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A Study on Time Scale Non-Shifted Hamiltonian Dynamics and Noether's Theorems

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Abstract: The time-scale non-shifted Hamiltonian dynamics are investigated, including both general holonomic systems and nonholonomic systems. The time-scale non-shifted Hamilton principle is presented and extended to nonconservative system, and the dynamic equations in Hamiltonian framework are deduced. The Noether symmetry, including its definition and criteria, for time-scale non-shifted Hamiltonian dynamics is put forward, and Noether's theorems for both holonomic and nonholonomic systems are presented and proved. The non-shifted Noether conservation laws are given. The validity of the theorems is verified by two examples.
Key words: non-shifted Hamiltonian dynamics; Noether symmetry; non-shifted Noether conservation laws; time scales
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0 Introduction

Time-scale analysis originated from the work of Hilger [1], which unifies and generalizes the differential equation and difference equation theory within the framework of time-scale calculus, thus avoiding the repeated solution of the two different kinds of equations and revealing the essential differences between continuous and discrete systems [2-4]. Time scale theory has been widely applied and achieved many achievements in various fields [4-10]. In last two decades, some new advances have emerged on the study of time-scale dynamics and its symmetries, such as kinetic equations [11-13], optimal control problems [14,15], fractional variational problems [16-18], Noether theorems [18-22], Lie symmetries [23-25], Mei symmetries [26,27], canonical transformation and Hamilton-Jacobi method [28,29], time-delay dynamics [30], Herglotz variational problems [31], higher-order delta derivatives [32], etc. However, Ref. [33] indicated that, possibly affected by Bohner's original paper [11], most studies on time-scale variational problems were only shifted variational problems. In fact, the numerical calculation scheme for variational problems based on non-shifted action functionals on time scales has good properties [33,34]. Recently, three symmetries and conservation laws for non-shifted time-scales dynamic systems were studied in Refs. [25, 27, 35-37]. The study of non-shifted variational problems on time scales is a new but important research direction of analytical mechanics.

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Conservation law plays an irreplaceable role in the solution and reduction of differential equations and stability analysis, so it has always been a research hotspot in the field of analytical mechanics [38-44]. Using symmetry to find conservation laws is an effective method. However, the exploration of symmetry and conservation laws theory on time scales is still very preliminary. Moreover, the studies are mainly limited to Lagrangian systems or Birkhoffian systems so far, and non-conservative forces or non-holonomic constraints are not considered. Since the phase space of mechanical system has natural symplectic structure, it is easier to describe mathematically than Lagrangian mechanics. For some cases, the symmetry that is difficult to find in configuration space can be found in phase space, and the corresponding conserved quantity can be derived by using Noether's theorem in the canonical form [45]. Here we will explore the non-shifted Hamiltonian dynamics, Noether symmetry and non-shifted conservation laws on time scales under Hamiltonian framework.

1 Non-Shifted Time-Scale Hamiltonian Dynamics

For time scale calculus and its basic operations, please refer to Refs. [2, 3, 12].

The time-scale non-shifted Hamilton action reads

$$S(\gamma)=\int_{t_1}^{t_2}\left[p_s(t)q_s^{\Delta}(t)-H\left(t,q_s(t),p_s(t)\right)\right]\Delta t \tag{1}$$

where γ is a certain curve, $H{:}\mathbb{T}^k\times\mathbb{R}^n\times\mathbb{R}^n\to\mathbb{R}$ is the Hamiltonian on time scales, q_s and p_s are generalized coordinate and generalized momentum, q_s^{Δ} is the delta derivative of q_s with respect to t, where $s=1,2,\cdots,n$. All functions belong to $C_{\mathrm{rd}}^{1,\Delta}(\mathbb{T})$.

1.1 Hamilton System

The isochronous variational principle

$$\delta S=0 \tag{2}$$

with endpoint conditions

$$\delta q_s\big|_{t=t_1}=\delta q_s\big|_{t=t_2}=0,\quad \delta p_s\big|_{t=t_1}=\delta p_s\big|_{t=t_2}=0 \tag{3}$$

and commutative relations

$$\delta q_s^{\Delta}=\left(\delta q_s\right)^{\Delta},\quad \delta p_s^{\Delta}=\left(\delta p_s\right)^{\Delta} \tag{4}$$

is called the time-scale non-shifted Hamilton principle.

By carrying out the variational operation of Eq. (2) and using the relations (4), we can get

$$-\left(\delta q_s\int_{t_1}^{t}\frac{\partial H}{\partial q_s}\Delta\tau\right)\Bigg|_{t_1}^{t_2}-\left(\delta p_s\int_{t_1}^{t}\left(\frac{\partial H}{\partial p_s}-q_s^{\Delta}\right)\Delta\tau\right)\Bigg|_{t_1}^{t_2}+\int_{t_1}^{t_2}\left\{\delta q_s^{\Delta}\left(p_s+\int_{t_1}^{\sigma(t)}\frac{\partial H}{\partial q_s}\Delta\tau\right)+\delta p_s^{\Delta}\int_{t_1}^{\sigma(t)}\left(\frac{\partial H}{\partial p_s}-q_s^{\Delta}\right)\Delta\tau\right\}\Delta t=0 \tag{5}$$

Substituting the endpoint conditions (3) into Eq. (5), we get

$$\int_{t_1}^{t_2}\left\{\delta q_s^{\Delta}\left(p_s+\int_{t_1}^{\sigma(t)}\frac{\partial H}{\partial q_s}\Delta\tau\right)+\delta p_s^{\Delta}\int_{t_1}^{\sigma(t)}\left(\frac{\partial H}{\partial p_s}-q_s^{\Delta}\right)\Delta\tau\right\}\Delta t=0 \tag{6}$$

