Wuhan University Journal of Natural Sciences

2023, Vol.28 No.3, 192-200

Article ID 1007-1202(2023)03-0192-09

DOI https://doi.org/10.1051/wujns/2023283192

The Number of Perfect Matchings in (3,6)-Fullerene

YANG Rui, YUAN Mingzhu

School of Mathematics and Information Science, Henan Polytechnic University, Jiaozuo 454003, Henan, China

Abstract: A $(3,6)$-fullerene is a connected cubic plane graph whose faces are only triangles and hexagons, and has the connectivity 2 or 3. The $(3,6)$-fullerenes with connectivity 2 are the tubes consisting of l concentric hexagonal layers such that each layer consists of two hexagons, capped on each end by two adjacent triangles, denoted by $T_l(l \geqslant 1)$. A $(3,6)$-fullerene T_l with n vertices has exactly $2^{\frac{n}{4}}+1$ perfect matchings. The structure of a $(3,6)$-fullerene G with connectivity 3 can be determined by only three parameters r, s and t, thus we denote it by $G=(r,s,t)$, where r is the radius (number of rings), s is the size (number of spokes in each layer, $s \geqslant 4$, s is even), and t is the torsion ($0 \leqslant t < s$, $t \equiv r \bmod 2$). In this paper, the counting formula of the perfect matchings in $G=(n+1,4,t)$ is given, and the number of perfect matchings is obtained. Therefore, the correctness of the conclusion that every bridgeless cubic graph with p vertices has at least $2^{\frac{p}{3656}}$ perfect matchings proposed by Esperet *et al* is verified for $(3,6)$-fullerene $G=(n+1,4,t)$.

Key words: perfect matching; $(3,6)$-fullerene graph; recurrence relation; counting formula

CLC number: O157.6

0 Introduction

Let G be a graph. A perfect matching of G is also called a Kekulé structure in mathematical chemistry. Since the study of Kekulé structure of molecules helps to study the structural stability, aromaticity and other chemical properties of molecules, chemists and graph theorists have paid extensive attention to Kekulé structure and its related properties[1]. For example, studying the sextet pattern of Kekulé structures in hydrocarbons can predict the relative stability and aromaticity of compounds[2], and calculating the number of Kekulé structures in hydrocarbons can estimate the resonance energy of compounds[3]. Therefore, it is of great theoretical significance to study the enumeration of perfect matchings of molecular graphs. Hence the enumeration problem for perfect matchings has played an important role in the chemical graph theory. For general graphs, the problem is NP-hard even in bipartite graphs[4]. Therefore, it is very difficult to obtain the formula of the perfect matchings in a graph.

A $(k,6)$-fullerene $(k \geqslant 3)$ is a 3-connected cubic planar graph whose faces are only k-length or hexagons. Došlić[5] showed that $(k,6)$-fullerene only exists for $k=3$, 4 and 5. A $(5,6)$-fullerene is the ordinary carbon fullerene molecular graph for which the problem of establishing an exponential lower bound on the number of perfect matchings for all fullerene graphs was settled by

Received date: 2022-11-20

Foundation item: Supported by National Natural Science Foundation of China (11801148, 11801149 and 11626089) and the Foundation for the Doctor of Henan Polytechnic University (B2014-060)

Biography: YANG Rui, female, Ph. D., Associate professor, research direction: graph theory and its application. E-mail: yangrui@hpu.edu.cn

Kardoš *et al*[6]. However, there is no systematic study on the number of perfect matchings in $(4,6)$-fullerenes and $(3,6)$-fullerenes. In the 1970s, Lovász and Plummer[7] conjectured that the number of perfect matchings of a cubic bridgeless graph G should grow exponentially with its order. It was solved in the affirmative by Esperet *et al*[8]. That is, every cubic bridgeless graph G with n vertices has at least $2^{n/3656}$ perfect matchings. And many scholars have given the enumeration formulas of perfect matchings for some graphs with special structures[9,10].

