Wuhan University Journal of Natural Sciences

2023, Vol.28 No.3, 185-191

Article ID 1007-1202(2023)03-0185-07

DOI https://doi.org/10.1051/wujns/2023283185

A Class of Inverse Boundary Value Problems for (λ, 1) Bi-Analytic Functions

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Abstract: In this paper, a class of inverse boundary value problems for (λ, 1) bi-analytic functions is given. Using the method of Riemann boundary value problem for analytic functions, the conditions of solvability and the expression of the solutions for the inverse problems are obtained.

Key words: inverse problem; Riemann boundary value problem; (λ, k) bi-analytic functions; canonical function

CLC number: O175.8

0 Introduction

Boundary value problems for (λ, k) bi-analytic functions are discussed in Refs.[1-6]. In Ref.[4], the author used the Cauchy-type integral of (λ, 1) bi-analytic functions to study a class of inverse Riemann problems of (λ, 1) bi-analytic functions. In this article, we study the general case by using the theory of boundary value problems for analytic functions.

Let Γ be a simple and closed smooth contour, oriented counter-clockwisely. $D^+(D^-)$ denote respectively the interior (exterior) region bounded by Γ. And let $O=(0,\ 0)$ in D^+. Our inverse boundary value problem, simply called I-BVP, is to find functions $\omega(t)$, $\vartheta(t)$ and constants λ^+, λ^-, satisfying the following boundary conditions

$$\begin{cases} \dfrac{\partial f(z)}{\partial \bar{z}} = \dfrac{\lambda^\pm - 1}{4\lambda^\pm}\phi^\pm(z) + \dfrac{\lambda^\pm + 1}{4\lambda^\pm}\overline{\phi^\pm(z)}, & z \in D^\pm, \\ \dfrac{\partial \phi^+(z)}{\partial \bar{z}} = \theta(z), & z \in D^+, \\ \dfrac{\partial \phi^-(z)}{\partial \bar{z}} = \varpi(z) & z \in D^-, \\ f^+(t) = a_i(t) f^-(t) + b_i(t)\omega(t), & t \in \Gamma, i = 1, 2, \\ \phi^+(t) = \beta_i(t)\phi^-(t) + \gamma_i(t)\vartheta(t), & t \in \Gamma, i = 1, 2, \\ f^-(t_i) = c_i, & t_i \in \Gamma, i = 1, 2, \end{cases} \tag{1}$$

Received date: 2022-09-28

Foundation item: Supported by the Natural Science Foundation of Fujian Province (2020J01322)

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where $\theta(t)$, $\varpi(t)$, $a_i(t)$, $b_i(t)$, $\beta_i(t)$, $\gamma_i(t)(i=1,\ 2)\in H(\Gamma)$ (Hölder continuous). $c_i\ (i=1,\ 2)$ are known constants.

1 Inverse Riemann Boundary Value Problems

In this section, we will consider a class of inverse Riemann boundary value problems, simply called I-RBVP. It is to find $\phi(z)$ and $\vartheta(t)$ meeting the following conditions

$$\begin{cases} \dfrac{\partial\phi^+(z)}{\partial\bar{z}}=\theta(z), & z\in D^+ \\ \dfrac{\partial\phi^-(z)}{\partial\bar{z}}=\varpi(z), & z\in D^- \\ \phi^+(t)=\beta_1(t)\phi^-(t)+\gamma_1(t)\vartheta(t), & t\in\Gamma \\ \phi^+(t)=\beta_2(t)\phi^-(t)+\gamma_2(t)\vartheta(t), & t\in\Gamma \end{cases} \tag{2}$$

Suppose that $\gamma_1\neq\gamma_2$ and $\beta_1\gamma_2\neq\beta_2\gamma_1$, the solvable conditions and the representation of the solutions are obtained.

