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A Characterization of the Polarity Mapping for Convex Bodies

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Abstract: In this paper, we establish a characterization of the polarity mapping for 1-dimensional convex bodies, which is a supplement to the result for such a characterization obtained by Böröczky and Schneider.

Key words: duality of convex bodies; polar convex body; involution; order-reversion

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0 Introduction

Let $\mathbb{R}^n$ denote the n-dimensional Euclidean vector space, equipped with its standard scalar product $\langle \cdot, \cdot \rangle$. We denote the set of convex bodies (compact convex subsets with nonempty interior) in $\mathbb{R}^n$ which contain origin o in the interior by $\boldsymbol{K}^n_{(o)}$. For $K \in \boldsymbol{K}^n_{(o)}$, its dual or polar body K^* is defined by (see, e.g., Ref.[1])

$$K^* := \left\{ x \in \mathbb{R}^n : \left\langle x, y \right\rangle \leqslant 1 \text{ for all } y \in K \right\}.$$

It is again in $\boldsymbol{K}^n_{(o)}$. Polar body is an important and fundamental notion of the dual theory of convex bodies, and this duality is one of the central concepts both in geometry and in analysis (see, e.g., Refs.[2-10]).

Mahler's conjecture (see, e.g., Ref.[11]), a famous open problem, is related to polar bodies. By $\boldsymbol{K}^n$ we denote the class of all convex bodies in $\mathbb{R}^n$, and by $\boldsymbol{K}^n_e$ we denote the class of all n-dimensional origin-symmetric convex bodies in $\mathbb{R}^n$. Let $K \in \boldsymbol{K}^n_e$, the volume product of K and its polar body is defined by (see, e.g., Ref.[5]) $P(K) = V(K)V(K^*)$, where $V(K)$ denotes the n-dimensional volume of K. Along the volume product, there is the Mahler's conjecture that: for $K \in \boldsymbol{K}^n_e$,

$$P(K) \geqslant \frac{4^n}{n!}$$

where equality holds for parallelepipeds and their polars (and other bodies). It is easily checked that $P(K) = 4$ for all $K \in \boldsymbol{K}^1_e$. In 1939, Mahler[12] himself proved that $P(K) \geqslant 8$ for all $K \in \boldsymbol{K}^2_e$, and in 1986, Reisner[13] characterized that equality holds only for parallelograms. Later, in 1991, Meyer[14] used some alternative methods to give a complete proof for the case $n = 2$, including the characterization of equality. Recently, Iriyeh and Shibata[15] showed that the conjecture holds for the case $n = 3$ and equality holds if and only if K or K^* is a parallelepiped. For the case $n \geqslant 4$, Mahler's conjecture is still a challenging open problem.

Duality for convex functions can also be defined. For a convex function $f: \mathbb{R}^n \to \mathbb{R} \cup \{\pm\infty\}$, its conjugate function is defined by (see, e.g., Ref.[16])

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$$f^*(x):=\sup\left\{\left\langle x,y\right\rangle - f(y): y\in\mathbb{R}^n\right\} \text{ for } x\in\mathbb{R}^n.$$

If f is a lower semi-continuous convex function, then f^* is also a lower semi-continuous convex function, and $f^{**}=f$. This duality for lower semi-continuous convex functions can be characterized from two simple and natural properties: involution and order-reversion. Artstein-Avidan and Milman[17] showed that any involution on the class of lower semi-continuous convex functions which is order-reversing, must be, up to linear terms, the well-known Legendre transform. For more results on the characterizations of the duality for convex functions, s-concave functions and log-concave functions, we can refer to Refs.[18-22].

Recently, Böröczky and Schneider[1] made use of an excellent tool that is lattice endomorphism from Gruber[23,24], to characterize the duality mapping for convex bodies by interchanging the pairwise intersections and convex hulls of unions. Let $\vee$ denote the convex hull of unions (see Sect.1 Notations for details).

