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Global Existence and Extinction Behaviour for a Doubly Nonlinear Parabolic Equation with Logarithmic Nonlinearity

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Abstract: This paper is mainly focused on the global existence and extinction behaviour of the solutions to a doubly nonlinear parabolic equation with logarithmic nonlinearity. By making use of energy estimates method and a series of ordinary differential inequalities, the global existence of the solution is obtained. Moreover, we give the sufficient conditions on the occurrence (or absence) of the extinction behaviour.

Key words: global existence; extinction behaviour; doubly nonlinear parabolic equation; logarithmic nonlinearity

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0 Introduction

Let $m \geqslant 1$, $p > 1$ and $q \in (1,2)$ be three constants, $\Omega \subset \mathbb{R}^N (N > p)$ be a bounded domain with smooth boundary $\partial\Omega$, and $u_0(x)$ be a bounded non-trivial function with $|u_0|^{m-1} u_0 \in W_0^{1,p}(\Omega)$. We focus our attention here on dealing with the global existence, extinction and non-extinction phenomenon of the following doubly nonlinear parabolic equation

$$\begin{cases} u_t - \operatorname{div}\left(\left| \nabla\left(|u|^{m-1} u \right) \right|^{p-2} \nabla\left(|u|^{m-1} u \right) \right) = f(u), & (x,t) \in \Omega \times (0, +\infty) \\ u(x,t) = 0, & (x,t) \in \partial\Omega \times (0, +\infty) \\ u(x,0) = u_0(x), & x \in \Omega \end{cases} \tag{1}$$

with

$$f(u) = \begin{cases} |u|^{q-2} u \log|u|, & \text{if } u \neq 0 \\ 0, & \text{if } u = 0 \end{cases} \tag{2}$$

Nonlinear evolutionary problems with logarithmic nonlinearity like model (1) came from inflation cosmology, su-

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per symmetric field theories, quantum mechanics and nuclear physics[1-6].

In the past few decades, many mathematical researchers have devoted themselves to investigating the doubly nonlinear parabolic equations, and obtained many meaningful results, such as local and global well-posedness, regularity, blow-up in a finite time and extinction singularity [7-12]. Especially, the authors [13-15] studied the problem

$$\begin{cases} u_t - \operatorname{div}\left(\left|\nabla u^m\right|^{p-2}\nabla u^m\right) = \lambda u^q, & (x,t) \in \Omega \times (0,+\infty) \\ u(x,t) = 0, & (x,t) \in \partial\Omega \times (0,+\infty) \\ u(x,0) = u_0(x), & x \in \Omega \end{cases} \tag{3}$$

where $m>0$, $p>1$, $\lambda>0$, $q>0$ and $u_0(x)$ is a non-negative bounded non-trivial function with $u_0^m(x) \in W_0^{1,p}(\Omega)$. For the case $0<m(p-1)<1$, the authors [14,15] proved that the critical blow-up and extinction exponents are $q_{bc}=1$ and $q_{ec}=m(p-1)$, respectively. Compared with the case $0<m(p-1)<1$, the solution for the case $m(p-1)\geqslant 1$ exhibits completely different properties [13]. On the one hand, for any $q>0$, the non-trivial solution to problem (3) will never become extinct at a finite time. On the other hand, the critical blow-up exponent becomes $q_{bc}^{*}=m(p-1)$.

Recently, Le and Le [16,17] considered problem (1) with $m(p-1)>1$ and obtained the existence and non-existence results of the global weak solutions. Precisely speaking, they concluded that if $m(p-1)>q-1$, then for any $|u_0|^{m-1}u_0 \in W_0^{1,p}(\Omega)$, problem (1) admits a global solution; if $m(p-1)\leqslant q-1$, then there exists a weak solution to problem (1) which is global provided that $u_0(x)$ belongs to some specific stable sets, and the weak solution blows up in a finite time provided that $u_0(x)$ belongs to some specific unstable sets.