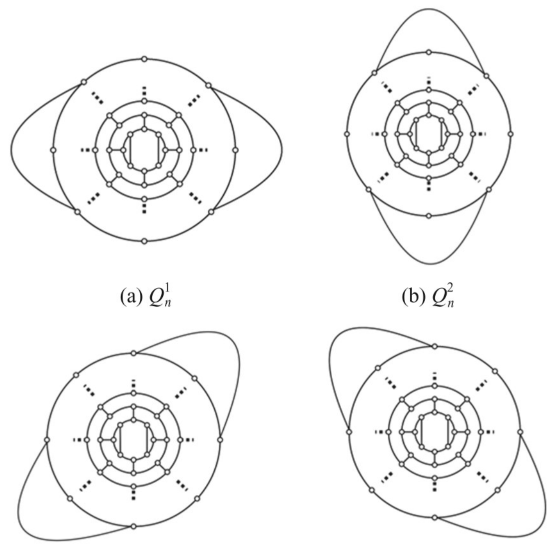
A $(3,6)$-fullerene graph is a plane cubic graph whose faces are only triangles and hexagons. It is known that a $(3,6)$-fullerene graph is 1-extendable[7] and has the connectivity 2 or 3. The $(3,6)$-fullerenes with connectivity 2 are the tubes consisting of $l\,(l\geqslant 1)$ concentric hexagonal layers such that each layer consists of two hexagons, capped on each end by two adjacent triangles, denoted by $T_l(l\geqslant 1)$. And a $(3,6)$-fullerene T_l with n vertices has exactly $2^{\frac{n}{4}}+1$ perfect matchings [11-13]. The structure of a $(3,6)$-fullerene G with connectivity 3 can be determined by only three parameters r, s and t, thus we denote it by $G=(r,s,t)$, where r is the radius (number of rings), s is the size (number of spokes in each layer, $s\geqslant 4$, s is even), and t is the torsion $(0\leqslant t<s,\ t\equiv r \bmod 2)$[14-16]. A set of edges M of a graph G is called a *matching* if no two edges of M have a vertex in common. A *perfect matching* of a graph G is a matching M that covers all vertices of G [17-19]. Let G be a graph with perfect matchings. If two perfect matchings M_1 and M_2 of G have a different edge, then M_1 and M_2 are said to be two different perfect matchings of G. A graph G is called a *planar graph* if G can be drawn in the plane so that its edges intersect only at their ends. Such a drawing is called a *planar embedding* of G in the plane. If a graph contains a vertex whose degree is exactly one, then such a vertex is called a *pendant vertex* of the graph [18,19].

The paper is organized as follows. In Section 1, we recall some notions, definitions and lemmas which will be used throughout the paper. In Section 2, we give the counting formulas of the perfect matchings in $G=(n+1,4,t)$.

1 Preliminaries

Let $G=(n+1,s,t)$ be a 3-connected $(3,6)$-fullerene graph, where $s=4$, $0\leqslant t<4$, $t\equiv(n+1)\bmod 2$. That is, G is a tubular $(3,6)$-fullerene consisting of n concentric hexagonal layers such that each layer consists of four hexagons, capped on each end by two opposite triangles and a hexagon. Since every simple 3-connected planar graph has a unique planar embedding, for the sake of simplification, G also represents its planar embedding graph. Since $0\leqslant t<4$, $t\equiv(n+1)\bmod 2$, G can be classified: (i) If n is odd and $t=0$, then $G\cong Q_n^1$ (see Fig. 1(a)); (ii) If n is odd and $t=2$, then $G\cong Q_n^2$ (see Fig. 1(b)); (iii) If n is even and $t=1$, then $G\cong Q_n^3$ (see Fig. 1(c)); (iv) If n is even and $t=3$, then $G\cong Q_n^4$ (see Fig. 1(d)). It is easy to verify that $Q_n^3\cong Q_n^4$. Therefore, there are three types of $G=(n+1,4,t)$ in the sense of isomorphism.

Proposition 1 If $G=(n+1,4,t)$ is a $(3,6)$-fullerene, then G is isomorphic to Q_n^1, Q_n^2, or Q_n^3(see Fig. 1).