Theorem 1 Let $n\geqslant 2$ and let $\phi:\boldsymbol{K}^n_{(o)}\to\boldsymbol{K}^n_{(o)}$ be a mapping satisfying

$$\phi(K\vee L)=\phi(K)\cap\phi(L) \tag{1}$$

$$\phi(K\cap L)=\phi(K)\vee\phi(L) \tag{2}$$

for all $K,L\in\boldsymbol{K}^n_{(o)}$. Then either ϕ is constant, or there exists a linear transformation $T\in\mathrm{GL}(n)$ such that $\phi(K)=TK^*$ for all $K\in\boldsymbol{K}^n_{(o)}$.

It is important to point out that the property (the duality interchanges the pairwise intersections and convex hulls of unions) is sufficient for a characterization, up to a trivial exception (the constant map) and the composition with a linear transformation. A mapping $\phi:\boldsymbol{K}^n_{(o)}\to\boldsymbol{K}^n_{(o)}$ is called involutive if

$$\phi(\phi(K))=K \tag{3}$$

for all $K\in\boldsymbol{K}^n_{(o)}$. If ϕ satisfies condition (3) and one of conditions (1) and (2), then ϕ satisfies conditions (1), (2) and (3), and ϕ is order-reversing. By replacing condition (1) with (3), Böröczky and Schneider[1] completely established a characterization of the duality mapping for convex bodies in $\mathbb{R}^n$ with $n\geqslant 2$. Condition (3) excludes the constant map and forces the linear map appearing in the theorem to be selfadjoint.

Theorem 2 Let $n\geqslant 2$ and let $\phi:\boldsymbol{K}^n_{(o)}\to\boldsymbol{K}^n_{(o)}$ be a mapping satisfying

$$\phi(\phi(K))=K,$$

$$\phi(K\cap L)=\phi(K)\vee\phi(L),$$

for all $K,L\in\boldsymbol{K}^n_{(o)}$. Then there exists a selfadjoint linear transformation $T\in\mathrm{GL}(n)$ such that $\phi(K)=TK^*$ for all $K\in\boldsymbol{K}^n_{(o)}$.

For more results on the characterization of duality and lattice endomorphism of the class of convex bodies and of convex sets, we can refer to Refs.[25-29].

The main purpose of this paper is to establish a characterization of the duality mapping for convex bodies on 1-dimensional Euclidean space with some additional assumptions. For simplicity, we will identify $x\in(0,+\infty)$ with $0<x<+\infty$ as follows. And, obviously, $K=[-x,y]$ if $K\in\boldsymbol{K}^1_{(o)}$, where $x,y\in(0,+\infty)$.

Theorem 3 Let $\phi:\boldsymbol{K}^1_{(o)}\to\boldsymbol{K}^1_{(o)}$ be a mapping satisfying

$$\phi(\phi(K))=K,$$

$$\phi(K\cap L)=\phi(K)\cup\phi(L),$$

$$\phi(rK)=\frac{1}{r}\phi(K),$$

for all $K,L\in\boldsymbol{K}^n_{(o)}$ and all real $r>0$. Then, there exist constants $c,d\in\mathbb{R}$ with $c,d>0$ such that

$$\phi([-x,y])=[-\frac{x}{c},\frac{y}{d}]^*$$

for all $x,y\in(0,+\infty)$; or there exists a constant $c\in\mathbb{R}$ with $c<0$ such that

$$\phi([-x,y])=c[-x,y]^*$$

for all $x,y\in(0,+\infty)$.

1 Notations

For reference, we collect some basic facts on convex sets and convex bodies. Excellent references are the books by Gardner[30], Gruber[31] and Schneider[16].

Let B stand for the unit ball $\{x\in\mathbb{R}^n: \left\langle x,x\right\rangle\leqslant 1\}$ and S^{n-1} the unit sphere of $\mathbb{R}^n$. A set $A\subset\mathbb{R}^n$ is convex if for any two points $x,y\in A$, the line segment $[x,y]$ joining them is contained in A, i.e.,

$$(1-\lambda)x+\lambda y\in A,\ 0\leqslant\lambda\leqslant 1.$$

If A,B are convex, then $A+B=\{x+y: \text{for all } x\in A, y\in B\}$ and $rA=\{rx: x\in A \text{ and } r\in\mathbb{R}\}$ are convex. A convex body is a compact convex subset of $\mathbb{R}^n$ with nonempty interior.