According to our knowledge, there is no extinction result of the solution to problem (1) with $0<m(p-1)<1$. Inspired by the above works, we naturally take the following two questions into consideration. Does the solution to problem (1) with $0<m(p-1)<1$ exist globally under some certain conditions? If so, is there a critical extinction exponent of the global solutions? In fact, in this paper, we work with the following equivalent formulation of problem (1), obtained by changing variable $v=|u|^{m-1}u$,

$$\begin{cases} \left(|v|^{\frac{1-m}{m}}v\right)_t - \operatorname{div}\left(|\nabla v|^{p-2}\nabla v\right) = f(v), & (x,t) \in \Omega \times (0,+\infty) \\ v(x,t) = 0, & (x,t) \in \partial\Omega \times (0,+\infty) \\ v(x,0) = v_0 = |u_0|^{m-1}u_0, & x \in \Omega \end{cases} \tag{4}$$

with

$$f(v) = \begin{cases} \dfrac{1}{m}|v|^{\frac{q-1-m}{m}}v\log|v|, & \text{if } v \neq 0 \\ 0, & \text{if } v = 0 \end{cases} \tag{5}$$

It is clear that the first equation in problem (4) has degeneracy or singularity at the points where $v(x,t)=0$ or $|\nabla v(x,t)|=0$, and hence problem (4) might not have classical solution in general. We introduce the definition of the weak solution to problem (4) as follows.

Definition 1 Let $T>0$. A measurable function $v(x,t)$ defined in $\Omega\times[0,T)$ is called a weak solution to problem (4) if $v \in L^{\infty}(0,T;W_0^{1,p}(\Omega))$, $\left(|v|^{\frac{1-m}{m}}v\right)_t \in L^2(0,T;L^2(\Omega))$, $v(x,0)=|u_0|^{m-1}u_0 \in W_0^{1,p}(\Omega)$ and

$$\int_{\Omega}\left[\left(|v|^{\frac{1-m}{m}}v\right)_t\varphi + |\nabla v|^{p-2}\nabla v\cdot\nabla\varphi\right]\mathrm{d}x = \frac{1}{m}\int_{\Omega}|v|^{\frac{q-1-m}{m}}v\log|v|\varphi\,\mathrm{d}x \tag{6}$$

holds for a.e. $t\in(0,T)$ and any $\varphi \in W_0^{1,p}(\Omega)$.

Similar to the proof of Theorem 3.3 [17], by Faedo-Galerkin method, we can prove the local existence result of the weak solution to problem (4). Now, we state the main results of this paper as follows.

Theorem 1 Assume that $0<m(p-1)<1$. Then the weak solution $u(x,t)$ of problem (1) exists globally.

Theorem 2 Assume that $0<m(p-1)<q-1<1$. If

$$\max\left\{\left(\int_\Omega\left(|u_0|^{m-1}u_0\right)_+^{\frac{mpa+m+1}{m}}\mathrm{d}x\right)^\theta,\left(\int_\Omega\left(|u_0|^{m-1}u_0\right)_-^{\frac{mpa+m+1}{m}}\mathrm{d}x\right)^\theta\right\}\leqslant\frac{\kappa_6}{2\kappa_7} \tag{7}$$

with $\theta=\dfrac{q-1-m(p-1)+m\beta}{mpa+m+1}$, $\beta\in\left(0,\dfrac{2-q}{m}\right)$ and $a>\max\left\{-\dfrac{1}{p},-\dfrac{1}{mp},\dfrac{(N-p)(m+1)-Npm}{mp^2}\right\}$. Then the weak solution $u(x,t)$ to problem (1) will vanish in finite time, where κ_6 and κ_7 are two positive constants, given by (23) and (24), respectively.

Theorem 3 Assume that $0<q-1\leqslant m(p-1)<1$. If

$$\begin{cases}\int_\Omega|u_0|^{m+1}\mathrm{d}x>0\ \text{and}\ E\left(|u_0|^{m-1}u_0\right)\leqslant 0, & m(p-1)=q-1\\ \int_\Omega|u_0|^{m+1}\mathrm{d}x>0\ \text{and}\ E\left(|u_0|^{m-1}u_0\right)<-\dfrac{m|\Omega|\left[m(p-1)-(q-1)\right]}{ep(q+m-1)^2}, & m(p-1)>q-1\end{cases} \tag{8}$$

Then the weak solution to problem (1) cannot vanish in finite time, where

$$E\left(|u_0|^{m-1}u_0\right)=\frac{1}{p}\int_\Omega\left|\nabla\left(|u_0|^{m-1}u_0\right)\right|^p\mathrm{d}x-\frac{m}{q+m-1}\int_\Omega|u_0|^{q+m-1}\log|u_0|\mathrm{d}x+\frac{m}{(q+m-1)^2}\int_\Omega|u_0|^{q+m-1}\mathrm{d}x \tag{9}$$

Remark 1 From Theorems 2 and 3, we know that the critical extinction exponent of the global solutions to problem (1) is $q_{ec}=m(p-1)+1$.