Assume that $\Lambda\in H^v(\Gamma)(0<v\leqslant 1)$. Let the Cauchy integral operators

$$\left(S_\Gamma[\Lambda]\right)(z)=\begin{cases} \dfrac{1}{2\pi\mathrm{i}}\displaystyle\int_\Gamma \frac{\Lambda(\tau)}{\tau-z}\,\mathrm{d}\tau, & z\notin\Gamma \\ \dfrac{1}{\pi\mathrm{i}}\displaystyle\int_\Gamma \frac{\Lambda(\tau)}{\tau-t}\,\mathrm{d}\tau, & z=t\in\Gamma \end{cases} \tag{3}$$

and its projection operator

$$\left(S_\Gamma^\pm[\Lambda]\right)(z)=\begin{cases} \dfrac{1}{2\pi\mathrm{i}}\displaystyle\int_\Gamma \frac{\Lambda(\tau)}{\tau-z}\,\mathrm{d}\tau, & z\in D^\pm \\ \dfrac{1}{2}\left[\left(S_\Gamma[\Lambda]\right)(t)\pm\Lambda(t)\right], & z=t\in\Gamma \end{cases} \tag{4}$$

Let

$$\phi_1(z)=\begin{cases} \phi(z)-\bar{z}\,\theta(z), & z\in D^+ \\ \phi(z)-\bar{z}\,\varpi(z), & z\in D^- \end{cases} \tag{5}$$

then we have

$$\begin{cases} \phi^+(t)=\phi_1^+(t)+\bar{t}\,\theta(t) \\ \phi^-(t)=\phi_1^-(t)+\bar{t}\,\varpi(t) \end{cases} \tag{6}$$

By (6) and (2), we obtain

$$\begin{cases} \phi_1^+(t)=G(t)\phi_1^-(t)+\bar{t}\,(G(t)\varpi(t)-\theta(t)), & t\in\Gamma \\ \dfrac{\partial\phi_1(z)}{\partial\bar{z}}=0, & z\notin\Gamma \end{cases} \tag{7}$$

and

$$\vartheta(t)=\frac{\beta_2(t)-\beta_1(t)}{\gamma_1(t)-\gamma_2(t)}\left(\phi_1^-(t)+\bar{t}\,\varpi(t)\right) \tag{8}$$

where

$$G(t)=\frac{\beta_1(t)\gamma_2(t)-\beta_2(t)\gamma_1(t)}{\gamma_2(t)-\gamma_1(t)},\ G(t)\in H(\Gamma)\text{ and }G(t)\neq 0 \tag{9}$$

Now we consider the Riemann boundary value problem, simply called RBVP. It is to find $\phi_1(z)$ meeting the conditions given in (7). Denote $\kappa=\dfrac{1}{2\pi}\left[\arg G(t)\right]_\Gamma$, then the index of RBVP (7) is κ. Suppose $\phi_1(\infty)=0$, thus from Refs.[7, 8], we have the following results.

Lemma 1 If $\kappa\geqslant 0$, then the solutions of RBVP (7) are

$$\phi_1(z)=X(z)\left(S_\Gamma\left[\frac{\bar{t}\,(G\varpi-\theta)}{X^+}\right]\right)(z)+P_{\kappa-1}(z)X(z), \qquad z\notin\Gamma \tag{10}$$

where

$$X(z)=\begin{cases} X^+(z)=\exp\left\{\left(S_\Gamma\left[\log((t-z_0)^{-\kappa}G)\right]\right)(z)\right\}, & z\in D^+ \\ X^-(z)=(z-z_0)^{-\kappa}\exp\left\{\left(S_\Gamma\left[\log((t-z_0)^{-\kappa}G)\right]\right)(z)\right\}, & z_0\in D^+,\ z\in D^- \end{cases} \tag{11}$$

$X(z)$ is the canonical, $P_{\kappa-1}(z)$ is arbitrary polynomial whose degree does not exceed $\kappa-1$.