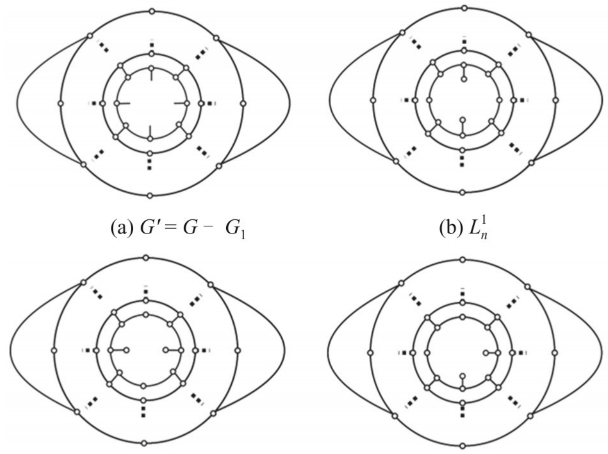


(a) Q_n^1 (b) Q_n^2

(c) Q_n^3 (d) Q_n^4

Fig. 1 A $(3,6)$-fullerene $G=(n+1,4,t)$

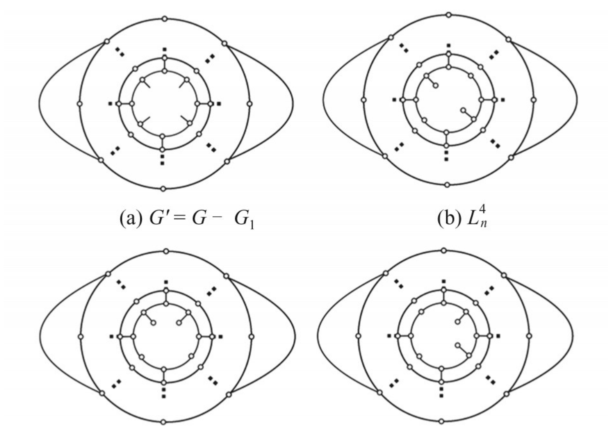
Let $G=(n+1,4,t)$. Denote by $C_1, C_2, \cdots, C_{n+1}$ the $n+1$ concentric rings of G from the inside to outside. The edge that is not on the concentric rings is called *transversal edge*. Let $G'=G-C_1$, then there are exactly one 8-length face, two 3-length faces, and the rest are 6-length faces in G'. Let f be the 8-length face in G', then there are four vertices of degree 2 and four vertices of degree 3 appearing on the boundary of f alternately. Denote by u, v the any two vertices of degree 2 on the boundary of f. Let u be adjacent to a pendant vertex, say u', and v adjacent to a pendant vertex, say v', such that $u'\neq v'$. Let $L_n=G'\cup\{uu', vv'\}$, then L_n is a subgraph of G. If n is odd, by Proposition 1, then G is isomorphic to Q_n^1 or Q_n^2. Therefore, $G'=G-C_1$ is isomorphic to the graph shown in Fig. 2(a). Thus, L_n is isomorphic to L_n^1, L_n^2, or L_n^3 according to the position of u', v' (see Fig. 2).



(a) $G' = G - G_1$ (b) L_n^1 (c) L_n^2 (d) L_n^3

Fig. 2 G', $L_n^i(i=1,2,3)$ when n is odd

Similarly, if n is even, by Proposition 1, then $G \cong Q_n^3$. Therefore, $G' = G - C_1$ is isomorphic to the graph shown in Fig. 3(a). Thus, L_n is isomorphic to L_n^4, L_n^5, or L_n^6 according to the position of u', v' (see Fig. 3).



(a) $G' = G - G_1$ (b) L_n^4 (c) L_n^5 (d) L_n^6

Fig. 3 G', $L_n^i(i=4,5,6)$ when n is even

Next, we give the number of perfect matchings in $L_n^i(i=1,2,\cdots,6)$.

Lemma 1 Let $G=(n+1,4,t)$ be a $(3,6)$-fullerene, where $0 \leqslant t < 4$, $t \equiv (n+1) \bmod 2$. Let L_n be a subgraph of G formed by removing the inner cap of G and adding two pendant vertices (see Figs. 2, 3 the subgraphs L_n^i, $i=1,2,\cdots,6$). Denote by $f_i(n)$ the number of perfect matchings in $L_n^i(i=1,2,\cdots,6)$.