1 Proofs of the Main Results

Proof of Theorem 1 Multiplying both sides of the first equality in (4) by $v(x,t)$ and integrating over Ω, one gets

$$\frac{1}{m+1}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega|v|^{\frac{m+1}{m}}\mathrm{d}x+\int_\Omega|\nabla v|^p\mathrm{d}x=\frac{1}{m}\int_\Omega|v|^{\frac{m+q-1}{m}}\log|v|\mathrm{d}x \tag{10}$$

Remembering that $q\in(1,2)$ and $m\geqslant 1$, we can select $\beta\in\left(0,\dfrac{2-q}{m}\right)$ such that $q+m\beta\in(1,2)$. For this chosen β, we know $\log|v|\leqslant\dfrac{1}{\mathrm{e}\beta}|v|^\beta$. Then, from (10), it holds that

$$\frac{1}{m+1}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega|v|^{\frac{m+1}{m}}\mathrm{d}x+\int_\Omega|\nabla v|^p\mathrm{d}x\leqslant\frac{1}{\mathrm{e}m\beta}\int_\Omega|v|^{\frac{m+q-1}{m}+\beta}\mathrm{d}x \tag{11}$$

By using Hölder's inequality, (11) leads to

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega|v|^{\frac{m+1}{m}}\mathrm{d}x\leqslant\frac{m+1}{\mathrm{e}m\beta}\int_\Omega|v|^{\frac{m+q-1}{m}+\beta}\mathrm{d}x\leqslant\kappa_1\left(\int_\Omega|v|^{\frac{m+1}{m}}\mathrm{d}x\right)^{\frac{m\beta+m+q-1}{m+1}} \tag{12}$$

where $\kappa_1=\dfrac{m+1}{\mathrm{e}m\beta}|\Omega|^{\frac{2-q-m\beta}{m+1}}$. By a simple calculation, we get

$$\int_\Omega|v|^{\frac{m+1}{m}}\mathrm{d}x\leqslant\left[\left(\int_\Omega|v_0|^{\frac{m+1}{m}}\mathrm{d}x\right)^{\frac{2-q-m\beta}{m+1}}+\frac{(2-q-m\beta)\kappa_1}{m+1}t\right]^{\frac{m+1}{2-q-m\beta}} \tag{13}$$

It follows from (13) and Hölder's inequality that

$$\int_\Omega|v|^{\frac{m\beta+m+q-1}{m}}\mathrm{d}x\leqslant|\Omega|^{\frac{2-q-m\beta}{m+1}}\left(\int_\Omega|v|^{\frac{m+1}{m}}\mathrm{d}x\right)^{\frac{m\beta+m+q-1}{m+1}}\leqslant|\Omega|^{\frac{2-q-m\beta}{m+1}}\left[\left(\int_\Omega|v_0|^{\frac{m+1}{m}}\mathrm{d}x\right)^{\frac{2-q-m\beta}{m+1}}+\frac{(2-q-m\beta)\kappa_1}{m+1}t\right]^{\frac{m\beta+m+q-1}{2-q-m\beta}} \tag{14}$$

On the other hand, multiplying both sides of the first equality in (4) by $v_t=\left(|u|^{m-1}u\right)_t$ and integrating over Ω, then with the help of Hölder's inequality and Cauchy's inequality with ε, we get

$$\frac{4m^2}{(m+1)^2}\int_\Omega\left[\left(|v|^{\frac{m+1}{2m}}\right)_t\right]^2\mathrm{d}x+\frac{m}{p}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega|\nabla v|^p\mathrm{d}x=\int_\Omega|v|^{\frac{m+q-1}{m}-2}v\log|v|v_t\mathrm{d}x$$