If $\kappa<0$, then the RBVP (7) has a unique solution

$$\phi_1(z)=X(z)\left(S_\Gamma\left[\frac{\bar{t}(G\varpi-\theta)}{X^+}\right]\right)(z),\qquad z\notin\Gamma \tag{12}$$

if and only if

$$\int_\Gamma \frac{t^j\bar{t}(G(t)\varpi(t)-\theta(t))}{X^+(t)}\mathrm{d}t=0,\quad j=0,\ 1,\ \cdots,\ -\kappa-1 \tag{13}$$

Hence, we obtain

Theorem 1 Under the requirement $\phi(\infty)=\overline{\infty}\ \varpi(\infty)$, when $\kappa\geqslant 0$, the solutions of I-RBVP (2) are

$$\phi(z)=\begin{cases} X(z)\left(S_\Gamma\left[\frac{\bar{t}(G\varpi-\theta)}{X^+}\right]\right)(z)+P_{\kappa-1}(z)X(z)+\bar{z}\,\theta(z), & z\in D^+ \\ X(z)\left(S_\Gamma\left[\frac{\bar{t}(G\varpi-\theta)}{X^+}\right]\right)(z)+P_{\kappa-1}(z)X(z)+\bar{z}\,\varpi(z), & z\in D^- \end{cases} \tag{14}$$

where $X(z)$ is given in (11), $P_{\kappa-1}(z)$ is arbitrary polynomial whose degree does not exceed $\kappa-1$, and

$$\vartheta(t)=\frac{\beta_2(t)-\beta_1(t)}{\gamma_1(t)-\gamma_2(t)}\left\{X^-(t)\left(S_\Gamma^-\left[\frac{\bar{\tau}(G\varpi-\theta)}{X^+}\right]\right)(t)+P_{\kappa-1}(t)X^-(t)+\bar{t}\,\varpi(t)\right\} \tag{15}$$

When $\kappa<0$, the solutions of I-RBVP (2) are

$$\phi(z)=\begin{cases} X(z)\left(S_\Gamma\left[\frac{\bar{t}(G\varpi-\theta)}{X^+}\right]\right)(z)+\bar{z}\,\theta(z), & z\in D^+ \\ X(z)\left(S_\Gamma\left[\frac{\bar{t}(G\varpi-\theta)}{X^+}\right]\right)(z)+\bar{z}\,\varpi(z), & z\in D^- \end{cases} \tag{16}$$

if and only if

$$\int_\Gamma \frac{t^j\bar{t}(G(t)\varpi(t)-\theta(t))}{X^+(t)}\mathrm{d}t=0,\quad j=0,\ 1,\ \cdots,\ -\kappa-1 \tag{17}$$

and

$$\vartheta(t)=\frac{\beta_2(t)-\beta_1(t)}{\gamma_1(t)-\gamma_2(t)}\left\{X^-(t)\left(S_\Gamma^-\left[\frac{\bar{\tau}(G\varpi-\theta)}{X^+}\right]\right)(t)+\bar{t}\,\varpi(t)\right\} \tag{18}$$

2 Inverse Boundary Value Problems for (λ, 1) Bi-Analytic Functions

In this section, we will consider the I-BVP for (λ, 1) bi-analytic functions. It is to find $\omega(t)$ and constants $\lambda^\pm$ meeting the following conditions

$$\begin{cases} \frac{\partial f(z)}{\partial\bar{z}}=\frac{\lambda^\pm-1}{4\lambda^\pm}\phi(z)+\frac{\lambda^\pm+1}{4\lambda^\pm}\overline{\phi(z)}, & z\in D^\pm \\ f^+(t)=a_1(t)f^-(t)+b_1(t)\omega(t), & t\in\Gamma \\ f^+(t)=a_2(t)f^-(t)+b_2(t)\omega(t), & t\in\Gamma \\ f^-(t_i)=c_i, & t_i\in\Gamma,\ i=1,\ 2 \end{cases} \tag{19}$$

where $a_i(t), b_i(t)(i=1,\ 2)\in H(\Gamma)$, constants $c_i(i=1,\ 2)$ are known.