1) If n is odd and $L_n \cong L_n^1$, then

$$f_1(n) = -\frac{\sqrt{2}}{4}\left(2-\sqrt{2}\right)^n + \left(\frac{1}{2}+\frac{\sqrt{2}}{2}\right)\left(2+\sqrt{2}\right)^{n-1} - 2^{\frac{n-1}{2}}.$$

2) If n is odd and $L_n \cong L_n^2$, then

$$f_2(n) = -\frac{\sqrt{2}}{4}\left(2-\sqrt{2}\right)^n + \left(\frac{1}{2}+\frac{\sqrt{2}}{2}\right)\left(2+\sqrt{2}\right)^{n-1} + 2^{\frac{n-1}{2}}.$$

3) If n is odd and $L_n \cong L_n^3$, then

$$f_3(n) = \frac{1}{4}\left(2-\sqrt{2}\right)^n + \frac{1}{4}\left(2+\sqrt{2}\right)^n.$$

4) If n is even and $L_n \cong L_n^4$, then

$$f_4(n) = \left(\frac{1}{2}-\frac{\sqrt{2}}{2}\right)\left(2-\sqrt{2}\right)^{n-1} + \left(2+\frac{3\sqrt{2}}{2}\right)\left(2+\sqrt{2}\right)^{n-2}.$$

5) If n is even and $L_n \cong L_n^5$, then

$$f_5(n) = \left(\frac{1}{2}-\frac{\sqrt{2}}{4}\right)\left(2-\sqrt{2}\right)^{n-1} + \left(\frac{3}{2}+\sqrt{2}\right)\left(2+\sqrt{2}\right)^{n-2} - 2^{\frac{n-2}{2}}.$$

6) If n is even and $L_n \cong L_n^6$, then

$$f_6(n) = \left(\frac{1}{2}-\frac{\sqrt{2}}{4}\right)\left(2-\sqrt{2}\right)^{n-1} + \left(\frac{3}{2}+\sqrt{2}\right)\left(2+\sqrt{2}\right)^{n-2} + 2^{\frac{n-2}{2}}.$$

Proof Let the planar embedding of G be shown in Fig. 1. Let $\mathcal{M}_i$ be the perfect matchings set of $L_n^i(i=1,2,\cdots,6)$. Denote by $C_2, C_3, \cdots, C_{n+1}$ the n concentric rings of L_n from the inside to outside. Denote by $u_{i1}, u_{i2}, \cdots, u_{i8}$ the vertices of C_i along the clockwise direction of C_i such that u_{ij} and $u_{i,j+4}$ are arranged on the same line for $i=2,3,\cdots,n+1$, $j=1,2,3,4$.

If $L_n \cong L_n^1$, then denote by u_{13}, u_{17} the two pendant vertices such that u_{13} and u_{17} are adjacent to u_{23} and u_{27} respectively. Then the labelling of the vertices of the graph L_n^1 is shown in Fig. 4. Thus, $\mathcal{M}_1$ can be expressed as four types: denote by $\mathcal{M}_1^1$ the perfect matchings set containing edges $u_{22}u_{32}, u_{24}u_{34}$; denote by $\mathcal{M}_1^2$ the perfect matchings set containing edges $u_{22}u_{32}, u_{26}u_{36}$; denote by $\mathcal{M}_1^3$ the perfect matchings set containing edges $u_{28}u_{38}$, $u_{24}u_{34}$; denote by $\mathcal{M}_1^4$ the perfect matchings set containing edges $u_{28}u_{38}$, $u_{26}u_{36}$. Thus $\mathcal{M}_1^i \cap \mathcal{M}_1^j = \varnothing \left(1 \leqslant i < j \leqslant 4\right)$, $\mathcal{M}_1 = \bigcup_{i=1}^{4} \mathcal{M}_1^i$. That is, $|\mathcal{M}_1| = \sum_{i=1}^{4} |\mathcal{M}_1^i|$. Since $|\mathcal{M}_1^2| = |\mathcal{M}_1^3| = f_4(n-1)$, $|\mathcal{M}_1^1| = |\mathcal{M}_1^4| = f_5(n-1)$, $f_1(n) =$