From the independence of δq_s^{Δ}, $\delta p_s^{\Delta}(s=1,2,\cdots,n)$, according to Dubois-Reymond lemma [11], we get

$$p_s+\int_{t_1}^{\sigma(t)}\frac{\partial H}{\partial q_s}\Delta\tau=C_s,\int_{t_1}^{\sigma(t)}\left(\frac{\partial H}{\partial p_s}-q_s^{\Delta}\right)\Delta\tau=D_s,\ s=1,2,\cdots,n \tag{7}$$

where C_s and D_s are some constants. Taking the nabla derivative of (7), we have

$$p_s^{\nabla}+\sigma^{\nabla}(t)\frac{\partial H}{\partial q_s}=0,\ \ \sigma^{\nabla}(t)\left(\frac{\partial H}{\partial p_s}-q_s^{\Delta}\right)=0 \tag{8}$$

So there are

$$q_s^{\Delta}=\frac{\partial H}{\partial p_s},\ \ p_s^{\nabla}=-\sigma^{\nabla}(t)\frac{\partial H}{\partial q_s} \tag{9}$$

Eqs.(9) are the Hamilton canonical equations for the non-shifted Hamilton system on time scales.

If take $\mathbb{T}=\mathbb{R}$, then $\sigma(t)=t$, Eqs.(9) become

$$\dot{q}_s = \frac{\partial H}{\partial p_s},\ \dot{p}_s = -\frac{\partial H}{\partial q_s} \tag{10}$$

Eqs.(10) are the classical Hamilton canonical equations.

1.2 General Holonomic Mechanical System in Phase Space

For a general holonomic mechanical system, we extend principle (2) as follows

$$\int_{t_1}^{t_2} \left\{ \delta \left[p_s(t) q_s^\Delta(t) - H\left(t, q_s(t), p_s(t)\right) \right] + Q_s'' \delta q_s \right\} \Delta t = 0 \tag{11}$$

where $Q_s'' = Q_s''\left(t, q_k(t), p_k(t)\right)$ are non-potential generalized forces.

Similar to the derivation of Eq. (6), from principle (11), we get

$$\int_{t_1}^{t_2} \left\{ \delta q_s^\Delta \left(p_s + \int_{t_1}^{\sigma(t)} \left(\frac{\partial H}{\partial q_s} - Q_s'' \right) \Delta \tau \right) + \delta p_s^\Delta \int_{t_1}^{\sigma(t)} \left(\frac{\partial H}{\partial p_s} - q_s^\Delta \right) \Delta \tau \right\} \Delta t = 0 \tag{12}$$

From the independence of $\delta q_s^\Delta,\ \delta p_s^\Delta (s = 1, 2, \cdots, n)$, according to Dubois-Reymond lemma [11], we get

$$p_s + \int_{t_1}^{\sigma(t)} \left(\frac{\partial H}{\partial q_s} - Q_s'' \right) \Delta \tau = C_s', \int_{t_1}^{\sigma(t)} \left(\frac{\partial H}{\partial p_s} - q_s^\Delta \right) \Delta \tau = D_s' \tag{13}$$

where C_s' and D_s' are some constants. Hence, we have

$$q_s^\Delta = \frac{\partial H}{\partial p_s},\ p_s^\nabla = -\sigma^\nabla(t) \frac{\partial H}{\partial q_s} + \sigma^\nabla(t) Q_s'' \tag{14}$$

Eqs. (14) are the time-scale dynamic equations of the general holonomic system. When $Q_s'' = 0$, Eqs. (14) are reduced to Eqs. (9), which are the non-shifted Hamilton canonical equations.

1.3 Nonholonomic Mechanical System in Phase Space

Consider the system is subject to g bilateral ideal nonholonomic constraints

$$f_\beta\left(t, q_s, q_s^\Delta\right) = 0 \tag{15}$$

and the virtual displacements δq_s need to meet the conditions

$$F_{\beta s}\left(t, q_j, q_j^\Delta\right) \delta q_s = 0 \tag{16}$$

where $\beta = 1, 2, \cdots, g$. In general, $F_{\beta s}$ is independent of $\frac{\partial f_\beta}{\partial q_s^\Delta}$, and the constraints are of non-Chetaev. If $F_{\beta s} = \frac{\partial f_\beta}{\partial q_s^\Delta}$, then the constraints are of Chetaev.

If the non-shifted Lagrangian is $L = L\left(t, q_s, q_s^\Delta\right)$, then

$$p_s = \frac{\partial L}{\partial q_s^\Delta} \tag{17}$$

According to Eq. (17), $q_s^\Delta = q_s^\Delta\left(t, q_j, p_j\right)$ can be solved, and then substituted into Eqs. (15) and (16), thus the constraints (15) and restriction conditions (16) can be written as

$$\tilde{f}_\beta\left(t, q_s, p_s\right) = 0 \tag{18}$$

$$\tilde{F}_{\beta s}\left(t, q_j, p_j\right) \delta q_s = 0 \tag{19}$$

By introducing the constraint multiplier λ_β multiplied by each of Eqs. (19) and summing over β, and integrating the equation on the interval $\left[t_1, t_2\right]$, and by integration by parts, we get

$$\left(\delta q_s \int_{t_1}^{t} \lambda_\beta \tilde{F}_{\beta s} \Delta \tau \right) \Bigg|_{t_1}^{t_2} - \int_{t_1}^{t_2} \delta q_s^\Delta \left(\int_{t_1}^{\sigma(t)} \lambda_\beta \tilde{F}_{\beta s} \Delta \tau \right) \Delta t = 0 \tag{20}$$