In the following discussion, we assume that $b_1(t)\neq b_2(t)$ and $a_1(t)b_2(t)\neq a_2(t)b_1(t)$. By (5), we get

$$\phi(z)=\begin{cases}\phi_1(z)+\bar{z}\,\theta(z), & z\in D^+\\ \phi_1(z)+\bar{z}\,\varpi(z), & z\in D^-\end{cases}\tag{20}$$

Substituting (20) into the first equation in (19) and letting

$$f_1(z)=\begin{cases}f(z)-\dfrac{\lambda^+-1}{8\lambda^+}(\bar{z})^2\theta(z)-\dfrac{\lambda^++1}{4\lambda^+}z\,\overline{\Theta(z)}, & z\in D^+\\ f(z)-\dfrac{\lambda^--1}{8\lambda^-}(\bar{z})^2\varpi(z)-\dfrac{\lambda^-+1}{4\lambda^-}z\,\overline{\Omega(z)}, & z\in D^-\end{cases}\tag{21}$$

where $\Theta'(z)=\theta(z)$, $\Omega'(z)=\varpi(z)$, we obtain

$$\frac{\partial f_1(z)}{\partial\bar{z}}=\frac{\lambda^\pm-1}{4\lambda^\pm}\phi_1(z)+\frac{\lambda^\pm+1}{4\lambda^\pm}\overline{\phi_1(z)},\quad z\in D^\pm\tag{22}$$

(22) and the second equation in (7) imply $f_1(z)$ is a $(\lambda, 1)$ bi-analytic function with associate function $\phi_1(z)$. By (22), we obtain

$$f_1(z)=\frac{\lambda^\pm-1}{4\lambda^\pm}\phi_1(z)\bar{z}+\frac{\lambda^\pm+1}{4\lambda^\pm}\overline{\Phi_1(z)}+\psi(z),\qquad z\in D^\pm\tag{23}$$

where $\Phi_1'(z)=\phi_1(z)$. Let $z\to t$ from the inner domain D^+ and the outer domain D^-, respectively, one get

$$f_1^+(t)=f^+(t)-\frac{\lambda^+-1}{8\lambda^+}(\bar{t})^2\theta(t)-\frac{\lambda^++1}{4\lambda^+}t\,\overline{\Theta(t)}=\frac{\lambda^+-1}{4\lambda^+}\phi_1^+(t)\bar{t}+\frac{\lambda^++1}{4\lambda^+}\overline{\Phi_1^+(t)}+\psi^+(t),\ \ t\in\Gamma\tag{24}$$

and

$$f_1^-(t)=f^-(t)-\frac{\lambda^--1}{8\lambda^-}(\bar{t})^2\varpi(t)-\frac{\lambda^-+1}{4\lambda^-}t\,\overline{\Omega(t)}=\frac{\lambda^--1}{4\lambda^-}\phi_1^-(t)\bar{t}+\frac{\lambda^-+1}{4\lambda^-}\overline{\Phi_1^-(t)}+\psi^-(t),\ \ t\in\Gamma\tag{25}$$

respectively.

By the second equation and the third equation in (19), we get

$$\begin{vmatrix}1 & b_1(t)\\ 1 & b_2(t)\end{vmatrix}f^+(t)=\begin{vmatrix}a_1(t) & b_1(t)\\ a_2(t) & b_2(t)\end{vmatrix}f^-(t),\quad t\in\Gamma\tag{26}$$

and

$$\omega(t)=\frac{a_2(t)-a_1(t)}{b_1(t)-b_2(t)}f^-(t)\tag{27}$$

Submitting $f^+(t)$ in (24) and $f^-(t)$ in (25) into (26), we obtain

$$\psi^+(t)=\frac{\begin{vmatrix}a_1(t) & b_1(t)\\ a_2(t) & b_2(t)\end{vmatrix}}{\begin{vmatrix}1 & b_1(t)\\ 1 & b_2(t)\end{vmatrix}}\psi^-(t)+\Xi(t),\quad t\in\Gamma\tag{28}$$