$$=\int_{\Omega_1=\{x\in\Omega:|v|\geqslant 1\}}|v|^{\frac{m+q-1}{m}-2}v\log|v|v_t\mathrm{d}x+\int_{\Omega_2=\{x\in\Omega:|v|<1\}}|v|^{\frac{m+q-1}{m}-2}v\log|v|v_t\mathrm{d}x$$

$$\leqslant\frac{2m}{\mathrm{e}\beta(m+1)}\int_{\Omega_1}|v|^{\frac{2m\beta+2q+m-3}{2m}}\left|\left(v^{\frac{m+1}{2m}}\right)_t\right|\mathrm{d}x+\frac{4m^2}{\mathrm{e}(m+1)(m+2q-3)}\int_{\Omega_2}\left|\left(v^{\frac{m+1}{2m}}\right)_t\right|\mathrm{d}x$$

$$\leqslant\frac{m}{2\varepsilon_1\mathrm{e}\beta(m+1)}\int_\Omega|v|^{\frac{2m\beta+2q+m-3}{2m}}\mathrm{d}x+\frac{2m\varepsilon_1}{\mathrm{e}\beta(m+1)}\int_\Omega\left|\left(v^{\frac{m+1}{2m}}\right)_t\right|^2\mathrm{d}x$$

$$+\frac{m^2|\Omega|}{\mathrm{e}\varepsilon_2(m+1)(m+2q-3)}+\frac{4m^2\varepsilon_2}{\mathrm{e}(m+1)(m+2q-3)}\int_\Omega\left|\left(v^{\frac{m+1}{2m}}\right)_t\right|^2\mathrm{d}x \tag{15}$$

If ε_1 and ε_2 are sufficiently small such that $\frac{\varepsilon_1}{\beta}+\frac{2m\varepsilon_2}{m+2q-3}\leqslant\frac{2m\mathrm{e}}{m+1}$, then from (15), it holds that

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega|\nabla v|^p\mathrm{d}x\leqslant\kappa_3+\kappa_2\left(\int_\Omega|v|^{\frac{m\beta+m+q-1}{m}}\mathrm{d}x\right)^{\frac{2m\beta+2q+m-3}{m\beta+m+q-1}} \tag{16}$$

where $\kappa_2=\frac{p}{2\mathrm{e}\beta\varepsilon_1(m+1)}|\Omega|^{\frac{2-q-m\beta}{m\beta+m+q-1}}$ and $\kappa_3=\frac{mp|\Omega|}{\mathrm{e}\varepsilon_2(m+1)(m+2q-3)}$. Combining (14) and (16) tells us that

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega|\nabla v|^p\mathrm{d}x\leqslant\kappa_3+\kappa_4\left[\left(\int_\Omega|v_0|^{\frac{m+1}{m}}\mathrm{d}x\right)^{\frac{2-q-m\beta}{m+1}}+\frac{(2-q-m\beta)\kappa_1}{m+1}t\right]^{\frac{2m\beta+2q+m-3}{2-q-m\beta}} \tag{17}$$

where $\kappa_4=\kappa_2|\Omega|^{\frac{(2-q-m\beta)(2m\beta+2q+m-3)}{(m+1)(m\beta+m+q-1)}}$. Integrating (17), we arrive at

$$\int_\Omega|\nabla v|^p\mathrm{d}x\leqslant\int_\Omega|\nabla v_0|^p\mathrm{d}x-\frac{(m+1)\kappa_4}{\kappa_1(m\beta+q+m-1)}\left(\int_\Omega|v_0|^{\frac{m+1}{m}}\mathrm{d}x\right)^{\frac{m\beta+q+m-1}{m+1}}+\kappa_3t$$

$$+\frac{(m+1)\kappa_4}{\kappa_1(m\beta+q+m-1)}\left(\left(\int_\Omega|v_0|^{\frac{m+1}{m}}\mathrm{d}x\right)^{\frac{2-q-m\beta}{m+1}}+\frac{(2-q-m\beta)\kappa_1}{m+1}t\right)^{\frac{m\beta+q+m-1}{2-q-m\beta}} \tag{18}$$

which implies that $\int_\Omega|\nabla v|^p\mathrm{d}x=\int_\Omega\left|\nabla\left(|u|^{m-1}u\right)\right|^p\mathrm{d}x$ is bounded for all $t\in[0,+\infty)$. The proof of Theorem 1 is complete.

