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Optimal Investment of Defined Contribution Pension Based on Self-Protection and Minimum Security

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Abstract: This paper mainly studies the optimal investment problem of defined contribution (DC) pension under the self-protection and minimum security. First, we apply Itô theorem to obtain the differential equation of the real stock price after discounting inflation. Then, under the constraint of external guarantee of DC pension terminal wealth, self-protection is introduced to study the maximization of the expected utility of terminal wealth at retirement time and any time before retirement. The explicit solution of the optimal investment strategy of DC pension at retirement time and any time before retirement should be derived by martingale method. Finally, the influence of self-protection on the optimal investment strategy of DC pension is numerically analyzed.

Key words: defined contribution pension; minimum guarantee; self-protection; martingale method; optimal investment
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0 Introduction

Endowment insurance is one of the five major social insurance types and occupies an important position in the social security system. With the increasing problem of population aging, how to manage pensions has become a hot topic. Pension plans can be divided into DB (defined benefit) and DC (defined contribution) plans according to different financing, operation modes, contributions and payment methods. DC pension participants regularly deposit a certain amount of money through the establishment of personal accounts, fund managers accumulate benefits for pension participants through investment behaviors in the financial market, and the risks are borne by the participants themselves. Since the investment income of pensions greatly affects the living standards of DC pension participants after retirement, research on the optimal investment portfolio of DC pensions has become a hot topic in academia. Vigna and Haberman[1] first studied the investment portfolio of pensions under the multi-period discrete time frame, considering the investment risk and annuitisation risk faced by individuals, and used dynamic programming method to find the optimal investment strategy of DC pension. Boulier *et al*[2] used the continuous time frame model to study the optimal investment problem of DC pension under constant interest rate with the CRRA(Constant Relative Risk Aversion) utility as the utility function. Gu *et al*[3] aimed to maximize the CRRA utility of the terminal wealth of DC pension participants, assuming that inves-

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tors invest in risk-free assets and a risky asset, and studied the optimal investment strategy of DC pension plans under the Ornstein-Uhlenbeck model. Guan *et al*[4] also assumed that the interest rate obeys the Ornstein-Uhlenbeck process, and the financial market consists of cash, bonds and stocks. Different from Gu *et al*[3], Guan *et al*[4] studied the optimal allocation of DC pensions under loss aversion and VaR(Value at Risk) constraints

Since employees have no additional income after retirement, pensions have become the main source of income for retired employees. The level of pensions greatly affects the retirement living standards of employees. Therefore, it is necessary to establish a minimum security mechanism to ensure the retirement life of employees. Boulier *et al*[2] studied the optimal asset allocation of DC pensions using the Vasicek interest rate model under the condition that the minimum guarantee is a random variable. Deelstra *et al*[5] studied the optimal investment of DC pension with minimum guarantee under the assumption that the interest rate is an affine interest rate, under the condition that the contribution rate is random and random discount factor is introduced. Romaniuk[6] first studied the rare minimum return guarantee mechanism under the asset allocation of standard investment funds, and used stochastic dynamic programming technology and standard option theory to solve the optimal asset portfolio properties under internal guarantee and external guarantee, but did not give an explicit solution. On top of this, Liu *et al*[7] analyzed the minimum income guarantee system of DC pension, and gave an analytical solution to the optimal investment strategy by using the martingale method, and finally put forward suggestions for the establishment of the country's minimum income guarantee system. Wang *et al*[8] considered random inflation factors on the basis of Ref. [6] and introduced European call options to obtain the optimal investment strategy of DC pension under the minimum guarantee.

In addition, in investment management, investment risk is a problem that cannot be ignored. The above studies only aim at maximizing returns, but do not consider the risks borne by investors. When investors make investments, they should consider the cost (also known as an effort) of reducing the risks faced by pension funds, namely self-protection or prevention. The first approach to the problem of self-protection was presented by Ehrlich and Becker[9] in their seminal paper, in which the work cost of self-preservation is assumed to be paid during the period when the decision maker is at risk of loss and in two natural states (when the loss occurs and when it does not). Jindapon and Neilson[10] studied the first variation of the above framework, replacing the acquisition of wealth in the two natural states in the Ehrlich and Becker[9] model with random variables. Liu *et al*[11] studied a different cycle model, which considers that decision makers, when faced with risk, only pay the cost of effort in self-protection if no loss occurs ("conditional payout"). Finally, Crainich and Menegatti[12] studied the propensity and impact of introducing random costs (as opposed to deterministic costs) on the implementation of self-protection actions. The analysis by Crainich and Menegatti[12] is carried out under the framework of four standard self-preservation, but they do not give an analytical solution.

This paper will introduce a self-protection framework in Ref. [12] on the basis of Ref. [8], and assume that the cost of self-protection is certain, and investors only pay the cost of self-protection when no loss occurs. The discounted stock equation is obtained by discounting inflation, the objective function considering self-protection and minimum security is established, and the explicit solution of the optimal investment strategy at the retirement time of DC pension and at any time before retirement is obtained.

The structure of the paper is arranged as follows: Section 1 builds the model and derives the expression of the DC pension wealth process; Section 2 establishes the minimum guarantee and self-protection, and uses the martingale method to obtain the optimal wealth process and investment strategy at the retirement time of DC pension and any time before retirement; Section 3 gives a numerical analysis of the impact of self-protection on optimal Investment strategy; Section 4 is the conclusion.

1 Financial Market

Let $\left(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\in[0,T]}, P\right)$ be a complete probability space with domain flow, where $\{\mathcal{F}_t\}_{t\in[0,T]}$ is the domain flow containing the information obtained before time t. The pension plan starts at time 0 and ends at time T. Assuming that the market is a continuous, frictionless and arbitrage-free complete market, it is assumed that all stochastic processes in this paper fit into this probability space. Suppose a pension manager can invest money in three assets: risk-free assets (cash), bonds, and risky assets (stocks). Suppose the price process of cash $S_0(t)$ at

time t satisfies the following differential equation

$$\mathrm{d}S_0(t)=S_0(t)r(t)\mathrm{d}t, S_0(0)=S_0 \tag{1}$$

Here the random interest rate $r(t)$ satisfies the Vasicek model,

$$\mathrm{d}r(t)=a(b-r(t))\mathrm{d}t-\sigma_r\mathrm{d}W_r(t),\ \ r(0)=r_0 \tag{2}$$

where $a>0, b>0, \sigma_r>0$ are constants, and $W_r(t)$ is the standard Brownian motion.

The price of a zero-coupon bond $B(t, T)$ with maturity at time T satisfies the following differential equation

$$\mathrm{d}B(t,T)=r(t)B(t,T)\mathrm{d}t+\sigma_r A(t,T)B(t,T)\left(\lambda_r\mathrm{d}t+\mathrm{d}W_r(t)\right),\ B(T,T)=1 \tag{3}$$

where $A(t, T)=\dfrac{1-\mathrm{e}^{-a(T-t)}}{a}$, λ_r represents the market risk price of $W_r(t)$, and $\sigma_r A(t, T)$ represents the volatility of $B(t, T)$.

Because it is unrealistic to find all zero-coupon bonds in the real market, on the basis of $B(t, T)$, we introduce a rolling bond $B_K(t)$, and the bond has a constant term K, and the rolling bond satisfies the following stochastic differential equation

$$\mathrm{d}B_K(t)=r(t)B_K(t)\mathrm{d}t+\sigma_r A(t,t+K)B_K(t)\left(\lambda_r\mathrm{d}t+\mathrm{d}W_r(t)\right) \tag{4}$$

And zero-coupon bonds and rolling bonds satisfy the following relationship

$$\frac{\mathrm{d}B(t,s)}{B(t,s)}=\left(1-\frac{A(t,s)}{A(t,s+K)}\right)\frac{\mathrm{d}S_0(t)}{S_0(t)}+\frac{A(t,s)}{A(t,s+K)}\frac{\mathrm{d}B_K(t)}{B_K(t)},\ \forall s>t \tag{5}$$

where $S_0(t)$ represents the price process of cash at time t and variable s represents the maturity date of the bond.