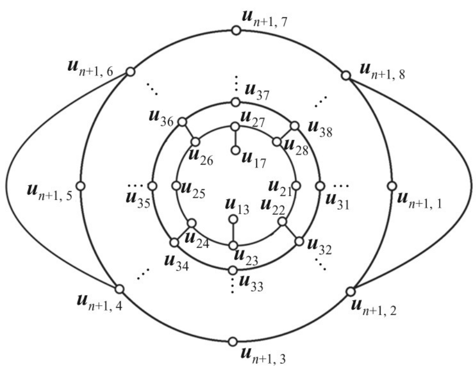


Fig. 4 The labelling of L_n^1

$\sum_{i=1}^{4}|\mathcal{M}_1^i|$, then

$$f_1(n)=2f_4(n-1)+2f_5(n-1) \tag{1}$$

If $L_n \cong L_n^2$, then denote by u_{11}, u_{15} the two pendant vertices such that u_{11} and u_{15} are adjacent to u_{21} and u_{25} respectively. Similarly, denote by $\mathcal{M}_2^1$ the perfect matchings set containing edges $u_{22}u_{32}, u_{26}u_{36}$; denote by $\mathcal{M}_2^2$ the perfect matchings set containing edges $u_{22}u_{32}, u_{28}u_{38}$; denote by $\mathcal{M}_2^3$ the perfect matchings set containing edges $u_{24}u_{34}, u_{26}u_{36}$; denote by $\mathcal{M}_2^4$ the perfect matchings set containing edges $u_{24}u_{34}, u_{28}u_{38}$. Thus $\mathcal{M}_2^i \cap \mathcal{M}_2^j = \varnothing\,(1 \leqslant i < j \leqslant 4)$, $\mathcal{M}_2 = \bigcup_{i=1}^{4}\mathcal{M}_2^i$. That is, $|\mathcal{M}_2| = \sum_{i=1}^{4}|\mathcal{M}_2^i|$. Since $|\mathcal{M}_2^1| = |\mathcal{M}_2^4| = f_4(n-1)$, $|\mathcal{M}_2^2| = |\mathcal{M}_2^3| = f_6(n-1)$, $f_2(n) = \sum_{i=1}^{4}|\mathcal{M}_2^i|$, then

$$f_2(n)=2f_4(n-1)+2f_6(n-1) \tag{2}$$

If $L_n \cong L_n^3$, then denote by u_{11}, u_{13} the two pendant vertices such that u_{11} and u_{13} are adjacent to u_{21} and u_{23} respectively. Similarly, denote by $\mathcal{M}_3^1$ the perfect matchings set containing edges $u_{22}u_{32}, u_{24}u_{34}$; denote by $\mathcal{M}_3^2$ the perfect matchings set containing edges $u_{22}u_{32}, u_{26}u_{36}$; denote by $\mathcal{M}_3^3$ the perfect matchings set containing edges $u_{22}u_{32}, u_{28}u_{38}$. Thus $\mathcal{M}_3^i \cap \mathcal{M}_3^j = \varnothing\,(1 \leqslant i < j \leqslant 3)$, $\mathcal{M}_3 = \bigcup_{i=1}^{3}\mathcal{M}_3^i$. That is, $|\mathcal{M}_3| = \sum_{i=1}^{3}|\mathcal{M}_3^i|$. Since $|\mathcal{M}_3^1| = f_5(n-1)$, $|\mathcal{M}_3^2| = f_4(n-1)$, $|\mathcal{M}_3^3| = f_6(n-1)$, $f_3(n) = \sum_{i=1}^{3}|\mathcal{M}_3^i|$, then

$$f_3(n)=f_4(n-1)+f_5(n-1)+f_6(n-1) \tag{3}$$

If $L_n \cong L_n^4$, then denote by u_{12}, u_{16} the two pendant vertices such that u_{12} and u_{16} are adjacent to u_{22} and u_{26} respectively. Similarly, denote by $\mathcal{M}_4^1$ the perfect matchings set containing edges $u_{23}u_{33}, u_{27}u_{37}$; denote by $\mathcal{M}_4^2$ the perfect matchings set containing edges $u_{23}u_{33}, u_{21}u_{31}$; denote by $\mathcal{M}_4^3$ the perfect matchings set containing edges $u_{25}u_{35}, u_{27}u_{37}$; denote by $\mathcal{M}_4^4$ the perfect matchings set containing edges $u_{25}u_{35}, u_{21}u_{31}$. Thus $\mathcal{M}_4^i \cap \mathcal{M}_4^j = \varnothing\,(1 \leqslant i < j \leqslant 4)$, $\mathcal{M}_4 = \bigcup_{i=1}^{4}\mathcal{M}_4^i$. That is, $|\mathcal{M}_4| = \sum_{i=1}^{4}|\mathcal{M}_4^i|$. Since $|\mathcal{M}_4^1| = f_1(n-1)$, $|\mathcal{M}_4^4| = f_2(n-1)$, $|\mathcal{M}_4^2| = |\mathcal{M}_4^3| = f_3(n-1)$, $f_4(n) = \sum_{i=1}^{4}|\mathcal{M}_4^i|$, then