Proof of Theorem 2 Multiplying both sides of the first equality in (4) by $\varphi=|v|^{pa}v_+=|u|^{mpa}\left(|u|^{m-1}u\right)_+$, we find that

$$\frac{1}{mpa+m+1}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega v_+^{\frac{mpa+m+1}{m}}\mathrm{d}x+\frac{pa+1}{(a+1)^p}\int_\Omega|\nabla v_+^{a+1}|^p\mathrm{d}x=\frac{1}{m}\int_\Omega|v|^{\frac{q-1-m}{m}}v|v|^{pa}v_+\log|v|\mathrm{d}x\leqslant\frac{1}{m\mathrm{e}\beta}\int_\Omega v_+^{\frac{m(pa+\beta+1)+q-1}{m}}\mathrm{d}x \tag{19}$$

By virtue of Hölder's inequality and Sobolev embedding inequality, we have

$$\int_\Omega v_+^{\frac{mpa+m+1}{m}}\mathrm{d}x\leqslant|\Omega|^{1-\frac{(N-p)(mpa+m+1)}{Npm(a+1)}}\left(\int_\Omega\left(v_+^{a+1}\right)^{\frac{mpa+m+1}{m(a+1)}\cdot\frac{Npm(a+1)}{(N-p)(mpa+m+1)}}\mathrm{d}x\right)^{\frac{(N-p)(mpa+m+1)}{Npm(a+1)}}$$

$$\leqslant|\Omega|^{1-\frac{(N-p)(mpa+m+1)}{Npm(a+1)}}\kappa_5^{\frac{(N-p)(mpa+m+1)}{Npm(a+1)}}\left(\int_\Omega|\nabla v_+^{a+1}|^p\mathrm{d}x\right)^{\frac{mpa+m+1}{pm(a+1)}} \tag{20}$$

which implies that

$$\int_\Omega|\nabla v_+^{a+1}|^p\mathrm{d}x\geqslant|\Omega|^{\frac{(N-p)(m+1)-mp(N+pa)}{N(mpa+m+1)}}\kappa_5^{\frac{p-N}{N}}\left(\int_\Omega v_+^{\frac{mpa+m+1}{m}}\mathrm{d}x\right)^{\frac{pm(a+1)}{mpa+m+1}} \tag{21}$$

where $\kappa_5=\kappa_5(p,N)$ is the optimal Sobolev embedding constant. Substituting (21) into (19) and using Hölder's inequality, we get

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega v_+^{\frac{mpa+m+1}{m}}\mathrm{d}x+\kappa_6\left(\int_\Omega v_+^{\frac{mpa+m+1}{m}}\mathrm{d}x\right)^{\frac{pm(a+1)}{mpa+m+1}}\leqslant\kappa_7\left(\int_\Omega v_+^{\frac{mpa+m+1}{m}}\mathrm{d}x\right)^{\frac{m(pa+\beta+1)+q-1}{mpa+m+1}} \tag{22}$$

where

$$\kappa_6=\frac{(mpa+m+1)(pa+1)}{(a+1)^p}\kappa_5^{\frac{p-N}{N}}|\Omega|^{\frac{(N-p)(m+1)-pm(N+pa)}{N(mpa+m+1)}} \tag{23}$$

and

$$\kappa_7=\frac{mpa+m+1}{me\beta}|\Omega|^{1-\frac{m(pa+\beta+1)+q-1}{mpa+m+1}} \tag{24}$$

Recalling that $0<m(p-1)<q-1$ and $q+m\beta<2$, we check that

$$0<\frac{pm(a+1)}{mpa+m+1}<\frac{m(pa+\beta+1)+q-1}{mpa+m+1}<1 \tag{25}$$

On the other hand, our assumption (7) tells us that

$$0<\kappa_7<\frac{1}{2}\kappa_6\left(\int_\Omega v_{0+}^{\frac{mpa+m+1}{m}}\mathrm{d}x\right)^{\frac{m(p-1)-(q-1)-m\beta}{mpa+m+1}} \tag{26}$$

Combining (22), (25), (26) and Lemma 1 [18], one can claim that there exist two positive constants ξ and η such that

$$0\leqslant y(t):=\int_\Omega v_+^{\frac{mpa+m+1}{m}}\mathrm{d}x\leqslant\xi\mathrm{e}^{-\eta t},\quad t\geqslant 0 \tag{27}$$