where

$$\begin{aligned}\Xi(t)=&\frac{\begin{vmatrix}a_1(t) & b_1(t)\\ a_2(t) & b_2(t)\end{vmatrix}}{\begin{vmatrix}1 & b_1(t)\\ 1 & b_2(t)\end{vmatrix}}\left\{\frac{\lambda^--1}{4\lambda^-}\left[\phi_1^-(t)\bar{t}+\frac{1}{2}(\bar{t})^2\varpi(t)\right]+\frac{\lambda^-+1}{4\lambda^-}\left[\overline{\Phi_1^-(t)}+t\,\overline{\Omega(t)}\right]\right\}\\ &-\left\{\frac{\lambda^+-1}{4\lambda^+}\left[\phi_1^+(t)\bar{t}+\frac{1}{2}(\bar{t})^2\theta(t)\right]+\frac{\lambda^++1}{4\lambda^+}\left[\overline{\Phi_1^+(t)}+t\,\overline{\Theta(t)}\right]\right\}\end{aligned}\tag{29}$$

From (23) and (22), obviously,

$$\frac{\partial\psi(z)}{\partial\bar{z}}=0,\quad z\notin\Gamma\tag{30}$$

Now we consider the following RBVP. It is to find $\psi(z)$ meeting the conditions (28) and (30), namely,

$$\begin{cases}\psi^+(t)=G_*(t)\psi^-(t)+\Xi(t), & t\in\Gamma\\ \dfrac{\partial\psi(z)}{\partial\bar{z}}=0, & z\notin\Gamma\end{cases}\tag{31}$$

where

$$G_*(t)=\frac{a_1(t)b_2(t)-a_2(t)b_1(t)}{b_2(t)-b_1(t)} \tag{32}$$

$G_*(t)\in H(\Gamma)$ and $G_*(t)\neq 0$. $\kappa_*=\frac{1}{2\pi}\left[\arg G_*(t)\right]_\Gamma$ is the index of RBVP (31). If $\psi(\infty)$ is to be finite, then we have the following results from Refs.[7, 8].

Lemma 2 1) If $\kappa_*+1\geqslant 0$, then the solutions of RBVP (31) are

$$\psi(z)=X_*(z)\left(S_\Gamma\left[\frac{\Xi}{X_*^+}\right]\right)(z)+P_{\kappa_*}(z)X_*(z), \qquad z\notin\Gamma \tag{33}$$

where $P_{\kappa_*}(z)$ is arbitrary polynomial whose degree does not exceed κ_*,

$$X_*(z)=\begin{cases} X_*^+(z)=\exp\left\{\left(S_\Gamma\left[\log((t-z_*)^{-\kappa_*}G_*)\right]\right)(z)\right\}, & z\in D^+ \\ X_*^-(z)=(z-z_*)^{-\kappa_*}\exp\left\{\left(S_\Gamma\left[\log((t-z_*)^{-\kappa_*}G_*)\right]\right)(z)\right\}, & z_*\in D^+,\ z\in D^- \end{cases} \tag{34}$$

2) If $\kappa_*+1<0$, then the RBVP (31) has a unique solution (33) ($P_{\kappa_*}\equiv 0$) if and only if

$$\int_\Gamma \frac{t^j\,\Xi(t)}{X_*^+(t)}\,\mathrm{d}t=0, \quad j=0,\ 1,\ \cdots,\ -\kappa_*-2 \tag{35}$$

3) $\omega(t)$ is written as following

$$\omega(t)=\frac{a_2(t)-a_1(t)}{b_1(t)-b_2(t)}\left\{\frac{\lambda^--1}{4\lambda^-}\left[\phi_1^-(t)\,\bar{t}+\frac{1}{2}(\bar{t})^2\varpi(t)\right]+\frac{\lambda^-+1}{4\lambda^-}\left[\overline{\Phi_1^-(t)}+t\,\overline{\Omega(t)}\right]+\psi^-(t)\right\}, \quad t\in\Gamma \tag{36}$$