The support function $h_K:\mathbb{R}^n\to\mathbb{R}$ of a compact, convex $K\subset\mathbb{R}^n$ is defined, for $x\in\mathbb{R}^n$, by

$$h_K(x)=\max\left\{\left\langle x,y\right\rangle: y\in K\right\}$$

It can be easily checked that the support function is sublinear, i.e., h_K has the positive homogeneity of degree 1 and satisfies subadditive. From the definition, it follows immediately that, for $g\in\mathrm{GL}(n)$, the support function of

$gK=\{gx: x\in K\}$ is given by $h_{gK}(x)=h_K(g^t x)$.

A boundary point $x\in\partial K$ is said to have $u\in S^{n-1}$ as an outer normal provided $\langle x,u\rangle=h_K(u)$. The convex body is equipped with the Hausdorff metric δ, which is defined for convex bodies K,L by

$$\delta(K,L):=\max_{u\in S^{n-1}}|h_K(u)-h_L(u)|.$$

Let $\vee$ denote the convex hull of unions, i.e., $A\vee B:=\operatorname{conv}(A\cup B)$ for all $A,B\in\boldsymbol{K}^n_{(o)}$. The set of convex bodies in $\mathbb{R}^n$ containing the origin o in their interior is denoted by $\boldsymbol{K}^n_{(o)}$. For $K\in\boldsymbol{K}^n_{(o)}$, its dual or polar body K^* is defined by

$$K^*:=\left\{x\in\mathbb{R}^n:\langle x,y\rangle\leqslant 1 \text{ for all } y\in K\right\}.$$

The duality mapping $K\to K^*$ has a number of remarkable properties, of which we list the following; they are valid for all $K,L\in\boldsymbol{K}^n_{(o)}$:

1) $(K^*)^*=K$;

2) $K\subset L$ implies $K^*\supset L^*$;

3) $(K\cap L)^*=K^*\vee L^*$;

4) $(K\vee L)^*=K^*\cap L^*$;

5) Continuity with respect to the Hausdorff metric;

6) If $g\in \mathrm{GL}(n)$, then $(gK)^*=g^{-t}K^*$.

If $K\cup L$ is convex, then $(K\cap L)^*=K^*\cup L^*$ and $(K\cup L)^*=K^*\cap L^*$. When $n=1$, for $K,L\in\boldsymbol{K}^1_{(o)}$, $K\vee L=K\cup L$ is convex. And for $K\in\boldsymbol{K}^1_{(o)}$ and real $r\neq 0$, $(rK)^*=r^{-1}K^*$ is the form of the case $n=1$ in 6) above. For more interesting properties of the duality mapping of convex body, we can refer to Ref.[16], §1.6.

2 Main Results

Lemma 1 Let $\phi:\boldsymbol{K}^1_{(o)}\to\boldsymbol{K}^1_{(o)}$ be a mapping satisfying

$$\phi(K\cup L)=\phi(K)\cap\phi(L) \tag{4}$$

$$\phi(K\cap L)=\phi(K)\cup\phi(L) \tag{5}$$

for all $K,L\in\boldsymbol{K}^1_{(o)}$. Then, for all $x,y\in(0,+\infty)$, the value of $\phi([-x,y])$ has the following four cases:

Case 1: $\phi([-x,y])=[-f(x),g(x)]$;

Case 2: $\phi([-x,y])=[-f(x),g(y)]$;

Case 3: $\phi([-x,y])=[-f(y),g(x)]$;

Case 4: $\phi([-x,y])=[-f(y),g(y)]$;

where f,g are two decreasing positive functions on $(0,+\infty)$.