Taking into account conditions (3), we get

$$-\int_{t_1}^{t_2} \delta q_s^\Delta \left(\int_{t_1}^{\sigma(t)} \lambda_\beta \tilde{F}_{\beta s} \Delta \tau \right) \Delta t = 0 \tag{21}$$

Add Eq. (21) to Eq. (12), and we get

$$\int_{t_1}^{t_2}\left\{\delta q_s^{\Delta}\left(p_s+\int_{t_1}^{\sigma(t)}\left(\frac{\partial H}{\partial q_s}-Q_s''-\lambda_\beta\tilde{F}_{\beta s}\right)\Delta\tau\right)+\delta p_s^{\Delta}\int_{t_1}^{\sigma(t)}\left(\frac{\partial H}{\partial p_s}-q_s^{\Delta}\right)\Delta\tau\right\}\Delta t=0 \tag{22}$$

According to the Lagrange multiplier method, without loss of generality, choose the multiplier λ_β such that $C_\beta''=0\left(\beta=1,2,\cdots,g\right)$, and using Dubois-Reymond lemma [11], from Eq. (22), we get

$$p_s+\int_{t_1}^{\sigma(t)}\left(\frac{\partial H}{\partial q_s}-Q_s''-\lambda_\beta\tilde{F}_{\beta s}\right)\Delta\tau=C_s'',\quad \int_{t_1}^{\sigma(t)}\left(\frac{\partial H}{\partial p_s}-q_s^{\Delta}\right)\Delta\tau=D_s'' \tag{23}$$

where $C_{g+1}'',\cdots,C_n'',D_s''$ are some constants. So there are

$$q_s^{\Delta}=\frac{\partial H}{\partial p_s},\quad p_s^{\nabla}=-\sigma^{\nabla}(t)\frac{\partial H}{\partial q_s}+\sigma^{\nabla}(t)Q_s''+\sigma^{\nabla}(t)\lambda_\beta\tilde{F}_{\beta s} \tag{24}$$

Assuming that the system is non-singular, by using Eqs. (24) and (18), λ_β can be solved as the function of q_s, p_s and t. Therefore, Eqs.(24) can be expressed as

$$q_s^{\Delta}=\frac{\partial H}{\partial p_s},\quad p_s^{\nabla}=-\sigma^{\nabla}(t)\frac{\partial H}{\partial q_s}+\sigma^{\nabla}(t)\left(Q_s''+\Lambda_s\right) \tag{25}$$

where $\Lambda_s=\lambda_\beta\tilde{F}_{\beta s}$ are the constraint forces corresponding to the nonholonomic constraints (18). Eqs. (25) can be regarded as a holonomic system corresponding to the nonholonomic system determined by Eqs.(18) and (24). If the initial values of q_s and p_s satisfy Eq. (18), namely

$$\tilde{f}_\beta\left(t_0,q_{s0},p_{s0}\right)=0 \tag{26}$$

then the solution of (25) is the desired solution of time-scale nonholonomic systems (18) and (24).

2 Noether Symmetry for Time-Scale Hamiltonian Dynamics

2.1 Noether Symmetry for Hamilton System

The infinitesimal transformations are

$$\bar{t}=t+\varepsilon\xi_0\left(t,q_j,p_j\right),\ \bar{q}_s(\bar{t})=q_s(t)+\varepsilon\xi_s\left(t,q_j,p_j\right),\ \bar{p}_s(\bar{t})=p_s(t)+\varepsilon\eta_s\left(t,q_j,p_j\right) \tag{27}$$

where ξ_0, ξ_s and η_s are the generating functions, ε is the infinitesimal parameter, and $s,j=1,2,\cdots,n$. Let the map $t\mapsto\alpha(t)=t+\varepsilon\xi_0+o(\varepsilon)$ be a strictly increasing $C_{\mathrm{rd}}^{1,\Delta}(\mathbb{T})$ function, whose image is denoted as $\bar{\mathbb{T}}$, delta derivative is $\bar{\Delta}$, forward jump operator $\bar{\sigma}$, and $\bar{\sigma}\circ\alpha=\alpha\circ\sigma$.

Under the transformation (27), the Hamilton action (1) reads

$$S\left(\bar{\gamma}\right)=\int_{\alpha(t_1)}^{\alpha(t_2)}\left[\bar{p}_s(\bar{t})\bar{q}_s^{\bar{\Delta}}(\bar{t})-H\left(\bar{t},\bar{q}_s(\bar{t}),\bar{p}_s(\bar{t})\right)\right]\bar{\Delta}\bar{t}=\int_{t_1}^{t_2}\left[\left(p_s+\varepsilon\eta_s\right)\frac{q_s^{\Delta}+\varepsilon\xi_s^{\Delta}}{1+\varepsilon\xi_0^{\Delta}}-H\left(t+\varepsilon\xi_0,q_s+\varepsilon\xi_s,p_s+\varepsilon\eta_s\right)\right]\left(1+\varepsilon\xi_0^{\Delta}\right)\Delta t$$

$$=\int_{t_1}^{t_2}\left\{p_sq_s^{\Delta}-H\left(t,q_s,p_s\right)+\varepsilon\left(p_s\xi_s^{\Delta}+q_s^{\Delta}\eta_s\right)-\varepsilon\left(\frac{\partial H}{\partial t}\xi_0+\frac{\partial H}{\partial q_s}\xi_s+\frac{\partial H}{\partial p_s}\eta_s+H\xi_0^{\Delta}\right)+o(\varepsilon)\right\}\Delta t \tag{28}$$