Suppose the price process of the stock satisfies the following stochastic differential equation

$$\mathrm{d}S'(t)=r(t)S'(t)\mathrm{d}t+\sigma_1 S'(t)\left(\lambda_r\mathrm{d}t+\mathrm{d}W_r(t)\right)+\sigma_2 S'(t)\left(\lambda_S\mathrm{d}t+\mathrm{d}W_S(t)\right),\ \ S'(0)=S_1' \tag{6}$$

where $S'(t)$ is the price of the stock, $W_S(t)$ is the one-dimensional standard Brownian motion defined in the probability space, $W_S(t)$ and $W_r(t)$ are independent of each other, and λ_S is the market risk price of $W_S(t)$.

Define the inflation rate $P(t)$ as a measurable process on a complete probability space $\left(\Omega,\mathcal{F},\{\mathcal{F}_t\}_{t\in[0,T]},P\right)$ that satisfies the following stochastic differential equation, Ω is a non empty set, $\mathcal{F}$ is a field.

$$\mathrm{d}P(t)=\mu_\pi P(t)\mathrm{d}t+\boldsymbol{\sigma}_{P_1}P(t)\mathrm{d}W_r(t)+\boldsymbol{\sigma}_{P_2}P(t)\mathrm{d}W_S(t),\ P(0)=1 \tag{7}$$

where μ_π is the expected spot inflation rate, $\boldsymbol{\sigma}_P$ is the inflation rate matrix, and they are both measurable and uniformly bounded processes on $[0, T]\times\Omega$.

As the pension investment period is longer, the impact of inflation becomes more and more obvious, and the inflation rate is considered here. We use the inflation rate to convert the stock price and obtain a discounted real stock price of $S(t)=\dfrac{S'(t)}{P(t)}$. Using the Itô formula, $S(t)$ gets to satisfy the following differential equation.

$$\mathrm{d}S(t)=\mu(t)S(t)\mathrm{d}t+\left(\sigma_1-\boldsymbol{\sigma}_{P_1}\right)S(t)\mathrm{d}W_r(t)+\left(\sigma_2-\boldsymbol{\sigma}_{P_2}\right)S(t)\mathrm{d}W_S(t) \tag{8}$$

where $\mu(t)=r(t)+\sigma_1\lambda_r+\sigma_2\lambda_S-\mu_\pi-\sigma_1\boldsymbol{\sigma}_{P_1}-\sigma_2\boldsymbol{\sigma}_{P_2}+\boldsymbol{\sigma}_{P_1}^2+\boldsymbol{\sigma}_{P_2}^2$.

Since employees pay pension insurance every year, the funds in the pension account will continue to increase with the contributions of employees. Therefore, the contribution of employees is an important factor that must be considered by pension managers when making optimal investments. The financial market often affects the wages of employees indirectly or directly, so it is assumed that the contribution rate is a random variable and satisfies the following stochastic differential equation

$$\mathrm{d}C(t)=\mu_C C(t)\mathrm{d}t+\sigma_{C_1}C(t)\mathrm{d}W_r(t)+\sigma_{C_2}C(t)\mathrm{d}W_S(t), C(0)=C_0 \tag{9}$$

where $\sigma_{C_1}=\sigma_{C_2}=0$, μ_C, σ_{C_1},σ_{C_2} and C_0 are all positive constants. At that time, the contribution rate is fixed.

Assuming that the initial wealth of the pension account is X_0, the pension managers invest their funds in cash, bonds, and stocks, respectively, and the funds invested in the three are represented by $u_0(t)$,$u_{\mathrm{B}}(t)$ and $u_{\mathrm{S}}(t)$, respectively, then the wealth of the pension $X'(t)$ satisfies

$$\mathrm{d}X'(t)=u_0(t)\frac{\mathrm{d}S_0(t)}{S_0(t)}+u_{\mathrm{B}}(t)\frac{\mathrm{d}B_K(t)}{B_K(t)}+u_{\mathrm{S}}(t)\frac{\mathrm{d}S(t)}{S(t)}+C(t)\mathrm{d}t \tag{10}$$

Bringing Eqs. (1), (4) and (6) into the above differential equation, from $X'(t)=u_0(t)+u_{\mathrm{B}}(t)+u_{\mathrm{S}}(t)$, the explicit ex-

pression of wealth can be deduced as

$$\mathrm{d}X'(t)=r(t)X'(t)\mathrm{d}t+u_{\mathrm{B}}(t)\sigma_r A(t,t+K)\left(\lambda_r\mathrm{d}t+\mathrm{d}W_r(t)\right)+u_{\mathrm{S}}(t)\sigma_1\lambda_r+\sigma_2\lambda_S-\mu_\pi-\sigma_1\boldsymbol{\sigma}_{P_1}-\sigma_2\boldsymbol{\sigma}_{P_2}+\boldsymbol{\sigma}_{P_1}^2$$
$$+\boldsymbol{\sigma}_{P_2}^2\mathrm{d}t+u_{\mathrm{S}}(t)\left(\sigma_1-\boldsymbol{\sigma}_{P_1}\right)\mathrm{d}W_r(t)+u_{\mathrm{S}}(t)\left(\sigma_2-\boldsymbol{\sigma}_{P_2}\right)\mathrm{d}W_S(t)+C(t)\mathrm{d}t,\ X(0)=X_0\geqslant 0 \tag{11}$$

In Ref. [12], when faced with a possible risk, sometimes people make efforts to reduce the loss. This behavior is called defense or self-protection. Pension wealth minus the cost of self-protection efforts $X(t)$ to satisfy

$$X(t)=X'(t)-ce_t \tag{12}$$

where e_t is the self-protection effort, $X'(t)$ and e_t are independent of each other, c is the unit effort cost.

Suppose e_t satisfies the following stochastic differential equation $\mathrm{d}e_t=\zeta e_t\mathrm{d}t$, where ζ is a constant. Then the explicit expression of pension wealth $X(t)$ after deducting self-protection costs is

$$\mathrm{d}X(t)=\mathrm{d}X'(t)-\zeta ce_t\mathrm{d}t$$
$$=r(t)\left(X(t)+ce_t\right)\mathrm{d}t+u_{\mathrm{B}}(t)\sigma_r A(t,t+K)\left(\lambda_r\mathrm{d}t+\mathrm{d}W_r(t)\right)+u_{\mathrm{S}}(t)\left(\sigma_1\lambda_r+\sigma_2\lambda_S-\mu_\pi-\sigma_1\boldsymbol{\sigma}_{P_1}-\sigma_2\boldsymbol{\sigma}_{P_2}+\boldsymbol{\sigma}_{P_1}^2+\boldsymbol{\sigma}_{P_2}^2\right)\mathrm{d}t$$
$$+u_{\mathrm{S}}(t)\left(\sigma_1-\boldsymbol{\sigma}_{P_1}\right)\mathrm{d}W_r(t)+u_{\mathrm{S}}(t)\left(\sigma_2-\boldsymbol{\sigma}_{P_2}\right)\mathrm{d}W_S(t)+C(t)\mathrm{d}t-\zeta ce_t\mathrm{d}t,\qquad X(0)=X_0\geqslant 0$$

2 Optimal Investment Strategy

2.1 DC Pensions under Minimum Guarantees

In order to guarantee the minimum benefit of pension plan participants, the government pension insurance profit-sharing plan is considered here. When the income of the pension plan is lower than the minimum guarantee level, the government provides the minimum guarantee for the employees participating in the pension plan. When the income is greater than the minimum guarantee level, the government enjoys $(1-\alpha)$ of the excess of the expected income.