Proof Let $K\subseteq L$, combining with (4), it follows that

$$\phi(L)=\phi(K\cup L)=\phi(K)\cap\phi(L),$$

which implies

$$K\subseteq L\Rightarrow\phi(K)\supseteq\phi(L) \tag{6}$$

for $K,L\in\boldsymbol{K}^1_{(o)}$. Note that if $K\subset L$, there is still $\phi(K)\supseteq\phi(L)$. Then (6) induces

$$\phi([-x,y])=[-f(x,y),g(x,y)] \tag{7}$$

for all $x,y\in(0,+\infty)$, where both $f(x,y)$ and $g(x,y)$ are decreasing positive functions with respect to x and y.

Suppose $a,b,c,d\in(0,+\infty)$ are arbitrary, it follows from (7) that

$$\phi([-c,b])=[-f(c,b),g(c,b)],$$

and

$$\phi([-a,d])=[-f(a,d),g(a,d)].$$

Without loss of generality, letting $a>c$, $b<d$. Then, we obtain from (4), (5) and (6) that

$$\begin{aligned}\phi([-c,b])&=\phi([-a,b]\cap[-c,d])\\&=\phi([-a,b])\cup\phi([-c,d])\\&=[-f(a,b),g(a,b)]\cup[-f(c,d),g(c,d)]\\&=[-\max\{f(a,b),f(c,d)\},\max\{g(a,b),g(c,d)\}]\\&=[-f(c,b),g(c,b)],\end{aligned}$$

and

$$\begin{aligned}\phi([-a,d])&=\phi([-a,b]\cup[-c,d])\\&=\phi([-a,b])\cap\phi([-c,d])\\&=[-f(a,b),g(a,b)]\cap[-f(c,d),g(c,d)]\\&=[-\min\{f(a,b),f(c,d)\},\min\{g(a,b),g(c,d)\}]\\&=[-f(a,d),g(a,d)].\end{aligned}$$

We now study the first result $f(c,b)=\max\{f(a,b),f(c,d)\}$ which implies the following three possibilities, and the others are similar.

(P1) If $f(a,b)>f(c,d)$, then we have $f(c,b)=f(a,b)$. Thus, $f(x,y)$ is independent of the first variable x. We obtain that $f(x,y)$ can be rewritten as $f(y)$, which is a decreasing positive function on $(0,+\infty)$.

(P2) If $f(a,b)<f(c,d)$, then we have $f(c,b)=f(c,d)$. Thus, $f(x,y)$ is independent of the second variable y. We obtain that $f(x,y)$ can be rewritten as $f(x)$, which is a decreasing positive function on $(0,+\infty)$.

(P3) If $f(a,b)=f(c,d)$, then we have $f(c,b)=f(a,b)=f(c,d)$. Thus, $f(x,y)$ is independent of the variable x and y. We obtain that $f(x,y)$ is a constant positive function on $(0,+\infty)$. Since the decreasing function contains the constant function as a special case, $f(x,y)$ can be rewritten as either $f(x)$ in (P2) or $f(y)$ in (P1).

Consequently, we may rewrite $f(x,y)$ as either $f(x)$ or $f(y)$, and similarly, we rewrite $g(x,y)$ as either $g(x)$ or $g(y)$, where f,g are two decreasing positive functions on $(0,+\infty)$. Then, we conclude from (7) that $\phi([-x,y])$ has 4 different cases as desired.

A mapping ϕ: $\boldsymbol{K}_{(o)}^1 \to \boldsymbol{K}_{(o)}^1$ is called order-reversing if

$$K \subset L \Rightarrow \phi(K) \supset \phi(L) \tag{8}$$

for $K, L \in \boldsymbol{K}_{(o)}^1$.

Lemma 2 Let ϕ:$\boldsymbol{K}_{(o)}^1 \to \boldsymbol{K}_{(o)}^1$ be a mapping satisfying (3) and (5) for all $K, L \in \boldsymbol{K}_{(o)}^1$. Then ϕ is order-reversing and satisfies (4).

Proof Let $K \subset L$, then, from (5), we obtain

$$\phi(K) = \phi(K \cap L) = \phi(K) \cup \phi(L)$$

which implies

$$K \subset L \Rightarrow \phi(K) \supseteq \phi(L)$$

for $K, L \in \boldsymbol{K}_{(o)}^1$.