Therefore, the nonisochronous variation Δ^*S, namely the main-line part of difference $S\left(\bar{\gamma}\right)-S\left(\gamma\right)$ relative to ε, is

$$\Delta^*S=\int_{t_1}^{t_2}\varepsilon\left[p_s\xi_s^{\Delta}+q_s^{\Delta}\eta_s-\left(\frac{\partial H}{\partial t}\xi_0+\frac{\partial H}{\partial q_s}\xi_s+\frac{\partial H}{\partial p_s}\eta_s+H\xi_0^{\Delta}\right)\right]\Delta t \tag{29}$$

By straightforward calculation, formula (29) can also be expressed as

$$\Delta^*S=\int_{t_1}^{t_2}\varepsilon\left\{\frac{\nabla}{\nabla t}\left(p_s\xi_s^{\sigma}-H\xi_0^{\sigma}\right)+\left(H^{\nabla}-\sigma^{\nabla}(t)\frac{\partial H}{\partial t}\right)\xi_0+\sigma^{\nabla}(t)\left(q_s^{\Delta}-\frac{\partial H}{\partial p_s}\right)\eta_s-\left(p_s^{\nabla}+\sigma^{\nabla}(t)\frac{\partial H}{\partial q_s}\right)\xi_s\right\}\frac{\Delta t}{\sigma^{\nabla}(t)} \tag{30}$$

Formulas (29) and (30) are two mutually equivalent nonisochronous variational formulas of non-shifted Hamilton action on time scales.

Definition 1 For the time-scale non-shifted Hamilton system (9), the transformation (27) is said to be Noether symmetric, if and only if $\Delta^*S=0$.

By using Eqs. (29) and (30), we obtain:

Criterion 1 For the time-scale non-shifted Hamilton system (9), if the Noether identity

$$p_s\xi_s^\Delta+q_s^\Delta\eta_s-\frac{\partial H}{\partial t}\xi_0-\frac{\partial H}{\partial q_s}\xi_s-\frac{\partial H}{\partial p_s}\eta_s-H\xi_0^\Delta=0 \tag{31}$$

holds, then the transformation (27) is Noether symmetric.

Criterion 2 If the generating functions ξ_0, ξ_s and η_s solve the equation

$$\frac{\nabla}{\nabla t}\left(p_s\xi_s^\sigma-H\xi_0^\sigma\right)+\left(H^\nabla-\sigma^\nabla(t)\frac{\partial H}{\partial t}\right)\xi_0+\sigma^\nabla(t)\left(q_s^\Delta-\frac{\partial H}{\partial p_s}\right)\eta_s-\left(p_s^\nabla+\sigma^\nabla(t)\frac{\partial H}{\partial q_s}\right)\xi_s=0 \tag{32}$$

then the transformation (27) is Noether symmetric for the time-scale Hamilton system (9).

Assume that H_1 is another Hamiltonian on time scales, if, considering the first-order approximation, the transformation (27) satisfies the following relation

$$\int_{t_1}^{t_2}\left[p_s(t)q_s^\Delta(t)-H\left(t,q_s(t),p_s(t)\right)\right]\Delta t=\int_{\alpha(t_1)}^{\alpha(t_2)}\left[\bar{p}_s(\bar{t})\bar{q}_s^{\bar{\Delta}}(\bar{t})-H_1\left(\bar{t},\bar{q}_s(\bar{t}),\bar{p}_s(\bar{t})\right)\right]\bar{\Delta}\bar{t} \tag{33}$$

then the action (1) is called quasi-invariant. Both H_1 and H satisfy the same time-scale Hamilton equations. So there is

$$H_1\left(t,q_s(t),p_s(t)\right)=H\left(t,q_s(t),p_s(t)\right)-\varepsilon G^\Delta \tag{34}$$

where $G=G\left(t,q_s,p_s\right)$ is the gauge function.

Definition 2 For the time-scale non-shifted Hamilton system (9), the transformation (27) is said to be Noether quasi-symmetric, if and only if

$$\Delta^* S=-\int_{t_1}^{t_2}\varepsilon G^\Delta\,\Delta t \tag{35}$$

By using Eqs.(29) and (30), we obtain:

Criterion 3 For the time-scale non-shifted Hamilton system (9), if the generalized Noether identity

$$p_s\xi_s^\Delta+q_s^\Delta\eta_s-\frac{\partial H}{\partial t}\xi_0-\frac{\partial H}{\partial q_s}\xi_s-\frac{\partial H}{\partial p_s}\eta_s-H\xi_0^\Delta+G^\Delta=0 \tag{36}$$

holds, then the transformation (27) is Noether quasi-symmetric.

Criterion 4 If the generating functions ξ_0, ξ_s and η_s solve the equation

$$\frac{\nabla}{\nabla t}\left(p_s\xi_s^\sigma-H\xi_0^\sigma+G^\sigma\right)+\left(H^\nabla-\sigma^\nabla(t)\frac{\partial H}{\partial t}\right)\xi_0+\sigma^\nabla(t)\left(q_s^\Delta-\frac{\partial H}{\partial p_s}\right)\eta_s-\left(p_s^\nabla+\sigma^\nabla(t)\frac{\partial H}{\partial q_s}\right)\xi_s=0 \tag{37}$$

then the transformation (27) is Noether quasi-symmetric for the time-scale Hamilton system (9).

2.2 Noether Symmetry for General Holonomic Mechanical System in Phase Space

For the general holonomic system, if the following relation

$$\int_{t_1}^{t_2}\left[p_s(t)q_s^\Delta(t)-H\left(t,q_s(t),p_s(t)\right)\right]\Delta t=\int_{\alpha(t_1)}^{\alpha(t_2)}\left[\bar{p}_s(\bar{t})\bar{q}_s^{\bar{\Delta}}(\bar{t})-H_1\left(\bar{t},\bar{q}_s(\bar{t}),\bar{p}_s(\bar{t})\right)\right]\bar{\Delta}\bar{t}+\int_{t_1}^{t_2}Q_s''\delta q_s\Delta t \tag{38}$$

is satisfied, then the action (1) is called generalized quasi-invariant.