(26) can be changed to

$$f^-(t)=\frac{b_2(t)-b_1(t)}{a_1(t)b_2(t)-a_2(t)b_1(t)}f^+(t) \tag{37}$$

Submitting $f^+(t)$ in (24) into (37), one get

$$f^-(t)=\frac{b_2(t)-b_1(t)}{a_1(t)b_2(t)-a_2(t)b_1(t)}\left\{\frac{\lambda^+-1}{4\lambda^+}\left[\phi_1^+(t)\,\bar{t}+\frac{1}{2}(\bar{t})^2\theta(t)\right]+\frac{\lambda^++1}{4\lambda^+}\left[\overline{\Phi_1^+(t)}+t\,\overline{\Theta(t)}\right]+\psi^+(t)\right\} \tag{38}$$

where $\psi^+(t)$ is from $\psi(z)$ (let $z\to t$ from the inner domain D^+) given in Lemma 2, namely,

$$\psi^+(t)=X_*^+(t)\left(S_\Gamma^+\left[\frac{\Xi}{X_*^+}\right]\right)(t)+\hbar\cdot P_{\kappa_*}(t)X_*^+(t) \tag{39}$$

so, we get

$$A(t)\frac{1}{\lambda^+}+B(t)\frac{1}{\lambda^-}=C(t) \tag{40}$$

where

$$\begin{aligned} A(t)=&\frac{b_2(t)-b_1(t)}{a_1(t)b_2(t)-a_2(t)b_1(t)}\left\{\phi_1^+(t)\,\bar{t}+\frac{1}{2}(\bar{t})^2\theta(t)-\overline{\Phi_1^+(t)}-t\,\overline{\Theta(t)}\right\} \\ &+\frac{b_2(t)-b_1(t)}{a_1(t)b_2(t)-a_2(t)b_1(t)}X_*^+(t)\left(S_\Gamma^+\left[\frac{-\phi_1^+\bar{\tau}-\frac{1}{2}(\bar{\tau})^2\theta+\overline{\Phi_1^+}+\tau\,\overline{\Theta}}{X_*^+}\right]\right)(t) \end{aligned} \tag{41}$$

$$B(t)=X_*^+(t)\left(S_\Gamma^+\left[\frac{\phi_1^-\bar{\tau}+\frac{1}{2}(\bar{\tau})^2\varpi-\overline{\Phi_1^-}-\tau\,\overline{\Omega}}{X_*^+}\right]\right)(t) \tag{42}$$

$$\begin{aligned} C(t)=&\frac{b_2(t)-b_1(t)}{a_1(t)b_2(t)-a_2(t)b_1(t)}\left\{\phi_1^+(t)\,\bar{t}+\frac{1}{2}(\bar{t})^2\theta(t)+\overline{\Phi_1^+(t)}+t\,\overline{\Theta(t)}+\hbar\cdot 4P_{\kappa_*}(t)X_*^+(t)\right\} \\ &+\frac{b_2(t)-b_1(t)}{a_1(t)b_2(t)-a_2(t)b_1(t)}X_*^+(t)\left(S_\Gamma^+\left[\frac{-\phi_1^+\bar{\tau}-\frac{1}{2}(\bar{\tau})^2\theta-\overline{\Phi_1^+}-\tau\,\overline{\Theta}}{X_*^+}\right]\right)(t) \end{aligned}$$

$$+X_*^+(t)\left(S_\Gamma^+\left[\frac{\phi_1^-\bar{\tau}+\frac{1}{2}(\bar{\tau})^2\varpi+\overline{\Phi_1^-}+\tau\,\overline{\Omega}}{X_*^+}\right]\right)(t)-4f^-(t) \tag{43}$$