Suppose ϕ is not order-reversing and we may assume that there exist two convex bodies $K_1, K_2 \in \boldsymbol{K}_{(o)}^1$ with $K_1 \subset K_2$ such that $\phi(K_1) = \phi(K_2)$. But, it follows from (3) that

$$K_1 = \phi(\phi(K_1)) = \phi(\phi(K_2)) = K_2$$

which is a contradiction. Thus, (3) and (5) induce the property of ϕ: order-reversion.

Moreover, (3) and (5) induce

$$\phi(\phi(K) \cap \phi(L)) = \phi(\phi(K)) \cup \phi(\phi(L)) = K \cup L$$

Applying (3) again, we conclude that

$$\phi(K) \cap \phi(L) = \phi(K \cup L)$$

as desired.

Lemma 3 Let ϕ:$\boldsymbol{K}_{(o)}^1 \to \boldsymbol{K}_{(o)}^1$ be a mapping satisfying (3) and (5) for all $K, L \in \boldsymbol{K}_{(o)}^1$. Then, there exist two strictly decreasing positive functions f,g on $(0, +\infty)$ with the properties that $f = f^{-1}$, $g = g^{-1}$ such that

$$\phi([-x, y]) = [-f(x), g(y)] \tag{9}$$

where f^{-1},g^{-1} denote the inverse function of f,g, respectively; or there exists a strictly decreasing positive function f on $(0, +\infty)$ such that

$$\phi([-x, y]) = [-f(y), f^{-1}(x)] \tag{10}$$

where f^{-1} denotes the inverse function of f.

Proof ϕ is order-reversing and satisfies (3), (4) and (5) via Lemma 2. Then, the functions $f(x, y)$, $g(x, y)$ must be strictly decreasing positive functions with respect to x and y on $(0, +\infty)$. Thus, together with Lemma 1, we obtain that the functions $f(x, y)$, $g(x, y)$ be rewritten as $f(x)$, $f(y)$ and $g(x)$,$g(y)$, where f,g are two strictly decreasing positive functions on $(0, +\infty)$ and that the forms of four cases in Lemma 1 are remained tentatively.

However, since ϕ is order-reversing, both Case 1 and Case 4 are removed when f,g are two strictly decreasing positive functions on $(0, +\infty)$. We show the contradiction for Case 1 (Case 4 is similar). Suppose $y_1 < y_2$ are arbitrary and x is fixed, then it follows from (8) that

$$\phi([-x, y_1]) \supset \phi([-x, y_2]).$$

And, combining with the form of Case 1, we obtain that

$$\phi([-x, y_1]) = [-f(x), g(x)],$$

and

$$\phi([-x, y_2]) = [-f(x), g(x)].$$

A contradiction occurs.

Now, we further study Case 2 and Case 3 with the conditions (3) and (5).

(i) From the form of Case 2 and above, we deduce that $\phi([-x, y]) = [-f(x), g(y)]$, where f, g are two strictly decreasing positive functions on $(0, +\infty)$. Then, from (3), we have

$$\begin{aligned}[-x, y] &= \phi(\phi([-x, y])) \\ &= \phi([-f(x), g(y)]) \\ &= [-f \circ f(x), g \circ g(y)]\end{aligned}$$

which implies

$$f \circ f(x) = x \text{ and } g \circ g(y) = y$$

for all $x, y \in (0, +\infty)$. Thus,

$$f = f^{-1} \text{ and } g = g^{-1}$$

on $(0, +\infty)$, where f^{-1}, g^{-1} denote the inverse function of f, g, respectively. Therefore, we obtain

$$\phi([-x, y]) = [-f(x), g(y)]$$

where f,g are two strictly decreasing positive functions on $(0, +\infty)$ with the properties that $f = f^{-1}$, $g = g^{-1}$.