Definition 3 For the time-scale non-shifted general holonomic system (14), the transformation (27) is said to be generalized Noether quasi-symmetric, if and only if

$$\Delta^* S+\int_{t_1}^{t_2}Q_s''\delta q_s\Delta t=-\int_{t_1}^{t_2}\varepsilon G^\Delta\,\Delta t \tag{39}$$

Criterion 5 For the time-scale non-shifted general holonomic system (14), if the generalized Noether identity

$$p_s\xi_s^\Delta+q_s^\Delta\eta_s-\frac{\partial H}{\partial t}\xi_0-\frac{\partial H}{\partial q_s}\xi_s-\frac{\partial H}{\partial p_s}\eta_s-H\xi_0^\Delta+Q_s''\left(\xi_s-q_s^\Delta\xi_0\right)+G^\Delta=0 \tag{40}$$

holds, then the transformation (27) is generalized Noether quasi-symmetric.

Criterion 6 If the generating functions ξ_0, ξ_s and η_s solve the equation

$$\frac{\nabla}{\nabla t}\left(p_s\xi_s^\sigma-H\xi_0^\sigma+G^\sigma\right)+\left(H^\nabla-\sigma^\nabla(t)\frac{\partial H}{\partial t}-\sigma^\nabla(t)Q_s''q_s^\Delta\right)\xi_0+\sigma^\nabla(t)\left(q_s^\Delta-\frac{\partial H}{\partial p_s}\right)\eta_s-\left(p_s^\nabla+\sigma^\nabla(t)\frac{\partial H}{\partial q_s}-\sigma^\nabla(t)Q_s''\right)\xi_s=0 \tag{41}$$

then the transformation (27) is generalized Noether quasi-symmetric for the time-scale general holonomic system (14).

2.3 Noether Symmetry for Nonholonomic Mechanical System in Phase Space

For the nonholonomic system, we have

Definition 4 For the corresponding holonomic system (25), the transformation (27) is said to be generalized Noether quasi-symmetric, if and only if

$$\Delta^* S+\int_{t_1}^{t_2}\left(Q_s''+\Lambda_s\right)\delta q_s\Delta t=-\int_{t_1}^{t_2}\varepsilon G^{\Delta}\Delta t \tag{42}$$

Criterion 7 For the corresponding holonomic system (25), if the generalized Noether identity

$$p_s\xi_s^{\Delta}+q_s^{\Delta}\eta_s-\frac{\partial H}{\partial t}\xi_0-\frac{\partial H}{\partial q_s}\xi_s-\frac{\partial H}{\partial p_s}\eta_s-H\xi_0^{\Delta}+\left(Q_s''+\Lambda_s\right)\left(\xi_s-q_s^{\Delta}\xi_0\right)+G^{\Delta}=0 \tag{43}$$

holds, then the transformation (27) is generalized Noether quasi-symmetric.

Criterion 8 If the generating functions ξ_0, ξ_s and η_s solve the equation

$$\frac{\nabla}{\nabla t}\left(p_s\xi_s^{\sigma}-H\xi_0^{\sigma}+G^{\sigma}\right)+\left(H^{\nabla}-\sigma^{\nabla}(t)\frac{\partial H}{\partial t}-\sigma^{\nabla}(t)\left(Q_s''+\Lambda_s\right)q_s^{\Delta}\right)\xi_0+\sigma^{\nabla}(t)\left(q_s^{\Delta}-\frac{\partial H}{\partial p_s}\right)\eta_s-\left(p_s^{\nabla}+\sigma^{\nabla}(t)\frac{\partial H}{\partial q_s}-\sigma^{\nabla}(t)\left(Q_s''+\Lambda_s\right)\right)\xi_s=0 \tag{44}$$

then the transformation (27) is generalized Noether quasi-symmetric for the corresponding holonomic system (25).

The restriction conditions of Eqs. (19) on the generating functions are

$$\tilde{F}_{\beta s}\left(\xi_s-q_s^{\Delta}\xi_0\right)=0 \tag{45}$$

Definition 5 For the time-scale non-shifted nonholonomic system determined by (18) and (24), if and only if the formula (42) and restriction conditions (45) hold, then the transformation (27) is said to be generalized Noether quasi-symmetric.

Criterion 9 For the time-scale non-shifted nonholonomic system determined by (18) and (24), if the generalized Noether identity (43) and the restriction conditions (45) hold, then the transformation (27) is generalized Noether quasi-symmetric.

Criterion 10 If the generating functions ξ_0, ξ_s and η_s solve the equation (44) and the restriction conditions (45), then the transformation (27) is generalized Noether quasi-symmetric for the time-scale nonholonomic system determined by (18) and (24).

3 Noether Theorems for Time-Scales Hamiltonian Dynamics

Noether symmetry is closely related to conservation laws. Here we establish and prove Noether's theorems for time-scale non-shifted holonomic and nonholonomic Hamiltonian dynamics.

3.1 Noether Theorems for Hamilton System

Theorem 1 For the time-scale non-shifted Hamilton system (9), if the transformation (27) is Noether symmetric, then

$$I_{\mathrm{N}}=p_s\xi_s^{\sigma}-H\xi_0^{\sigma}+\int_{t_1}^{t}\left\{\left[H^{\nabla}-\sigma^{\nabla}(t)\frac{\partial H}{\partial t}\right]\xi_0\right\}\nabla t \tag{46}$$

is a non-shifted Noether conserved quantity.