Choosing

$$T_0>\max\left\{0,\frac{1}{\eta}\ln\left[\xi\left(\frac{2\kappa_7}{k_6}\right)^{\frac{mpa+m+1}{q-1-m(p-1)+m\beta}}\right]\right\} \tag{28}$$

Then it follows from (22) and (27) that

$$\frac{\mathrm{d}}{\mathrm{d}t}y(t)+\frac{\kappa_6}{2}\big(y(t)\big)^{\frac{pm(a+1)}{mpa+m+1}}\leqslant 0,\quad t\geqslant T_0 \tag{29}$$

Integrating above inequality, one has

$$0\leqslant y^{1-\frac{pm(a+1)}{mpa+m+1}}(t)\leqslant y^{1-\frac{pm(a+1)}{mpa+m+1}}(T_0)-\frac{\kappa_6(m+1-pm)}{2(mpa+m+1)}(t-T_0) \tag{30}$$

which suggests that there exists a

$$T_0{}'\in\left[T_0,T_0+\frac{2(mpa+m+1)}{\kappa_6(m+1-pm)}y^{\frac{m+1-pm}{mpa+m+1}}(T_0)\right] \tag{31}$$

such that

$$\lim_{t\to T_0{}'^-}y(t)=\lim_{t\to T_0{}'^-}\int_\Omega v_+^{\frac{mpa+m+1}{m}}(t)\mathrm{d}x=0 \tag{32}$$

Moreover, it can be concluded that

$$\int_\Omega v_+(t)\mathrm{d}x\leqslant|\Omega|^{\frac{mpa+1}{mpa+m+1}}\left(\int_\Omega v_+^{\frac{mpa+m+1}{m}}\mathrm{d}x\right)^{\frac{m}{mpa+m+1}}\to 0\text{ as }t\to T_0{}'^- \tag{33}$$

On the other hand, by a similar way, it can be shown that $\int_\Omega v_-(t)\mathrm{d}x$ will vanish in finite time. Thus $\int_\Omega|v(t)|\mathrm{d}x=\int_\Omega v_+(t)\mathrm{d}x+\int_\Omega v_-(t)\mathrm{d}x$ vanishes in finite time. Recalling that $v=|u|^{m-1}u$, one can claim that the solution $u(x,t)$ to problem (1) possesses the extinction property. The proof of Theorem 2 is completed.

Proof of Theorem 3 Denoting

$$E\left(v(t)\right)=\frac{1}{p}\int_{\Omega}|\nabla v|^{p}\mathrm{d}x-\frac{1}{q+m-1}\int_{\Omega}|v|^{\frac{q+m-1}{m}}\log|v|\mathrm{d}x+\frac{m}{\left(q+m-1\right)^{2}}\int_{\Omega}|v|^{\frac{q+m-1}{m}}\mathrm{d}x \tag{34}$$

with $v=|u|^{m-1}u$, then a direct calculation shows that

$$\frac{\mathrm{d}E\left(v(t)\right)}{\mathrm{d}t}=-\frac{4m}{\left(m+1\right)^{2}}\int_{\Omega}\left[\left(v^{\frac{m+1}{2m}}\right)_{t}\right]^{2}\mathrm{d}x \tag{35}$$

which implies

$$E\left(v(t)\right)=E\left(v_{0}\right)-\frac{4m}{\left(m+1\right)^{2}}\int_{0}^{t}\int_{\Omega}\left[\left(v^{\frac{m+1}{2m}}\right)_{t}\right]^{2}\mathrm{d}x\mathrm{d}\tau \tag{36}$$

Set

$$M(t)=\frac{1}{m+1}\int_{\Omega}|v|^{\frac{m+1}{m}}\mathrm{d}x \tag{37}$$

A direct calculation tells us that

$$\begin{aligned}M'(t)=&-\int_{\Omega}|\nabla v|^{p}\mathrm{d}x+\frac{1}{m}\int_{\Omega}|v|^{\frac{q+m-1}{m}}\log|v|\mathrm{d}x=-pE\left(v(t)\right)+\frac{q-1-m\left(p-1\right)}{m\left(q+m-1\right)}\int_{\Omega}|v|^{\frac{q+m-1}{m}}\log|v|\mathrm{d}x\\&+\frac{mp}{\left(q+m-1\right)^{2}}\int_{\Omega}|v|^{\frac{q+m-1}{m}}\mathrm{d}x\geqslant-pE\left(v(t)\right)+\frac{q-1-m\left(p-1\right)}{m\left(q+m-1\right)}\int_{\Omega}|v|^{\frac{q+m-1}{m}}\log|v|\mathrm{d}x\end{aligned} \tag{38}$$