(ii) From the form of Case 3 and above, we deduce that $\phi([-x, y]) = [-f(y), g(x)]$, where f,g are two strictly decreasing positive functions on $(0, +\infty)$. Then, from (3), we have

$$\begin{aligned}[-x, y] &= \phi(\phi([-x, y])) \\ &= \phi([-f(y), g(x)]) \\ &= [-f \circ g(x), g \circ f(y)]\end{aligned}$$

which implies

$$f \circ g(x) = x \text{ and } g \circ f(y) = y$$

for all $x, y \in (0, +\infty)$. Thus,

$$g = f^{-1}$$

on $(0, +\infty)$, where f^{-1} denotes the inverse function of f. Therefore, we conclude that

$$\phi([-x, y]) = [-f(y), f^{-1}(x)],$$

where f is a strictly decreasing positive function on $(0, +\infty)$. Certainly, f^{-1} is also a strictly decreasing positive function on $(0, +\infty)$.

A mapping ϕ:$\boldsymbol{K}_{(o)}^1 \to \boldsymbol{K}_{(o)}^1$ is called homogeneous if there exists $p \in \mathbb{R}$ such that

$$\phi(\lambda K) = \lambda^p \phi(K)$$

for all $\lambda > 0$ and all $K \in \boldsymbol{K}_{(o)}^1$. With the additional assumption of homogeneity, we establish a characterization of polarity or duality mapping for 1-dimensional convex bodies.

Proof of Theorem 3 Recall that ϕ satisfies $\phi(\phi(K))=K$, $\phi(K\cap L)=\phi(K)\cup\phi(L)$ and $\phi(rK)=\frac{1}{r}\phi(K)$ for all $K,L\in\boldsymbol{K}^1_{(o)}$ and real $r>0$. Together with Lemmas above, we finish the proof of Theorem 3 with two cases.

(i) Due to Lemma 3, if (9) holds, then

$$\phi([-rx,ry])=[-f(rx),g(ry)]$$

for all $x,y\in(0,+\infty)$ and real $r>0$, where f,g are two strictly decreasing positive functions on $(0,+\infty)$ with the properties that $f=f^{-1}$, $g=g^{-1}$. Since ϕ satisfies the homogeneity of degree -1, we obtain

$$\phi([-rx,ry])=\frac{1}{r}\phi([-x,y])=[-\frac{f(x)}{r},\frac{g(y)}{r}]$$

for all $x,y\in(0,+\infty)$ and real $r>0$. Thus,

$$f(rx)=\frac{f(x)}{r}\text{ and }g(ry)=\frac{g(y)}{r}$$

for all $x,y\in(0,+\infty)$ and real $r>0$, which implies

$$f(t)=\frac{c}{t}\text{ and }g(t)=\frac{d}{t}$$

for all $t>0$, where $c,d>0$. Thus,

$$\phi([-x,y])=[-\frac{c}{x},\frac{d}{y}]=[-\frac{x}{c},\frac{y}{d}]^*$$

for all $x,y\in(0,+\infty)$, where $c,d>0$.

(ii) Due to Lemma 3, if (10) holds, then

$$\phi([-rx,ry])=[-f(ry),f^{-1}(rx)]$$

for all $x,y\in(0,+\infty)$ and real $r>0$, where f is a strictly decreasing positive function on $(0,+\infty)$. And, from the homogeneity of ϕ, we deduce that

$$\phi([-rx,ry])=\frac{1}{r}\phi([-x,y])=[-\frac{f(y)}{r},\frac{f^{-1}(x)}{r}]$$

for all $x,y\in(0,+\infty)$ and real $r>0$. Thus,

$$f(ry)=\frac{f(y)}{r}\text{ and }f^{-1}(rx)=\frac{f^{-1}(x)}{r}$$

for all $x,y\in(0,+\infty)$ and real $r>0$, which implies

$$f(t)=f^{-1}(t)=\frac{d}{t}$$

for all $t>0$, where $d>0$. Thus,

$$\phi([-x,y])=[-\frac{d}{y},\frac{d}{x}]=c[-x,y]^*$$

for all $x,y\in(0,+\infty)$, where $c=-d$.

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