_m\right) = 0$, $m > n$. So from definition we can get $\{x_n\}$ is a Cauchy sequence. And since X is complete, there exists $u \in X$, such that $\{x_n\}$ converges to some $u \in X$.

Step 3 We prove u is a fixed point of T.

From equation (1), we can get:

$d(u, Tu) \leqslant d(u, x_{n+1}) + d(x_{n+1}, Tu)$
$= d(u, x_{n+1}) + d(Tx_n, Tu)$
$= d(u, x_{n+1}) + (\varphi(x_n) - \varphi(x_{n+1})) \cdot K(x_n, u)$
$= d(u, x_{n+1}) + (\varphi(x_n) - \varphi(x_{n+1})) \max\{[d(x_n, x_{n+1}) + d(u, Tu)], [d(u, x_{n+1}) + d(x_n, u)]\}$.

We denote $(*) = \max\{[d(x_n, x_{n+1}) + d(u, Tu)], [d(u, x_{n+1}) + d(x_n, u)]\}$.

Since that $\{\varphi(x_n)\}$ converges to some $r \geqslant 0$ and $\{x_n\}$ converges to some $u \in X$, let $n \to +\infty$, $n \in \mathbb{N}$, we have

$d(u,Tu) \leqslant \lim\{d(u, x_{n+1}) + (\varphi(x_n) - \varphi(x_{n+1})) \cdot (*)\} \to 0.$

Consequently, we prove that $d(u, Tu) = 0$. That means $Tu = u$. We completed the proof.

Then, we also get some corollaries based on Theorem 4.

Firstly, we combine Theorem 4 with Theorem 3 to get Corollary 1 and Corollary 2.

Then, we respectively combine contraction fixed point with Caristi type fixed point theory of Refs.[16,17] to get Corollary 3-6 on the basis of Theorem 4.

Corollary 1 Suppose that T is a self-mapping on complete metric space (X, d). If there is a $\varphi : X \to [0, \infty)$ with $d(x, Tx) > 0$, then we get:

$$d(Tx, Ty) \leqslant (\varphi(x) - \varphi(Tx))K_1(x, y).$$

where

$$K_1(x, y) = \max\{d(x, y), d(x, Tx), d(y, Ty), [d(y, Tx) + d(x, Ty)]\}$$

for all $x, y \in X$. Then T has a fixed point.

Corollary 2 Suppose that T is a self-mapping on complete metric space (X, d). If there is a $\varphi : X \to [0, \infty)$ with $d(x, Tx) > 0$, then we get:

$$d(Tx, Ty) \leqslant (\varphi(x) - \varphi(Tx))K_2(x, y).$$

where

$$K_2(x, y) = \max\{d(x, y), [d(x, Tx) + d(y, Ty)], d(y, Tx), d(x, Ty)\}$$

for all $x, y \in X$. Then T has a fixed point.

Corollary 3 Suppose that T is a self-mapping on complete metric space (X, d). If there is a $\varphi : X \to [0, \infty)$ with $d(x, Tx) > 0$, then we get:

$$d(Tx, Ty) \leqslant (\varphi(x) - \varphi(Tx))K_3(x, y)$$

where

$$K_3(x, y) = \max\{d(x, y), \frac{1}{2}[d(x, Tx) + d(y, Ty)], \frac{1}{2}[d(y, Tx) + d(x, Ty)]\}$$

for all $x, y \in X$. Then T has a fixed point.

Corollary 4 Suppose that T is a self-mapping on complete metric space (X, d). If there is a $\varphi : X \to [0, \infty)$ with $d(x, Tx) > 0$, then we get:

$$d(Tx, Ty) \leqslant (\varphi(x) - \varphi(Tx))K_4(x, y)$$

where

$$K_4(x, y) = \max\{d(x, y), [\alpha_1 d(x, Tx) + \alpha_2 d(y, Ty)], [\alpha_3 d(y, Tx) + \alpha_4 d(x, Ty)]\}$$

for all $x, y \in X$ and $\alpha_i \in (0, 1)$, $i = 1, 2, 3, 4$. Then T has a fixed point.

Corollary 5 Suppose that T is a self-mapping on complete metric space (X, d). If there is a $\varphi : X \to [0, \infty)$ with $d(x, Tx) > 0$, then we get:

$$d(Tx, Ty) \leqslant (\varphi(x) - \varphi(Tx))K_5(x, y)$$

where

$$K_5(x, y) = h \max\{d(x, y), [d(x, Tx) + d(y, Ty)], [d(y, Tx) + d(x, Ty)]\}$$

for all $x, y \in X$ and $h \in (0, 1)$. Then T has a fixed point.

Corollary 6 Suppose that T is a self-mapping on complete metric space (X, d). If there is a $\varphi : X \to [0, \infty)$ with $d(x, Tx) > 0$, then we get:

$$d(Tx, Ty) \leqslant (\varphi(x) - \varphi(Tx))K_6(x, y)$$

where

$$K_6(x, y) = \max\{d(x, y), [a_1\left(d(x,y)\right)d(x, Tx) + a_2\left(d(x,y)\right) d(y, Ty)], [a_3\left(d(x,y)\right)d(y, Tx) + a_4\left(d(x,y)\right)d(x, Ty)]\}$$

for all $x, y \in X$ and there exists monotonically decreasing function $(0, \infty) \to (0, 1)$ satisfying $a_1(t) + a_2(t) + a_3(t) + a_4(t) < 1$. Then T has a fixed point.