Let $G(T)$ be the minimum guarantee level, for the convenience of calculation, let $G(T)$ be a certain constant G at the end time, then the present value of the minimum guarantee income of $G(T)$ at time $t\,(t\leqslant T)$ is

$$G(t)=G\mathrm{e}^{-r(t)(T-t)} \tag{14}$$

Let

$$W(T)=G+\alpha\max\left(X(T)-G,0\right) \tag{15}$$

Then $W(T)$ represents the terminal wealth value of the DC pension after considering the minimum guarantee. Remember

$$M(T)=\max\left(X(T)-G,0\right) \tag{16}$$

It can be seen from Eq. (16) that setting an external guarantee for terminal wealth is equivalent to granting a European call option $M(t)$ to the pension manager. $M(t)$ represents the option price based on pension asset X, the strike price at time t is G, and the expiration date is T. Using Itô formula for $M(t)$, we get

$$\mathrm{d}M(t)=M_t\mathrm{d}t+M_X\mathrm{d}X+\frac{1}{2}M_{XX}(\mathrm{d}X)^2$$
$$=M_t\mathrm{d}t+\left[r(t)\left(X(t)+ce_t\right)+C(t)-\zeta ce_t\right]M_X\mathrm{d}t+u_{\mathrm{B}}(t)\sigma_r A(t,t+K)\lambda_r M_X\mathrm{d}t$$
$$+u_{\mathrm{S}}(t)\left(\sigma_1\lambda_r+\sigma_2\lambda_S-\mu_\pi-\sigma_1\boldsymbol{\sigma}_{P_1}-\sigma_2\boldsymbol{\sigma}_{P_2}+\boldsymbol{\sigma}_{P_1}^2+\boldsymbol{\sigma}_{P_2}^2\right)M_X\mathrm{d}t+u_{\mathrm{B}}(t)\sigma_r A(t,t+K)M_X\mathrm{d}W_r(t)+u_{\mathrm{S}}(t)\left(\sigma_1-\boldsymbol{\sigma}_{P_1}\right)M_X\mathrm{d}W_r(t)$$
$$+\frac{1}{2}\left[\left(u_{\mathrm{B}}(t)\sigma_r A(t,t+K)+u_{\mathrm{S}}(t)\left(\sigma_1-\boldsymbol{\sigma}_{P_1}\right)\right)^2+u_{\mathrm{S}}{}^2(t)\left(\sigma_2-\boldsymbol{\sigma}_{P_2}\right)^2\right]M_{XX}\mathrm{d}t+u_{\mathrm{S}}(t)\left(\sigma_2-\boldsymbol{\sigma}_{P_2}\right)M_X\mathrm{d}W_S(t) \tag{17}$$

where $M_i(i=t,X)$ represents the first-order partial derivative of the option price with respect to i, and M_{XX} represents the second-order partial derivative with respect to X. From Eqs. (16) and (17), we can obtain

$$W(T)=G+\alpha M(T) \tag{18}$$

Then the wealth $W(t)$ of the DC pension manager at time t under the minimum guarantee satisfies the following equation

$$\mathrm{d}W(t)=\mathrm{d}G(t)+\alpha\mathrm{d}M(t)$$
$$=-G\mathrm{e}^{-r(t)(T-t)}(T-t)\left(a(b-r(t))\mathrm{d}t-\sigma_r\mathrm{d}W_r(t)\right)+G\mathrm{e}^{-r(t)(T-t)}r(t)\mathrm{d}t+\alpha M_t\mathrm{d}t+\alpha\left[r(t)\left(X(t)+ce_t\right)+C(t)-\zeta ce_t\right]M_X\mathrm{d}t$$

$$+\alpha u_{\mathrm{B}}(t)\sigma_r A(t,t+K)\lambda_r M_X \mathrm{d}t+\alpha u_{\mathrm{S}}(t)\left(\sigma_1\lambda_r+\sigma_2\lambda_S-\mu_\pi-\sigma_1\boldsymbol{\sigma}_{P_1}-\sigma_2\boldsymbol{\sigma}_{P_2}+\boldsymbol{\sigma}_{P_1}^2+\boldsymbol{\sigma}_{P_2}^2\right)M_X\mathrm{d}t+\alpha u_{\mathrm{B}}(t)\sigma_r A(t,t+K)M_X\mathrm{d}W_r(t)$$
$$+\alpha u_{\mathrm{S}}(t)\left(\sigma_1-\boldsymbol{\sigma}_{P_1}\right)M_X\mathrm{d}W_r(t)+\frac{1}{2}\alpha\left[\left(u_{\mathrm{B}}(t)\sigma_r A(t,t+K)+u_{\mathrm{S}}(t)\left(\sigma_1-\boldsymbol{\sigma}_{P_1}\right)\right)^2+u_{\mathrm{S}}{}^2(t)\left(\sigma_2-\boldsymbol{\sigma}_{P_2}\right)^2\right]M_X\mathrm{d}t$$
$$+\alpha u_{\mathrm{S}}(t)\left(\sigma_2-\boldsymbol{\sigma}_{P_2}\right)M_X\mathrm{d}W_S(t) \tag{19}$$

2.2 Self-Protection and Solutions under Minimum Safeguards

Let $u(t)=\left(u_{\mathrm{B}}(t),u_{\mathrm{S}}(t)\right)$, if there is a unique solution to Eq. (20), and $u(t)$ is asymptotically measurable with respect to $\left\{\mathcal{F}_t\right\}_{t\in[0,T]}$, then $u(t)$ is an acceptable strategy, and the goal of DC-type pension managers is finding the optimal investment strategy $u^y(t)=\left(u_{\mathrm{B}}^y(t),u_{\mathrm{S}}^y(t)\right)$ to maximize the expectation of terminal wealth with respect to $p(e)U\left(W(T)-L\right)+\left(1-p(e)\right)U\left(W(T)\right)$, that is

$$\max E\left[p(e)U\left(W(T)-L\right)+\left(1-p(e)\right)U\left(W(T)\right)\right],\ \text{s.t.}\ \left(W(t),u(t)\right)\text{ satisfies (19)} \tag{20}$$

where $U(X)$ is a strictly concave utility function that satisfies $U'(X)>0, U''(X)<0$, e is e_t, $p(e)$ is the probability of the loss, and L is the loss.

This paper assumes that the utility function of DC pension managers is in the form of CRRA utility, that is

$$U(x)=\frac{x^{1-\gamma}}{1-\gamma} \tag{21}$$

where $\gamma>0,\gamma\neq 1$ represent the risk aversion coefficient of $U(x)$.

To find out (21), the pricing kernel function, also known as the stochastic discount factor, is introduced first. From (3), (8) and Ref. [13], the pricing kernel $H(t)$ satisfies the following differential equation

$$\mathrm{d}H(t)=-r(t)H(t)\mathrm{d}t-\lambda_r H(t)\mathrm{d}W_r(t)-\left(\lambda_S+\frac{\boldsymbol{\sigma}_{P_1}^2+\boldsymbol{\sigma}_{P_2}^2-\mu_\pi}{\sigma_2-\boldsymbol{\sigma}_{P_2}}\right)H(t)\mathrm{d}W_S(t),\ \ H(0)=1 \tag{22}$$

Due to the completeness of the market, this paper can use the martingale method to solve the problem. It can be seen from Ref. [14] that the problem (21) can be transformed into an equivalent problem with budget constraints, that is

$$\max E[p(e)U\left(W(T)-L\right)+\left(1-p(e)\right)U\left(W(T)\right)],\ \text{s.t.}\ E\left[H(T)W(T)-\int_0^T H(s)C(s)\mathrm{d}s\right]\leqslant W(0),\ W(T)\geqslant 0 \tag{23}$$

In a complete market, the budget constraint of problem (23) is equivalent to Eq. (20), and its budget constraint can be understood as that the DC pension account is funded from initial wealth and cumulative contributions of employees.