Proof Due to

$$\begin{aligned}\frac{\nabla}{\nabla t}I_{\mathrm{N}}&=p_s^{\nabla}\xi_s+\sigma^{\nabla}(t)p_s\xi_s^{\Delta}-H^{\nabla}\xi_0-\sigma^{\nabla}(t)H\xi_0^{\Delta}+\left(H^{\nabla}-\sigma^{\nabla}(t)\frac{\partial H}{\partial t}\right)\xi_0\\&=\sigma^{\nabla}(t)\left(p_s\xi_s^{\Delta}+q_s^{\Delta}\eta_s-\frac{\partial H}{\partial t}\xi_0-\frac{\partial H}{\partial q_s}\xi_s-\frac{\partial H}{\partial p_s}\eta_s-H\xi_0^{\Delta}\right)+\sigma^{\nabla}(t)\left(-q_s^{\Delta}+\frac{\partial H}{\partial p_s}\right)\eta_s+\left(p_s^{\nabla}+\sigma^{\nabla}(t)\frac{\partial H}{\partial q_s}\right)\xi_s\end{aligned} \tag{47}$$

Substituting the non-shifted Hamilton equations (9) and the Noether identity (31) into (47), we get

$$\frac{\nabla}{\nabla t}I_{\mathrm{N}}=0 \tag{48}$$

Therefore, formula (46) is a non-shifted Noether conserved quantity.

Theorem 2 For the time-scale non-shifted Hamilton system (9), if the transformation (27) is Noether quasi-symmetric, then

$$I_{\mathrm{N}}=p_s\xi_s^{\sigma}-H\xi_0^{\sigma}+G^{\sigma}+\int_{t_1}^{t}\left\{\left[H^{\nabla}-\sigma^{\nabla}(t)\frac{\partial H}{\partial t}\right]\xi_0\right\}\nabla t \tag{49}$$

is a non-shifted Noether conserved quantity.

Proof Taking the nabla derivative of (49), and using Eqs.(9) and (36), we get the result immediately.

Theorem 1 and 2 are Noether's theorems for time-scale non-shifted Hamilton system.

3.2 Noether Theorems for General Holonomic Mechanical System in Phase Space

For the general holonomic mechanical system, we have

Theorem 3 For the time-scale non-shifted general holonomic system (14), if the transformation (27) is generalized Noether quasi-symmetric, then

$$I_{\mathrm{N}}=p_s\xi_s^\sigma-H\xi_0^\sigma+G^\sigma+\int_{t_1}^{t}\left\{\left[H^\nabla-\sigma^\nabla(t)\left(\frac{\partial H}{\partial t}+Q_s''q_s^\Delta\right)\right]\xi_0\right\}\nabla t \tag{50}$$

is a non-shifted Noether conserved quantity.

Proof Due to

$$\begin{aligned}\frac{\nabla}{\nabla t}I_{\mathrm{N}}&=p_s^\nabla\xi_s+\sigma^\nabla(t)p_s\xi_s^\Delta-H^\nabla\xi_0-\sigma^\nabla(t)H\xi_0^\Delta+\left(G^\sigma\right)^\nabla+\left(H^\nabla-\sigma^\nabla(t)\frac{\partial H}{\partial t}-\sigma^\nabla(t)Q_s''q_s^\Delta\right)\xi_0\\&=\sigma^\nabla(t)\left(p_s\xi_s^\Delta+q_s^\Delta\eta_s-\frac{\partial H}{\partial t}\xi_0-\frac{\partial H}{\partial q_s}\xi_s-\frac{\partial H}{\partial p_s}\eta_s-H\xi_0^\Delta+Q_s''\left(\xi_s-q_s^\Delta\xi_0\right)+G^\Delta\right)\\&\quad+\sigma^\nabla(t)\left(-q_s^\Delta+\frac{\partial H}{\partial p_s}\right)\eta_s+\left(p_s^\nabla+\sigma^\nabla(t)\frac{\partial H}{\partial q_s}-\sigma^\nabla(t)Q_s''\right)\xi_s=0\end{aligned} \tag{51}$$

This completes the proof.

Theorem 3 is Noether's theorem for time-scale non-shifted general holonomic system under Hamiltonian framework.

3.3 Noether Theorems for Nonholonomic Mechanical System in Phase Space

For the nonholonomic mechanical system, we have

Theorem 4 For the corresponding holonomic system (25), if the transformation (27) is generalized Noether quasi-symmetric, then

$$I_{\mathrm{N}}=p_s\xi_s^\sigma-H\xi_0^\sigma+G^\sigma+\int_{t_1}^{t}\left\{\left[H^\nabla-\sigma^\nabla(t)\left(\frac{\partial H}{\partial t}+Q_s''q_s^\Delta+\Lambda_s q_s^\Delta\right)\right]\xi_0\right\}\nabla t \tag{52}$$

is a non-shifted Noether conserved quantity.

Proof Taking the nabla derivative of (52), and using Eqs.(25) and (43), we get the results.

Theorem 5 For the time-scale non-shifted nonholonomic system determined by (18) and (24), if the transformation (27) is generalized Noether quasi-symmetric, then formula (52) is a non-shifted Noether conserved quantity.

Theorem 5 and Theorem 4 are Noether's theorems for time-scale non-shifted nonholonomic system and its corresponding holonomic system under Hamiltonian framework.