By the fourth equations in (19), we have the following system of equations

$$\begin{cases} A(t_1)\dfrac{1}{\lambda^+}+B(t_1)\dfrac{1}{\lambda^-}=C(t_1) \\ A(t_2)\dfrac{1}{\lambda^+}+B(t_2)\dfrac{1}{\lambda^-}=C(t_2) \end{cases} \tag{44}$$

Furthermore, we obtain

$$\begin{cases} \lambda^+(t)=\dfrac{A(t_1)B(t_2)-B(t_1)A(t_2)}{C(t_1)B(t_2)-B(t_1)C(t_2)} \\ \lambda^-(t)=\dfrac{A(t_1)B(t_2)-B(t_1)A(t_2)}{A(t_1)C(t_2)-C(t_1)A(t_2)} \end{cases} \tag{45}$$

Remark 1 Take $\hbar=1$ while $\kappa_*+1\geqslant 0$. Take $\hbar=0$ while $\kappa_*+1<0$.

Summarizing the above discussion, we have the following result.

Theorem 2 For the I-BVP of (λ, 1) bi-analytic functions, the solutions are given in Theorem 1, (36) and (45), namely

$$\vartheta(t)=\frac{\beta_2(t)-\beta_1(t)}{\gamma_1(t)-\gamma_2(t)}\left\{X^+(t)\left(S_\Gamma^+\left[\frac{\bar{\tau}(G\varpi-\theta)}{X^+}\right]\right)(t)+\hbar\cdot P_{\kappa-1}(t)X^+(t)+\bar{t}\,\varpi(t)\right\},\quad t\in\Gamma,$$

($\hbar=1$ while $\kappa\geqslant 0$, $\hbar=0$ while $\kappa<0$.)

$$\omega(t)=\frac{a_2(t)-a_1(t)}{b_1(t)-b_2(t)}\left\{\frac{\lambda^--1}{4\lambda^-}\left[\phi_1^-(t)\,\bar{t}+\frac{1}{2}(\bar{t})^2\varpi(t)\right]+\frac{\lambda^-+1}{4\lambda^-}\left[\overline{\Phi_1^-(t)}+t\,\overline{\Omega(t)}\right]+\psi^-(t)\right\},\quad t\in\Gamma,$$

$$\lambda^+(t)=\frac{A(t_1)B(t_2)-B(t_1)A(t_2)}{C(t_1)B(t_2)-B(t_1)C(t_2)},$$

$$\lambda^-(t)=\frac{A(t_1)B(t_2)-B(t_1)A(t_2)}{A(t_1)C(t_2)-C(t_1)A(t_2)},$$

where $A(t)$, $B(t)$, $C(t)$ are given in (41), (42), (43), respectively. $\phi_1^\pm(t)$ is from $\phi_1(z)$ given in Lemma 1. $\psi^-(t)$ is from $\psi(z)$ given in Lemma 2. For $\phi_1(z)$ and $\psi(z)$, there exist four cases.

1) When $\kappa\geqslant 0$ and $\kappa_*+1\geqslant 0$, $\phi_1(z)$ and $\psi(z)$ are given in (10) and (33), respectively.

2) When $\kappa\geqslant 0$ and $\kappa_*+1<0$, $\phi_1(z)$ is given in (10) and $\psi(z)$ is given in (33)($P_{\kappa_*}\equiv 0$) if and only if the condition (35) is satisfied.

3) When $\kappa<0$ and $\kappa_*+1\geqslant 0$, $\phi_1(z)$ is given in (12) if and only if the condition (13) is satisfied and $\psi(z)$ is given in (33).

4) When $\kappa<0$ and $\kappa_*+1<0$, $\phi_1(z)$ and $\psi(z)$ are given in (12) and (33)($P_{\kappa_*}\equiv 0$) if and only if the conditions (13) and (35) are satisfied, respectively.

Remark 2 The upper conclusions may be applied to interface problems of the elastic system in plane, for instance, the welding problems and the quasi-static system of thermoelasticity, etc.Refs.[9-14].

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