$$f_4(n)=f_1(n-1)+f_2(n-1)+2f_3(n-1) \tag{4}$$

If $L_n \cong L_n^5$, then denote by u_{16}, u_{18} the two pendant vertices such that u_{16} and u_{18} are adjacent to u_{26} and u_{28} respectively. Similarly, denote by $\mathcal{M}_5^1$ the perfect matchings set containing edges $u_{27}u_{37}, u_{21}u_{31}$; denote by $\mathcal{M}_5^2$ the perfect matchings set containing edges $u_{27}u_{37}, u_{23}u_{33}$; denote by $\mathcal{M}_5^3$ the perfect matchings set containing edges $u_{27}u_{37}, u_{25}u_{35}$. Thus $\mathcal{M}_5^i \cap \mathcal{M}_5^j = \varnothing\,(1 \leqslant i < j \leqslant 3)$, $\mathcal{M}_5 = \bigcup_{i=1}^{3}\mathcal{M}_5^i$. That is, $|\mathcal{M}_5| = \sum_{i=1}^{3}|\mathcal{M}_5^i|$. Since $|\mathcal{M}_5^2| = f_1(n-1)$, $|\mathcal{M}_5^1| = |\mathcal{M}_5^3| = f_3(n-1)$, $f_5(n) = \sum_{i=1}^{3}|\mathcal{M}_5^i|$, then

$$f_5(n)=f_1(n-1)+2f_3(n-1) \tag{5}$$

If $L_n \cong L_n^6$, then denote by u_{12}, u_{18} the two pendant vertices such that u_{12} and u_{18} are adjacent to u_{22} and u_{28} respectively. Similarly, denote by $\mathcal{M}_6^1$ the perfect matchings set containing edges $u_{21}u_{31}$, $u_{23}u_{33}$; denote by $\mathcal{M}_6^2$ the perfect matchings set containing edges $u_{21}u_{31}, u_{25}u_{35}$; denote by $\mathcal{M}_6^3$ the perfect matchings set containing edges $u_{21}u_{31}, u_{27}u_{37}$. Thus $\mathcal{M}_6^i \cap \mathcal{M}_6^j = \varnothing\,(1 \leqslant i < j \leqslant 3)$, $\mathcal{M}_6 = \bigcup_{i=1}^{3}\mathcal{M}_6^i$. That is, $|\mathcal{M}_6| = \sum_{i=1}^{3}|\mathcal{M}_6^i|$. Since $|\mathcal{M}_6^2| = f_2(n-1)$, $|\mathcal{M}_6^1| = |\mathcal{M}_6^3| = f_3(n-1)$, $f_6(n) = \sum_{i=1}^{3}|\mathcal{M}_6^i|$, then

$$f_6(n)=f_2(n-1)+2f_3(n-1) \tag{6}$$

According to (1)-(6), we can obtain

$$f_1(n)=4f_1(n-2)+2f_2(n-2)+8f_3(n-2) \tag{7}$$

$$f_2(n)=2f_1(n-2)+4f_2(n-2)+8f_3(n-2) \tag{8}$$

$$f_3(n)=2f_1(n-2)+2f_2(n-2)+6f_3(n-2) \tag{9}$$

By the definition of $f_i(1)$, we can get $f_1(1) = 0$, $f_2(1) = 2$ and $f_3(1) = 1$. Equations (7) and (8) imply that

$$f_2(n)=f_1(n)+2^{\frac{n+1}{2}} \tag{10}$$

According to (10), combining (8) with (9), we can obtain

$$f_2(n)=6f_2(n-2)+8f_3(n-2)-2^{\frac{n+1}{2}} \tag{11}$$

$$f_3(n)=4f_2(n-2)+6f_3(n-2)-2^{\frac{n+1}{2}} \tag{12}$$

Then it follows that $f_2(n)+\sqrt{2}\,f_3(n)=\left(6+4\sqrt{2}\right)$

$\left[f_2(n-2)+\sqrt{2}\,g(n-2)\right]-\left(1+\sqrt{2}\right)\cdot 2^{\frac{n+1}{2}}$. Let $f(n)=f_2(n)+\sqrt{2}\,f_3(n)$, where n is odd. Then $f(1)=f_2(1)+\sqrt{2}\,f_3(1)=2+\sqrt{2}$. By canceling the items, we can get $f(n)=\left(1+\sqrt{2}\right)\left(2+\sqrt{2}\right)^{n-1}+2^{\frac{n-1}{2}}$. That is,

$$f_2(n)+\sqrt{2}\,f_3(n)=\left(1+\sqrt{2}\right)\left(2+\sqrt{2}\right)^{n-1}+2^{\frac{n-1}{2}} \quad (13)$$