By virtue of (36), (38) and Hölder's inequality, one derives

$$\begin{aligned}M'(t)\geqslant&-pE\left(v_{0}\right)-\frac{m\left(p-1\right)-\left(q-1\right)}{q+m-1}\int_{\Omega_{1}=\{x\in\Omega:|v|\geqslant1\}\cup\Omega_{2}=\{x\in\Omega:|v|<1\}}|v|^{\frac{q+m-1}{m}}\left|\log|v|\right|\mathrm{d}x\\\geqslant&-pE\left(v_{0}\right)-\frac{m\left(p-1\right)-\left(q-1\right)}{q+m-1}\left(\frac{1}{e\beta}\int_{\Omega_{1}}|v|^{\frac{m\beta+m+q-1}{m}}\mathrm{d}x+\frac{m|\Omega|}{e\left(m+q-1\right)}\right)\\\geqslant&-pE\left(v_{0}\right)-\frac{m\left(p-1\right)-\left(q-1\right)}{e\beta\left(q+m-1\right)}|\Omega|^{\frac{2-q-m\beta}{m+1}}\left(\int_{\Omega}|v|^{\frac{m+1}{m}}\mathrm{d}x\right)^{\frac{m\beta+m+q-1}{m+1}}-\frac{\left[m\left(p-1\right)-\left(q-1\right)\right]m|\Omega|}{\mathrm{e}\left(q+m-1\right)^{2}}\end{aligned} \tag{39}$$

If $m\left(p-1\right)=q-1$. Then from (39), one can see that, for any $t\geqslant0$,

$$M(t)\geqslant M(0)-pE\left(v_{0}\right)t \tag{40}$$

Keeping in mind that

$$M(0)=\frac{1}{m+1}\int_{\Omega}|v_{0}|^{\frac{m+1}{m}}\mathrm{d}x=\frac{1}{m+1}\int_{\Omega}|u_{0}|^{m+1}\mathrm{d}x>0 \tag{41}$$

and

$$E\left(v_{0}\right)=E\left(|u_{0}|^{m-1}u_{0}\right)\leqslant0 \tag{42}$$

then (40) gives us that $M(t)>0$, which means that the solution $u(x,t)$ to problem (1) cannot vanish in finite time.

If $m\left(p-1\right)\geqslant q-1$. Remembering that

$$M(0)=\frac{1}{m+1}\int_{\Omega}|v_{0}|^{\frac{m+1}{m}}\mathrm{d}x=\frac{1}{m+1}\int_{\Omega}|u_{0}|^{m+1}\mathrm{d}x>0 \tag{43}$$

and

$$E\left(v_{0}\right)=E\left(|u_{0}|^{m-1}u_{0}\right)<-\frac{m|\Omega|\left[m\left(p-1\right)-\left(q-1\right)\right]}{\mathrm{e}p\left(q+m-1\right)^{2}} \tag{44}$$

combining (39) and Lemma 1.2 [19] gives us that, for any $t\geqslant0$,

$$M(t)\geqslant\min\left\{M(0),\left(\frac{\kappa_{8}}{\kappa_{9}}\right)^{\frac{m+1}{m\beta+q+m-1}}\right\}>0 \tag{45}$$

which means that the solution $u(x,t)$ to problem (1) cannot vanish in finite time, where

$$\kappa_8 = -\left(pE\left(v_0\right) + \frac{\left[m\left(p-1\right) - \left(q-1\right)\right]m|\Omega|}{\mathrm{e}\left(q+m-1\right)^2} \right) > 0 \tag{46}$$

$$\kappa_9 = \frac{m\left(p-1\right) - \left(q-1\right)}{\mathrm{e}\beta\left(q+m-1\right)} |\Omega|^{\frac{2-q-m\beta}{m+1}} > 0 \tag{47}$$

The proof of Theorem 3 is completed.

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