Proposition 1 The optimal wealth of DC-type pension managers in problem (23) under self-protection and minimum security is

$$W^y(T)=\begin{cases}\dfrac{L}{2(\gamma+1)}+\dfrac{L}{2\gamma(\gamma+1)}\sqrt{\gamma^2+2\gamma(\gamma+1)\left[\dfrac{yH(T)}{1-2p(e)}\left(\dfrac{L}{2}\right)^{\gamma}-1\right]}, & H(T)\leqslant\bar{H}\\ G, & H(T)>\bar{H}\end{cases} \tag{24}$$

where $\bar{H}=\left[\frac{1}{2}\left(G^2\gamma(\gamma+1)-GL\gamma\right)+\left(\frac{L}{2}\right)^2\right]\frac{1-2p(e)}{y}\left(\frac{L}{2}\right)^{-\gamma-2}$; $y>0$ is a Lagrangian multiplier on the budget constraint that satisfies

$$E\left[H(T)W^y(T)-\int_0^T H(s)C(s)\mathrm{d}s\right]=W(0)$$

Proof First, we can solve (24) using the Lagrangian dual theory, and the Lagrangian operator for this problem is defined as

$$\mathcal{L}(W(T),y)=E[p(e)U\left(W(T)-L\right)+\left(1-p(e)\right)U\left(W(T)\right)]-yE[H(T)W(T)]+yW(0)+yE\left[\int_0^T H(s)C(s)\mathrm{d}s\right]$$

Then, problem (24) is equivalent to

$$\inf_{y\geqslant 0}\sup_{W(T)}\mathcal{L}(W(T),y),\ W(T)\geqslant 0$$

Fix the Lagrange multipliers, that is, solve the following problems

$$\sup_{W(T)} \mathcal{L}(W(T), y), W(T) \geqslant 0 \tag{25}$$

Since the solution to (25) can be found, the problem is an optimization problem with respect to y:

$$\inf_{y \geqslant 0} \mathcal{L}(W^y(T), y)$$

Since the above problem is an optimization problem about $W(T)$, ignoring the part irrelevant to $W(T)$, the above problem can be simplified to the problem of fixing y,

$$\max_{W(T)} \left\{ E[p(e)U\left(W(T)-L\right) + \left(1-p(e)\right)U\left(W(T)\right)] - yE[H(T)W(T)] \right\}, \quad W(T) \geqslant 0 \tag{26}$$

Assuming that the utility function form of the DC pension manager is CRRA, $U(x) = \dfrac{x^{1-\gamma}}{1-\gamma}$, then (26) is a concave optimization problem, and the calculation (26) gets

$$p(e)\left(W(T)-L\right)^{-\gamma} + \left(1-p(e)\right)\left(W(T)\right)^{-\gamma} = yH(T) \tag{27}$$

Using Taylor expansion at $W_0(T) = \dfrac{L}{2}$ for (27), we get

$$\frac{1}{2}a(a-1)W(T)^2 + a\frac{L}{2}W(T) + \left(\frac{L}{2}\right)^2 = \frac{yH(T)}{1-2p(e)}\left(\frac{L}{2}\right)^{2-a} \tag{28}$$

Applying the root-finding formula to (27), the solution of (26) is

$$W^y(T) = \frac{L}{2(\gamma+1)} + \frac{L}{2\gamma(\gamma+1)}\sqrt{\gamma^2 + 2\gamma(\gamma+1)\left[\frac{yH(T)}{1-2p(e)}\left(\frac{L}{2}\right)^{\gamma} - 1\right]}$$

In fact, the general optimal solution to problem (26) is

$$W_1^y(T) = \frac{L}{2(\gamma+1)} + \frac{L}{2\gamma(\gamma+1)}\sqrt{\gamma^2 + 2\gamma(\gamma+1)\left[\frac{yH(T)}{1-2p(e)}\left(\frac{L}{2}\right)^{\gamma} - 1\right]} \text{ or } W_2^y(T) = G.$$

Let

$$\begin{aligned} f(H(T)) = &\left(p(e)U\left(W_1^y(T)-L\right) + \left(1-p(e)\right)U\left(W_1^y(T)\right)\right) - yH(T)W_1^y(T) \\ &- [\left(p(e)U\left(W_2^y(T)-L\right) + \left(1-p(e)\right)U\left(W_2^y(T)\right)\right) - yH(T)W_2^y(T)]. \end{aligned}$$

Obviously, when $H(T) \leqslant \bar{H}, f(H(T)) > 0$; when $H(T) > \bar{H}, f(H(T)) < 0$.

So the optimal solution to problem (26) is

$$W^y(T) = \begin{cases} \dfrac{L}{2(\gamma+1)} + \dfrac{L}{2\gamma(\gamma+1)}\sqrt{\gamma^2 + 2\gamma(\gamma+1)\left[\dfrac{yH(T)}{1-2p(e)}\left(\dfrac{L}{2}\right)^{\gamma} - 1\right]}, & H(T) \leqslant \bar{H} \\ G, & H(T) > \bar{H} \end{cases}$$

Proposition 2 The optimal wealth of $W^y(T)$ at time $t\,(0 \leqslant t < T)$ is derived as follows

$$\begin{aligned} W^y(t) = &\frac{1}{H(t)}E\left[H(T)W^y(T)|\mathcal{F}_t\right] - \int_t^T E\left[\frac{H(s)}{H(t)}C(s)|\mathcal{F}_t\right]\mathrm{d}s \\ = &G\exp\left[\frac{1}{2}\operatorname{Var}\{N_t\} + E\{N_t\}\right]\left[1-\Phi\left(d_1(\bar{H})\right)\right] + \frac{b'H(t)L}{4\sqrt{a'}\gamma(\gamma+1)}\exp\left[\frac{1}{2}\operatorname{Var}\{N_t\} + E\{N_t\}\right]\Phi\left(d_1(\bar{H})\right) \\ &- \int_t^T C(t)\exp\left[\left(\mu_C - \frac{1}{2}\sigma_{C_1}^2 - \frac{1}{2}\sigma_{C_2}^2 - \frac{1}{2}\lambda_r^2 - \frac{1}{2}\left(\lambda_s + \frac{\boldsymbol{\sigma}_{P_1}^2 + \boldsymbol{\sigma}_{P_2}^2 - \mu_\pi}{\sigma_2 - \boldsymbol{\sigma}_{P_2}}\right)^2\right)(s-t)\right]\exp\left[E\{Q(t,s)\} + \frac{1}{2}\operatorname{Var}\{Q(t,s)\}\right]\mathrm{d}s \\ &+ \frac{\sqrt{a'}L + \gamma L}{2\gamma(\gamma+1)}\Phi\left(d_2(\bar{H})\right) \end{aligned} \tag{29}$$

where $\Phi(\cdot)$ represents the cumulative distribution function of the standard normal variable, and

$$a' = \gamma^2 - 2\gamma(\gamma-1), \; b' = \frac{2\gamma(\gamma+1)y}{1-2p(e)}\left(\frac{L}{2}\right)^{\gamma},$$

$$N_t=-\int_t^T r(s)\mathrm{d}s-\frac{1}{2}\lambda_r^2(T-t)-\frac{1}{2}\left(\lambda_S+\frac{\boldsymbol{\sigma}_{P_1}^2+\boldsymbol{\sigma}_{P_2}^2-\mu_\pi}{\sigma_2-\boldsymbol{\sigma}_{P_2}}\right)^2(T-t)-\lambda_r\left[W_r(T)-W_r(t)\right]-\left(\lambda_S+\frac{\boldsymbol{\sigma}_{P_1}^2+\boldsymbol{\sigma}_{P_2}^2-\mu_\pi}{\sigma_2-\boldsymbol{\sigma}_{P_2}}\right)\left[W_S(T)-W_S(t)\right],$$