4 Examples

Example 1 Consider a non-shifted holonomic non-conservative system on time scale $\mathbb{T}=\left\{2^m: m\in\mathbb{N}_0\right\}$, and let the Lagrangian function be

$$L=\frac{1}{2}\left[\left(q_1^\Delta\right)^2+\left(q_2^\Delta\right)^2\right]-\frac{1}{2}q_2^2 \tag{53}$$

The generalized forces are

$$Q_1''=-q_2^\Delta,\quad Q_2''=0 \tag{54}$$

The generalized momenta and the Hamiltonian are

$$p_1=\frac{\partial L}{\partial q_1^\Delta}=q_1^\Delta,p_2=\frac{\partial L}{\partial q_2^\Delta}=q_2^\Delta,H=\frac{1}{2}\left(p_1^2+p_2^2\right)+\frac{1}{2}q_2^2 \tag{55}$$

According to equation (14), the Hamilton equations are

$$q_1^\Delta=p_1,q_2^\Delta=p_2,p_1^\nabla=-\sigma^\nabla(t)p_2,p_2^\nabla=-\sigma^\nabla(t)q_2 \tag{56}$$

If we take $\mathbb{T}=\mathbb{R}$, then Eqs.(56) are reduced to

$$\dot{q}_1=p_1,\dot{q}_2=p_2,\dot{p}_1+p_2=0,\dot{p}_2+q_2=0 \tag{57}$$

This is the classic Hojman-Urrutia problem [46].

The generalized Noether identity (40) reads

$$p_1\xi_1^\Delta+p_2\xi_2^\Delta+\eta_1q_1^\Delta+\eta_2q_2^\Delta-q_2\xi_2-p_1\eta_1-p_2\eta_2-\frac{1}{2}\left(p_1^2+p_2^2\right)\xi_0^\Delta-\frac{1}{2}q_2^2\xi_0^\Delta-p_2\left(\xi_1-q_1^\Delta\xi_0\right)+G^\Delta=0 \tag{58}$$

Since $\sigma(t)=2t$, Eq.(58) has a solution

$$\xi_0^1=0,\xi_1^1=1,\xi_2^1=0,\eta_1^1=1,\eta_2^1=0,G^1=q_2 \tag{59}$$

$$\xi_0^2=0,\xi_1^2=2t,\xi_2^2=1,\eta_1^2=0,\eta_2^2=0,G^2=-2q_1+q_2t \tag{60}$$

By Theorem 3, we obtain

$$I_{\mathrm{N}}^1=p_1+q_2^\sigma=\text{const.} \tag{61}$$

$$I_{\mathrm{N}}^2=4p_1t+p_2+2q_2^\sigma t-2q_1^\sigma=\text{const.} \tag{62}$$

The conserved quantity (61) and (62) correspond to the quasi-symmetry (59) and (60). Assume the initial conditions $q_1(0)=1$, $q_2(0)=1$, $p_1(0)=1$, $p_2(0)=1$. Let $m\in[0,6]$. The numerical simulation results of conserved quantities (61) and (62) on $t\in[2,64]$ are shown in Fig. 1.

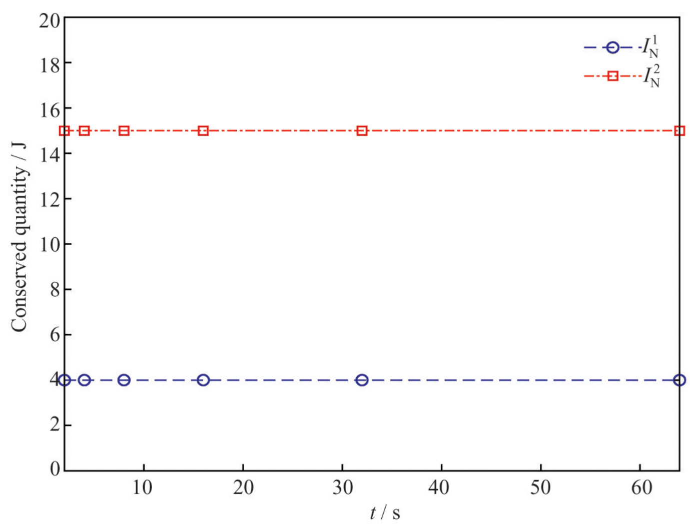

Fig. 1 The values of conserved quantities (61) and (62) on $t\in[2,64]$

From Fig. 1, it can be seen intuitively that conserved quantities obtained from formulae (61) and (62) are both constants, which shows the correctness of Theorem 3.

Example 2 Let us study Appell-Hamel problem [47] on time scales. The non-shifted Lagrangian and nonholonomic constraint are respectively

$$L=\frac{1}{2}m\left[\left(q_1^\Delta\right)^2+\left(q_2^\Delta\right)^2+\left(q_3^\Delta\right)^2\right]-mgq_3 \tag{63}$$

$$f=q_3^\Delta-\left[\left(q_1^\Delta\right)^2+\left(q_2^\Delta\right)^2\right]^{1/2}=0 \tag{64}$$

The constraint (64) is of Chetaev type. The generalized momenta and the Hamiltonian are

$$p_1=\frac{\partial L}{\partial q_1^\Delta}=mq_1^\Delta,p_2=\frac{\partial L}{\partial q_2^\Delta}=mq_2^\Delta,p_3=\frac{\partial L}{\partial q_3^\Delta}=mq_3^\Delta \tag{65}$$

$$H=\frac{1}{2m}\left(p_1^2+p_2^2+p_3^2\right)+mgq_3 \tag{66}$$

In canonical coordinates, the constraint (64) can be shown as

$$\tilde{f}=p_3-\left(p_1^2+p_2^2\right)^{1/2}=0 \tag{67}$$

The time-scale dynamical equations can be expressed as

$$q_1^{\Delta}=\frac{1}{m}p_1, q_2^{\Delta}=\frac{1}{m}p_2, q_3^{\Delta}=\frac{1}{m}p_3, p_1^{\nabla}=-\sigma^{\nabla}(t)\lambda p_1\left(p_1^2+p_2^2\right)^{-1/2},\ p_2^{\nabla}=-\sigma^{\nabla}(t)\lambda p_2\left(p_1^2+p_2^2\right)^{-1/2}, p_3^{\nabla}=\sigma^{\nabla}(t)\left(-mg+\lambda\right) \quad (68)$$