2 Example

In this section, we give an example (Example 1) to show that our conclusion contains fixed point.

Example 1 Let $X = \{0, 1, 2, \cdots, n\}$ $(n \in \mathbb{N}^*)$, endowed with the following metric:

$$d(x, y) = |x - y|, \quad \forall x, y \in X.$$

Define $T : X \to X$ by $T0 = 0$, $T1 = 2$, $T2 = 3, \cdots$, $Tk = k + 1, \cdots$, $T(n-1) = n$, $Tn = 0$ and $\varphi : X \to [0, +\infty)$ by

$$\varphi(t)=\begin{cases}0, & \text{if } t=0,\\ \frac{2}{3}(n-t), & \text{if } t=1,2,\cdots,n.\end{cases}$$

If $x\in X$ and $d(x, Tx)>0$, then $x\neq 0$. We have

$d(Tn, T0)\leqslant(\varphi(n)-\varphi(0))K(n, 0)$.

$d(Tk, T0)\leqslant(\varphi(k)-\varphi(k+1))K(k, 0)$, $0<k\leqslant n-1$.

$d(Tp, Tq)\leqslant(\varphi(p)-\varphi(p+1))K(p, q)$, $0<p<q\leqslant n-1$.

$d(Tp, Tq)\leqslant(\varphi(p)-\varphi(p+1))K(p, q)$, $0<q<p\leqslant n-1$.

We can easily confirm that this condition satisfies Theorem 4, and T really has a fixed point.

However, if we apply this example into Theorem 3, we find that:

$d(Tk, T0)>(\varphi(k)-\varphi(k+1))N(k, 0)$, $0<k\leqslant n-1$.

$d(Tp, Tq)>(\varphi(p)-\varphi(p+1))N(p, q)$, $0<p<q\leqslant n-1$.

$d(Tp, Tq)>(\varphi(p)-\varphi(p+1))N(p, q)$, $0<q<p\leqslant n-1$.

It shows that Example 1 does not satisfy the condition of Theorem 3.

But T really has a fixed point $T0=0$.

Remark 1 We can easily see that $N(x, y)\leqslant K(x, y)$. So, Theorem 4 is a real generalization of Theorem 3.

References

[1] Banach S. Sur les opérations dans les ensembles abstraits et leur application Aux équations intégrales[J]. *Fundamenta Mathematicae*, 1922, **3**: 133-181.

[2] Ciric L B. A generalization of Banach contraction principle [J]. *Proc Am Math Soc*, 1974, **45**: 267-273.

[3] Ciric L B. Generalized contractions and fixed-point theorems [J]. *Publ Inst Math* , 1971, **12**(26): 19-26.

[4] Rhoades B E. Fixed point iterations using infinite matrices [J]. *Transactions of the American Mathematical Society*, 1974, **196**: 161-176.

[5] Rhoades B E. Some fixed point theorems in a Banach space [J]. *Comment Math Univ St Pauli*, 1976, **24**: 13-16.

[6] Caristi J. Fixed point theorems for mappings satisfying inwardness conditions[J]. *Transactions of the American Mathematical Society*, 1976, **215**: 241-251.

[7] Kirk W A, Caristi J. Mapping theorems in metric and Banach spaces[J]. *Bulletin of the Polish Academy of Sciences*, 1975, **25**: 891-894.

[8] Ekeland I. On the variational principle[J]. *J Math Anal Appl*, 1974, **47**(2): 324- 353.

[9] Ekeland I. Nonconvex minimization problems[J]. *Bull Am Math Soc*, 1979, **1**(3): 443-474.

[10] Kimura Y, Toyoda M. Fixed point theorem in ball spaces and Caristis fixed point theorem[J]. *Journal of Nonlinear and Convex Analysis*, 2022, **23**:185-189.

[11] Bakery A A, El Dewaik M H. A generalization of Caristi's fixed point theorem in the variable exponent weighted formal power series space[J]. *Journal of Function Spaces*, 2021, **2021**: 1-18.

[12] Romaguera S. On the correlation between Banach contraction principle and caristi's fixed point theorem in b-metric spaces[J]. *Mathematics*, 2022, **10**(1): 136.

[13] Karapınar E, Khojasteh F, Mitrović Z. A proposal for revisiting Banach and caristi type theorems in b-metric spaces[J]. *Mathematics*, 2019, **7**(4): 308.

[14] Du W S, Karapinar E. A note on Caristi-type cyclic maps: Related results and applications[J]. *Fixed Point Theory and Applications*, 2013, **2013**: 344.

[15] Karapiner E, Khojasteh F, Shatanawi W. Revisiting ćirić -type contraction with caristi's approach[J]. *Symmetry*, 2019, **11**(6): 726.

[16] Zhang S. *Fixed Point Theory and Application*[M]. Chongqing: Chongqing Press, 1984(Ch).

[17] Vasile I S. *Fixed Point Theory*, *An Introduction*[M]. Dordrecht: D. Reidel Pulishing Company, 1981.

□