$$E\left\{N_t\right\}=-(r(t)-b)\frac{1-\exp(-a(T-t))}{a}-b(T-t)-\frac{1}{2}\left[\lambda_r^2+\left(\lambda_S+\frac{\boldsymbol{\sigma}_{P_1}^2+\boldsymbol{\sigma}_{P_2}^2-\mu_\pi}{\sigma_2-\boldsymbol{\sigma}_{P_2}}\right)\right](T-t),$$

$$\mathrm{Var}\left[N_t\right]=\frac{\sigma_r^2}{a^2}\left[(T-t)+\frac{2\exp(-a(T-t))}{a}-\frac{\exp(-2a(T-t))}{2a}-\frac{3}{2a}\right]$$
$$+\left[\lambda_r^2+\left(\lambda_S+\frac{\boldsymbol{\sigma}_{P_1}^2+\boldsymbol{\sigma}_{P_2}^2-\mu_\pi}{\sigma_2-\boldsymbol{\sigma}_{P_2}}\right)^2\right](T-t)-2\frac{\lambda_r}{a}\left(\sigma_r(T-t)-\sigma_r A(t,T)\right),$$

$$d_1(\bar{H})=\frac{\ln\left(\frac{\bar{H}}{H(t)}\right)-E\left\{N_1\right\}-\mathrm{Var}\left\{N_t\right\}}{\sqrt{\mathrm{Var}\left\{N_t\right\}}},\quad d_2(\bar{H})=\frac{\ln\left(\frac{\bar{H}}{H(t)}\right)-E\left\{N_t\right\}}{\sqrt{\mathrm{Var}\left\{N_t\right\}}}$$

$$C(s)=C(t)\exp\left[\left(\mu_C-\frac{1}{2}\sigma_{C_1}^2-\frac{1}{2}\sigma_{C_2}^2\right)(s-t)+\sigma_{C_1}\left(W_r(s)-W_r(t)\right)+\sigma_{C_2}\left(W_S(s)-W_S(t)\right)\right],\ \forall s\geqslant t$$

$$E\{Q(t,s)\}=-(r(t)-b)\frac{1-\exp(-a(s-t))}{a}-b(s-t),$$

$$\mathrm{Var}\{Q(t,s)\}=\int_t^s\sigma_r^2A(u,s)^2\mathrm{d}u+\left(\sigma_{C_1}-\lambda_r\right)^2(s-t)+\left(\sigma_{C_2}-\lambda_S-\frac{\boldsymbol{\sigma}_{P_1}^2+\boldsymbol{\sigma}_{P_2}^2-\mu_\pi}{\sigma_2-\boldsymbol{\sigma}_{P_2}}\right)^2(s-t)+2\left(\sigma_{C_1}-\lambda_r\right)\int_t^s\sigma_r A(u,s)\mathrm{d}u$$

Proof

1) $\frac{1}{H(t)}E\left[H(T)W^y(T)\,|\,\mathcal{F}_t\right]=$

$$\frac{1}{H(t)}E\left[H(T)\left[\frac{L}{2(\gamma+1)}+\frac{L}{2\gamma(\gamma+1)}\sqrt{\gamma^2+2\gamma(\gamma+1)\left[\frac{yH(T)}{1-2p(e)}\left(\frac{L}{2}\right)^\gamma-1\right]}\right]1_{\{H(T)\leqslant\bar{H}\}}\,|\,\mathcal{F}_t\right]+\frac{1}{H(t)}E\left[GH(T)1_{\{H(T)>\bar{H}\}}\,|\,\mathcal{F}_t\right]\quad(30)$$

From (2), we can get $r(t)=(r_0-b)\exp(-at)+b-\sigma_r\exp(-at)\int_0^t\exp(as)\mathrm{d}W_r(s)$. So,

$$\int_t^T r(s)\mathrm{d}s=\int_t^T\left[(r(t)-b)\exp(-a(s-t))+b-\sigma_r\exp(-as)\int_0^t\exp(au)\mathrm{d}W_r(u)\right]\mathrm{d}s$$
$$=(r(t)-b)\frac{1-\exp(-a(T-t))}{a}+b(T-t)-\sigma_r\int_t^T\exp(-as)\int_t^s\exp(au)\mathrm{d}W_r(u)\mathrm{d}s$$
$$=(r(t)-b)\frac{1-\exp(-a(T-t))}{a}+b(T-t)-\sigma_r\int_t^T\frac{1-\exp(-a(T-s))}{a}\mathrm{d}W_r(s)$$
$$=(r(t)-b)\frac{1-\exp(-a(T-t))}{a}+b(T-t)-\int_0^t\sigma_r A(s,T)\mathrm{d}W_r(s)$$

In addition, $\int_t^T r(s)\mathrm{d}s$ is a normally distributed random variable, i.e.

$$\int_t^T r(s)\mathrm{d}s\sim N\left((r(t)-b)\frac{1-\exp(-a(T-t))}{a}+b(T-t),\ \frac{\sigma_r^2}{a^2}[(T-t)+\frac{2\exp(-a(T-t))}{a}-\frac{\exp(-2a(T-t))}{2a}-\frac{3}{2a}]\right)$$

From the differential Eq. (23) of the pricing kernel $H(t)$, $H(T)=H(t)\exp(N_t)$, where

$$N_t=-\int_t^T r(s)\mathrm{d}s-\frac{1}{2}\lambda_r^2(T-t)-\frac{1}{2}(\lambda_S+\frac{\boldsymbol{\sigma}_{P_1}^2+\boldsymbol{\sigma}_{P_2}^2-\mu_\pi}{\sigma_2-\boldsymbol{\sigma}_{P_2}})^2(T-t)-\lambda_r[W_r(T)-W_r(t)]-(\lambda_S+\frac{\boldsymbol{\sigma}_{P_1}^2+\boldsymbol{\sigma}_{P_2}^2-\mu_\pi}{\sigma_2-\boldsymbol{\sigma}_{P_2}})[W_S(T)-W_S(t)]$$

So N_t is also a normally distributed random variable, and

$$E\left\{N_t\right\}=-(r(t)-b)\frac{1-\exp(-a(T-t))}{a}-b(T-t)-\frac{1}{2}\left[\lambda_r^2+(\lambda_S+\frac{\boldsymbol{\sigma}_{P_1}^2+\boldsymbol{\sigma}_{P_2}^2-\mu_\pi}{\sigma_2-\boldsymbol{\sigma}_{P_2}})^2\right](T-t),$$

$$\mathrm{Var}\left\{N_t\right\}=\frac{\sigma_r^2}{a^2}\left[(T-t)+\frac{2\exp(-a(T-t))}{a}-\frac{\exp(-2a(T-t))}{2a}-\frac{3}{2a}\right]$$

$$+\left[\lambda_r^2+(\lambda_S+\frac{\boldsymbol{\sigma}_{P_1}^2+\boldsymbol{\sigma}_{P_2}^2-\mu_\pi}{\sigma_2-\boldsymbol{\sigma}_{P_2}})^2\right](T-t)-2\frac{\lambda_r}{a}(\sigma_r(T-t)-\sigma_r A(t,T))$$

Therefore, the next step is to calculate (30), and the Taylor expansion of $\sqrt{\gamma^2+2\gamma(\gamma+1)\left[\frac{yH(T)}{1-2p(e)}\left(\frac{L}{2}\right)^\gamma-1\right]}$ in (30) at $H_0(T)=0$ is $\sqrt{a'}+\frac{b'}{2\sqrt{a'}}H(T)$, where $a'=\gamma^2-2\gamma(\gamma-1), b'=\frac{2\gamma(\gamma+1)y}{1-2p(e)}\left(\frac{L}{2}\right)^\gamma$.