Taking the nabla derivative of (67), we get

$$p_3^{\nabla}-\frac{1}{2}\left(p_1^2+p_2^2\right)^{-1/2}\left(p_1+p_1^{\rho}\right)p_1^{\nabla}-\frac{1}{2}\left(\left(p_1^{\rho}\right)^2+p_2^2\right)^{-1/2}\left(p_2+p_2^{\rho}\right)p_2^{\nabla}=0 \quad (69)$$

Substituting (68) into (69), we can get

$$\lambda=\frac{2mg\left(p_1^2+p_2^2\right)^{1/2}}{2\left(p_1^2+p_2^2\right)^{1/2}+p_1\left(p_1+p_1^{\rho}\right)\left(p_1^2+p_2^2\right)^{-1/2}+p_2\left(p_2+p_2^{\rho}\right)\left(\left(p_1^{\rho}\right)^2+p_2^2\right)^{-1/2}} \quad (70)$$

Therefore, the nonholonomic constraint forces are

$$\Lambda_1=-\lambda p_1\left(p_1^2+p_2^2\right)^{-1/2}, \Lambda_2=-\lambda p_2\left(p_1^2+p_2^2\right)^{-1/2}, \Lambda_3=\lambda \quad (71)$$

From (43), the generalized Noether identity for the system is

$$p_1\xi_1^{\Delta}+p_2\xi_2^{\Delta}+p_3\xi_3^{\Delta}+\eta_1q_1^{\Delta}+\eta_2q_2^{\Delta}+\eta_3q_3^{\Delta}-mg\xi_3-\frac{1}{m}p_1\eta_1-\frac{1}{m}p_2\eta_2-\frac{1}{m}p_3\eta_3$$
$$-\frac{1}{2m}\left(p_1^2+p_2^2+p_3^2\right)\xi_0^{\Delta}-mgq_3\xi_0^{\Delta}+\Lambda_1\left(\xi_1-q_1^{\Delta}\xi_0\right)+\Lambda_2\left(\xi_2-q_2^{\Delta}\xi_0\right)+\Lambda_3\left(\xi_3-q_3^{\Delta}\xi_0\right)+G^{\Delta}=0 \quad (72)$$

The restriction condition (45) reads

$$-p_1\left(p_1^2+p_2^2\right)^{-1/2}\left(\xi_1-q_1^{\Delta}\xi_0\right)-p_2\left(p_1^2+p_2^2\right)^{-1/2}\left(\xi_2-q_2^{\Delta}\xi_0\right)+\xi_3-q_3^{\Delta}\xi_0=0 \quad (73)$$

The equations (72) and (73) have a solution

$$\xi_0^1=-1, \xi_1^1=0, \xi_2^1=0, \xi_3^1=0, \eta_1^1=0, \eta_2^1=0, \eta_3^1=0, G^1=0 \quad (74)$$

From Theorem 5, we get the Noether conserved quantity

$$I_{\mathrm{N}}=\frac{1}{2m}\left(p_1^2+p_2^2+p_3^2\right)+mgq_3+\int_{t_1}^{t}\left\{\frac{\lambda\sigma^{\nabla}}{2m}\left[\left(p_1p_1^{\rho}+p_2p_2^{\rho}\right)\left(p_1^2+p_2^2\right)^{-1/2}-p_3^{\rho}\right]+\frac{1}{2}\sigma^{\nabla}g\left(p_3+p_3^{\rho}\right)-mgq_3^{\Delta}\right\}\nabla t=\text{const.} \quad (75)$$

If $\mathbb{T}=\mathbb{R}$, then $\sigma(t)=t$, and formula (75) becomes

$$I_{\mathrm{N}}=\frac{1}{2m}\left(p_1^2+p_2^2+p_3^2\right)+mgq_3=\text{const.} \quad (76)$$

This is the classical conservation law of energy, which has been given in Ref.[47].

If $\mathbb{T}=\mathbb{Z}$, then $\sigma(t)=t+1, \rho(t)=t-1, \mu(t)=1$, and formula (75) becomes

$$I_{\mathrm{N}}=\frac{1}{2m}\left[p_1^2(t)+p_2^2(t)+p_3^2(t)\right]+mgq_3(t)+\sum_{\tau=t_1+1}^{t}\left\{\frac{\lambda(\tau)}{2m}\left[\left(p_1(\tau)p_1(\tau-1)+p_2(\tau)p_2(\tau-1)\right)\left(p_1^2(\tau)+p_2^2(\tau)\right)^{-1/2}-p_3(\tau-1)\right]\right.$$
$$\left.+\frac{1}{2}g\left[p_3(\tau)+p_3(\tau-1)\right]-mg\left[q_3(\tau+1)-q_3(\tau)\right]\right\} \quad (77)$$

This is the Noether conserved quantity of the discrete version with unit step size.

5 Conclusion

The time-scale calculus provides an excellent mathematical tool for exploring the dynamics of continuous and discrete systems or their mixtures, and has attracted extensive attentions. In this paper, we proposed the time-scale non-shifted Hamilton principle and extended it to non-conservative systems, and derived the dynamic equations for non-shifted Hamilton systems, non-shifted general holonomic systems and non-shifted nonholonomic systems. We defined Noether symmetries and gave their criteria. We proved Noether's theorems for non-shifted Hamilton systems, non-shifted general holonomic systems and non-shifted nonholonomic systems, and obtained the non-shifted Noether conserved quantities. The ideas presented here can be applied to solving the symmetries of complex dynamics under time-scale framework, such as nonlinear problems.

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