Therefore

$$\frac{1}{H(t)}E\left[H(T)\left[\frac{L}{2(\gamma+1)}+\frac{L}{2\gamma(\gamma+1)}\left(\sqrt{a'}+\frac{b'}{2\sqrt{a'}}H(T)\right)\right]1_{\{H(T)\leqslant\bar{H}\}}\middle|\mathcal{F}_t\right]$$

$$=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\ln\left(\frac{\bar{H}}{H(t)}\right)}\frac{1}{\sqrt{\mathrm{Var}\{N_t\}}}\exp\left(-\frac{\left(x-E\{N_t\}\right)^2}{2\mathrm{Var}\{N_t\}}\right)[\left(\sqrt{a'}+\frac{b'}{2\sqrt{a'}}H(t)\exp(x)\right)\frac{L}{2\gamma(\gamma+1)}+\frac{L}{2(\gamma+1)}]\mathrm{d}x$$

$$=\frac{\sqrt{a'}L+\gamma L}{2\gamma(\gamma+1)}\int_{-\infty}^{\frac{\ln\left(\frac{\bar{H}}{H(t)}\right)-E\{N_t\}}{\sqrt{\mathrm{Var}\{N_t\}}}}\exp\left(-\frac{y^2}{2}\right)\frac{1}{\sqrt{2\pi}}\mathrm{d}y+\frac{b'H(t)L}{4\sqrt{a'}\gamma(\gamma+1)}$$

$$\cdot\int_{-\infty}^{\frac{\ln\left(\frac{\bar{H}}{H(t)}\right)-E\{N_t\}-\mathrm{Var}\{N_t\}}{\sqrt{\mathrm{Var}\{N_t\}}}}\exp\left(-\frac{y^2}{2}\right)\frac{1}{\sqrt{2\pi}}\exp\left[\frac{1}{2}\mathrm{Var}\{N_t\}+E\{N_t\}\right]\mathrm{d}y$$

$$=\frac{b'H(t)L}{4\sqrt{a'}\gamma(\gamma+1)}\exp\left[\frac{1}{2}\mathrm{Var}\{N_t\}+E\{N_t\}\right]\Phi\left(d_1(\bar{H})\right)++\frac{\sqrt{a'}L+\gamma L}{2\gamma(\gamma+1)}\Phi\left(d_2(\bar{H})\right)$$

Similarly

$$\frac{1}{H(t)}E\left[GH(T)1_{\{H(T)>\bar{H}\}}\middle|\mathcal{F}_t\right]=G\exp\left[\frac{1}{2}\mathrm{Var}\{N_t\}+E\{N_t\}\right]\left[1-\Phi\left(d_1(\bar{H})\right)\right]$$

2) From Eq. (9),

$$C(s)=C(t)\exp\left[\left(\mu_C-\frac{1}{2}\sigma_{C_1}-\frac{1}{2}\sigma_{C_2}\right)(s-t)\right]+\sigma_{C_1}(W_r(s)-W_r(t))+\sigma_{C_2}(W_S(s)-W_S(t)),\ \forall s\geqslant t$$

Therefore

$$E\left[\frac{H(s)}{H(t)}C(s)\middle|\mathcal{F}_t\right]=E\left[C(t)\exp\left(\left(\mu_C-\frac{1}{2}\sigma_{C_1}^2-\frac{1}{2}\sigma_{C_2}^2-\frac{1}{2}\lambda_r^2-\frac{1}{2}(\lambda_S+\frac{\boldsymbol{\sigma}_{P_1}^2+\boldsymbol{\sigma}_{P_2}^2-\mu_\pi}{\sigma_2-\boldsymbol{\sigma}_{P_2}})^2\right)(s-t)\right)\exp(Q(t,s))\right]$$

$$=C(t)\exp\left[\left(\mu_C-\frac{1}{2}\sigma_{C_1}^2-\frac{1}{2}\sigma_{C_2}^2-\frac{1}{2}\lambda_r^2-\frac{1}{2}(\lambda_S+\frac{\boldsymbol{\sigma}_{P_1}^2+\boldsymbol{\sigma}_{P_2}^2-\mu_\pi}{\sigma_2-\boldsymbol{\sigma}_{P_2}})^2\right)(s-t)\right]\exp\left[E\{Q(t,s)\}+\frac{1}{2}\mathrm{Var}\{Q(t,s)\}\right]$$

Let $D(t,s)=E\left[\frac{H(s)}{H(t)}C(s)\middle|\mathcal{F}_t\right]$, then $D(t,s)$ satisfies the following backward stochastic differential equation

$$\begin{cases}\frac{\mathrm{d}D(t,s)}{D(t,s)}=r(t)\mathrm{d}t+(\sigma_{C_1}+\sigma_r A(t,s))[\lambda_r\mathrm{d}t+\mathrm{d}W_r(t)]+\sigma_{C_2}\left[(\lambda_S+\frac{\boldsymbol{\sigma}_{P_1}^2+\boldsymbol{\sigma}_{P_2}^2-\mu_\pi}{\sigma_2-\boldsymbol{\sigma}_{P_2}})\mathrm{d}t+\mathrm{d}W_S(t)\right]\\ D(s,s)=C(s),\quad s\geqslant t.\end{cases}$$

Let $F(t,T)=\int_t^T D(t,s)\mathrm{d}s$, $F(t,T)$ also satisfies the following backward stochastic differential equation

$$\begin{cases}\mathrm{d}F(t,T)=-C(t)\mathrm{d}t+r(t)F(t,T)\mathrm{d}t+F_1(t)[\lambda_r\mathrm{d}t+\mathrm{d}W_r(t)]+F_2(t)\left[(\lambda_S+\frac{\boldsymbol{\sigma}_{P_1}^2+\boldsymbol{\sigma}_{P_2}^2-\mu_\pi}{\sigma_2-\boldsymbol{\sigma}_{P_2}})\mathrm{d}t+\mathrm{d}W_S(t)\right]\\ F(T,T)=0\end{cases}\tag{31}$$

where $F_1(t)=\sigma_{C_1}F(t,T)+\int_t^T D(t,s)\sigma_r A(t,s)\mathrm{d}s$, $F_2(t)=\sigma_{C_2}F(t,T)$.

The result of Proposition 2 shows that the optimal wealth of a DC pension manager at time t consists of the price of $W^y(T)$ at time t and the cumulative contributions from time t to T.

Proposition 3 Given the explicit form of $W^y(t)$, the differential form of $W^y(t)$ can be obtained from Eq. (26), and the optimal investment strategy can be obtained by comparing the differential form of $W^y(t)$ with Eq. (20).

Proposition 4 The optimal investment capital to invest in bonds and stocks is

$$\begin{aligned}
u_{\mathrm{S}}^y(t) &= -\left(\lambda_S + \frac{\boldsymbol{\sigma}_{P_1}^2 + \boldsymbol{\sigma}_{P_2}^2 - \mu_\pi}{\sigma_2 - \boldsymbol{\sigma}_{P_2}}\right)\frac{\partial G(t, r(t), H(t))}{\partial H(t)}\frac{H(t)}{\alpha\left(\sigma_2 - \boldsymbol{\sigma}_{P_2}\right)M_X} - \frac{F_2(t)}{\alpha\left(\sigma_2 - \boldsymbol{\sigma}_{P_2}\right)M_X} \\
u_{\mathrm{B}}^y(t) &= -\frac{\partial G(t, r(t), H(t))}{\partial r(t)}\frac{1}{\alpha A(t, t+K)M_X} - \frac{G(T-t)\exp(-r(t)(T-t))}{\alpha A(t, t+K)M_X} \\
&\quad -\lambda_r\frac{\partial G(t, r(t), H(t))}{\partial H(t)}\frac{H(t)}{\alpha\sigma_r A(t, t+k)M_X} \\
&\quad +\left(\lambda_S + \frac{\boldsymbol{\sigma}_{P_1}^2 + \boldsymbol{\sigma}_{P_2}^2 - \mu_\pi}{\sigma_2 - \boldsymbol{\sigma}_{P_2}}\right)\frac{\partial G(t, r(t), H(t))}{\partial H(t)}\frac{\left(\sigma_1 - \boldsymbol{\sigma}_{P_2}\right)H(t)}{\alpha\sigma_r\left(\sigma_2 - \boldsymbol{\sigma}_{P_2}\right)A(t, t+K)M_X} \\
&\quad -\frac{F_1(t)}{\alpha\sigma_r A(t, t+K)M_X} + \frac{\left(\sigma_1 - \boldsymbol{\sigma}_{P_1}\right)F_2(t)}{\alpha\sigma_r\left(\sigma_2 - \boldsymbol{\sigma}_{P_2}\right)A(t, t+K)M_X}
\end{aligned} \tag{32}$$

where

$$\begin{aligned}
\frac{\partial G(t, r(t), H(t))}{\partial H(t)} &= \frac{G}{H(t)\sqrt{\mathrm{Var}\{N_t\}}}\exp\left[\frac{1}{2}\mathrm{Var}\{N_t\} + E\{N_t\}\right]\varphi\left(d_1(\bar{H})\right) \\
&\quad + \frac{b'L}{4\sqrt{a'}\gamma(\gamma+1)}\exp\left[\frac{1}{2}\mathrm{Var}\{N_t\} + E\{N_t\}\right]\Phi\left(d_1(\bar{H})\right) \\
&\quad - \frac{b'H(t)L}{4\sqrt{a'}\gamma(\gamma+1)}\frac{1}{H(t)\sqrt{\mathrm{Var}\{N_t\}}}\exp\left[\frac{1}{2}\mathrm{Var}\{N_t\} + E\{N_t\}\right]\varphi\left(d_1(\bar{H})\right) \\
&\quad - \frac{1}{H(t)\sqrt{\mathrm{Var}\{N_t\}}}\frac{\sqrt{a'}L + \gamma L}{2\gamma(\gamma+1)}\varphi\left(d_2(\bar{H})\right)
\end{aligned} \tag{33}$$

$$\begin{aligned}
\frac{\partial G(t, r(t), H(t))}{\partial r(t)} &= -GA(t, T)\exp\left[\frac{1}{2}\mathrm{Var}\{N_t\} + E\{N_t\}\right]\left[1 - \Phi\left(d_1(\bar{H})\right)\right] \\
&\quad - G\frac{A(t, T)}{\sqrt{\mathrm{Var}\{N_t\}}}\exp\left[\frac{1}{2}\mathrm{Var}\{N_t\} + E\{N_t\}\right]\varphi\left(d_1(\bar{H})\right) \\
&\quad - \frac{A(t, T)b'H(t)L}{4\sqrt{a'}\gamma(\gamma+1)}\exp\left[\frac{1}{2}\mathrm{Var}\{N_t\} + E\{N_t\}\right]\Phi\left(d_1(\bar{H})\right)
\end{aligned} \tag{34}$$

where $\varphi(\cdot)$ represents the density function of a standard normal variable, $\varphi(x) = \frac{1}{\sqrt{2\pi}}\exp\left(-\frac{x^2}{2}\right)$, and

$$F_1(t) = \sigma_{C_1}F(t, T) + \int_t^T D(t, s)\sigma_r A(t, s)\mathrm{d}s, \quad F_2(t) = \sigma_{C_2}F(t, T).$$

Proof Calculate the differential form of $W^y(t)$ and compare it with the diffusion term part of (19) to get the optimal investment strategy.

Let $G(t, r(t), H(t)) = \frac{1}{H(t)}E\left[H(T)W^y(T)\,\middle|\,\mathcal{F}_t\right]$, so $\mathrm{d}W^y(t) = \mathrm{d}G(t, r(t),\ H(t)) - \mathrm{d}F(t, T)$.

According to Eqs, (2), (22) and (31), the explicit form of $\mathrm{d}W^y(t)$ is

$$\begin{aligned}
\mathrm{d}W^y(t) &= (\cdot)\mathrm{d}t - \left[\sigma_r\frac{\partial G(t, r(t), H(t))}{\partial r(t)} + \lambda_r\frac{\partial G(t, r(t), H(t))}{\partial H(t)}H(t) + F_1(t)\right]\mathrm{d}W_r(t) \\
&\quad - \left[(\lambda_S + \frac{\boldsymbol{\sigma}_{P_1}^2 + \boldsymbol{\sigma}_{P_2}^2 - \mu_\pi}{\sigma_2 - \boldsymbol{\sigma}_{P_2}})\frac{\partial G(t, r(t), H(t))}{\partial H(t)}H(t) + F_2(t)\right]\mathrm{d}W_S(t)
\end{aligned}$$

Since the part of the non-diffusion term in (·) is irrelevant to our solution to the optimal investment strategy, we only need to compare the diffusion term of $\mathrm{d}W^{y}(t)$ with Eq. (19) to solve it. The proof is completed.

3 Numerical Analysis

This section analyzes the impact of self-protection on the optimal investment strategy. For the convenience of analysis, fixed inflation rate and minimum guarantee are considered here, and the contribution rate is fixed. The parameters used in this paper refer to Ref. [8], and the parameters are as follows: $a=0.2$; $b=0.02$; $\sigma_r=0.02$; $r_0=0.04$; $A=2.25$; $B=1$; $K=20$; $T=40$; $\lambda_r=0.15$; $\lambda_S=0.2$; $\sigma_1=0.2$; $\sigma_2=0.4$; $C_0=0.15$; $\mu_C=0.02$; $\sigma_{C_1}=\sigma_{C_2}=0$; $\mu_\pi=0.06$; $\boldsymbol{\sigma}_{P_1}=0.2$; $\boldsymbol{\sigma}_{P_2}=0.2$; $X_0=1$; $\gamma=0.4$; $\alpha=0.2$. The program is simulated 1 000 times in Matlab, and the results are shown in Figs. 1 to 3. From Fig. 1(a) and (b), when the loss L is 10 000 and the probability of loss p is 0.2, the funds invested in stocks and bonds will increase to 55%; when the probability of loss p is high,with a value of 0.6, investors appear more cautious at the time, and the funds invested in bonds are slowly reduced from 100% to 10%. When approaching retirement, savings and stocks are reduced to nearly 0%. In the case where the losses are both high at 100 000, when the probability of loss is 0.2, the trend in Fig. 1(c) is roughly the same as that in Fig.1(a). Due to the higher possible losses, the funds invested in stocks and bonds increase to 50% at retirement time, which is 5% less than that in Fig.1(a); when the probability of loss is high, investors are more cautious, and the proportion of investing in stocks and bonds when they retire in Fig.1(d) is 30% less than that in Fig.1(c). Figure 1(b) and (d) illustrate that when the probability of loss is the same and the possible loss is high, investors are more willing to invest money into stocks and bonds to reduce risk.

Figure 2 show the impact of different minimum guarantees on optimal investment under the same probability and magnitude of loss. It can be seen from Fig.2(a) and (c) that, when the probability of loss is small, the trend and proportion of the optimal investment change little compared with that of Fig.1(a) and (c); it can be seen from Fig.2(b) and (d) that when the probability of loss is high, the initial investment ratio in the optimal investment ratio becomes lower than that in Fig.1(b) and Fig.1(d), which reduces the capital pressure of investors in the initial stage of investment.

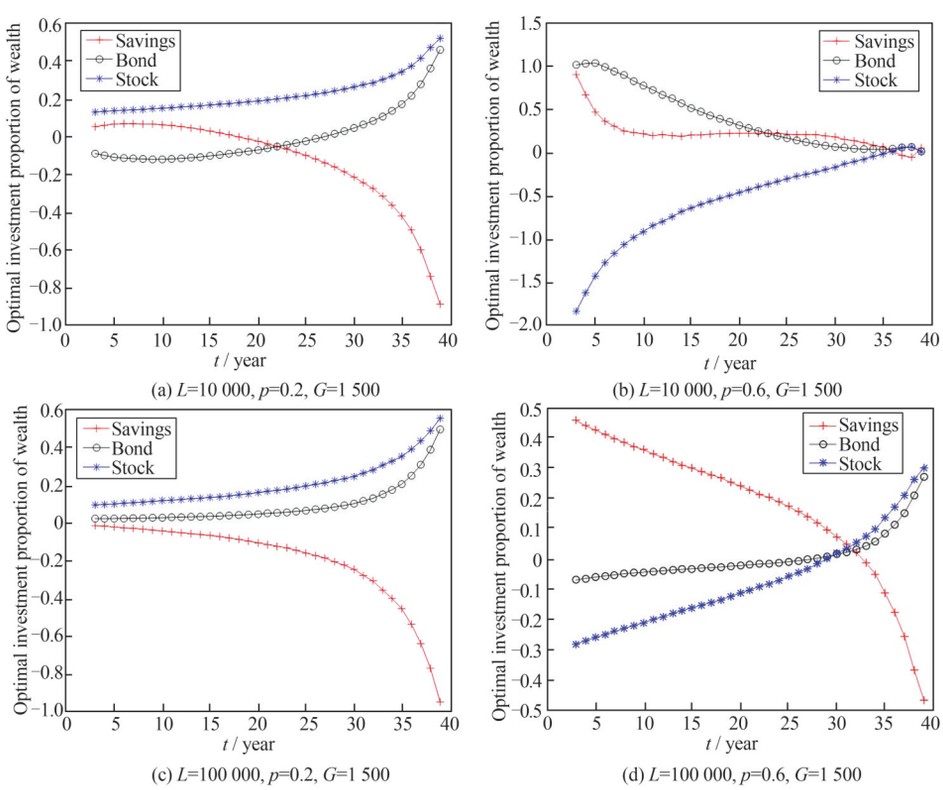

Fig. 1 Optimal investment proportion of wealth with different loss and probability when *G*=1 500

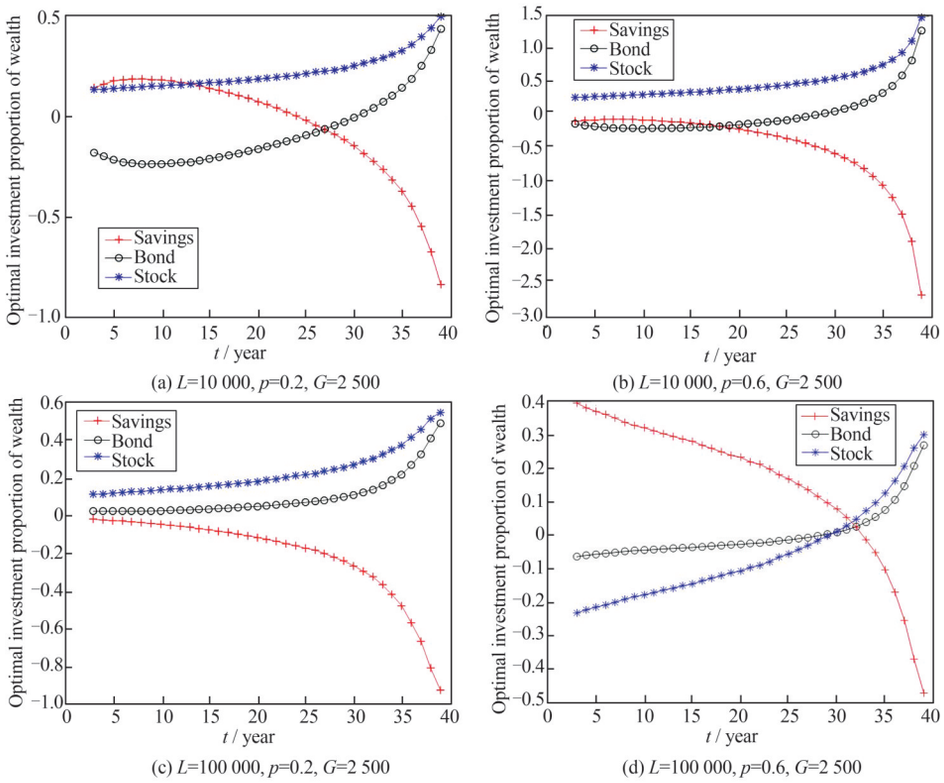

Fig. 2 Optimal investment proportion of wealth with different loss and probability when *G*=2 500

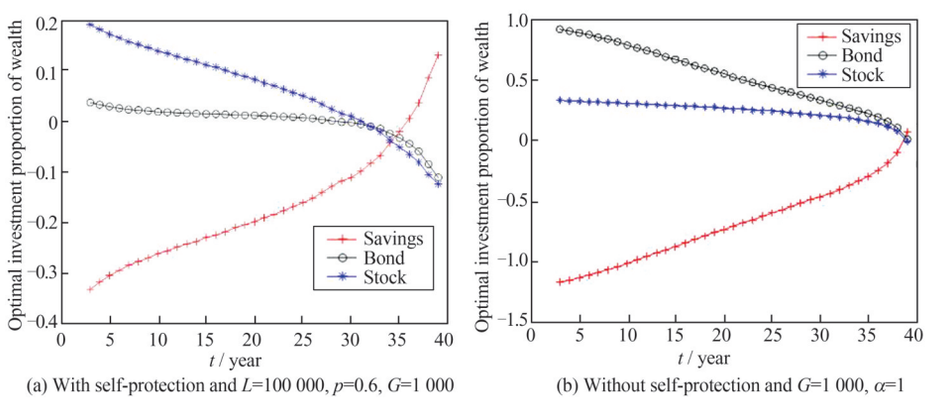

Fig. 3 Optimal investment proportion of wealth with different model

Figure 3 shows the optimal investment ratio with and without self-protection using the parameters in Ref.[8] and the minimum guarantee is 1 000. In Fig. 3(a), the optimal investment ratios for bonds and stocks gradually decrease from approximately 5% and 20%, respectively, and the proportion of savings has slowly increased from −32%. Since sufficient funds have been accumulated after initial investment, by the 35th year, investors began to reduce stocks and bonds, and invested funds instead. Therefore, savings with better stability and security increased faster (from 0% to 15%), while bonds and stocks fell to −10%. The investment ratio of bonds and stocks in Fig. 3(b) has dropped from 100% and 30% to 0%, respectively. The savings keep a short position in the funds during the accumulation phase, and it slowly increases from −120% to 5%. Compared with those without self-protection, the initial investment ratio and final ratio of bonds and stocks under self-protection are lower, and the savings at the time of retirement are higher, which makes investors face less capital pressure at the initial moment and can save money at the time of retirement to get more stable savings protection.

4 Conclusion

In this paper, the random inflation factor is considered by the discount method. Under the CRRA utility function, the minimum guarantee is established, self-protection is introduced, and the martingale method is used to calculate the optimal investment strategy of the DC pension at the time of retirement and the optimal investment ratio of stocks, bonds and cash at any time. And then Matlab is used to carry out numerical analysis. From the analysis, it can be seen that when the loss of funds or the probability of loss is high, investors will be more cautious, and the proportion of funds invested in bonds and stocks will be relatively reduced. When the loss of funds and the probability of loss are high at the same time, investors will increase their investment in stocks and bonds to reduce losses. When the minimum guarantee is high, investors will reduce the initial investment ratio. Compared with not adding self-protection, adding self-protection allows investors to face less financial pressure at the initial moment and obtain more stable savings protection at the time of retirement.

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