Equations (12) and (13) imply that $f_3(n)$ satisfies the recurrence relation

$$f_3(n)=\left(6-4\sqrt{2}\right)f_3(n-2)+\left(4+4\sqrt{2}\right)\left(2+\sqrt{2}\right)^{n-3} \quad (14)$$

The homogeneous relation is $f_3(n)=\left(6-4\sqrt{2}\right)f_3(n-2)$, and the characteristic roots are $q_1=2-\sqrt{2}$, $q_2=\sqrt{2}-2$. Therefore, the general solution is

$$f_3(n)=c_1\left(2-\sqrt{2}\right)^n+c_2\left(\sqrt{2}-2\right)^n.$$

We now seek a particular solution of the recurrence relation (14). We try $f_3(n)=p\left(2+\sqrt{2}\right)^n$ as a particular solution. Substituting $f_3(n)$ into (14), we now get

$$p\left(2+\sqrt{2}\right)^n=\left(6-4\sqrt{2}\right)\cdot p\cdot\left(2+\sqrt{2}\right)^{n-2}+\left(4+4\sqrt{2}\right)\left(2+\sqrt{2}\right)^{n-3},$$

the above equation gives $p=\frac{1}{4}$. Hence

$$f_3(n)=c_1\left(2-\sqrt{2}\right)^n+c_2\left(\sqrt{2}-2\right)^n+\frac{1}{4}\left(2+\sqrt{2}\right)^n \quad (15)$$

is a solution for each choice of constants c_1 and c_2. To satisfy the initial condition $f_3(1)=1$, then $c_1-c_2=\frac{1}{4}$. Since n is odd, then $\left(\sqrt{2}-2\right)^n=-\left(2-\sqrt{2}\right)^n$. Substituting it into (15), we get

$$f_3(n)=\frac{1}{4}\left(2-\sqrt{2}\right)^n+\frac{1}{4}\left(2+\sqrt{2}\right)^n \quad (16)$$

where n is odd. Combining (13) with (16), we obtain

$$f_2(n)=-\frac{\sqrt{2}}{4}\left(2-\sqrt{2}\right)^n+\left(\frac{1}{2}+\frac{\sqrt{2}}{2}\right)\left(2+\sqrt{2}\right)^{n-1}+2^{\frac{n-1}{2}} \quad (17)$$

where n is odd. Combining (10) with (17), we obtain

$$f_1(n)=-\frac{\sqrt{2}}{4}\left(2-\sqrt{2}\right)^n+\left(\frac{1}{2}+\frac{\sqrt{2}}{2}\right)\left(2+\sqrt{2}\right)^{n-1}-2^{\frac{n-1}{2}} \quad (18)$$

where n is odd. Substituting formulas (16), (17) and (18) into formulas (4), (5), (6), respectively, we conclude that

$$f_4(n)=\left(\frac{1}{2}-\frac{\sqrt{2}}{2}\right)\left(2-\sqrt{2}\right)^{n-1}+\left(2+\frac{3\sqrt{2}}{2}\right)\left(2+\sqrt{2}\right)^{n-2},$$

$$f_5(n)=\left(\frac{1}{2}-\frac{\sqrt{2}}{4}\right)\left(2-\sqrt{2}\right)^{n-1}+\left(\frac{3}{2}+\sqrt{2}\right)\left(2+\sqrt{2}\right)^{n-2}-2^{\frac{n-2}{2}},$$

$$f_6(n)=\left(\frac{1}{2}-\frac{\sqrt{2}}{4}\right)\left(2-\sqrt{2}\right)^{n-1}+\left(\frac{3}{2}+\sqrt{2}\right)\left(2+\sqrt{2}\right)^{n-2}+2^{\frac{n-2}{2}},$$

where n is even.

2 Main Results

By Proposition 1 and Lemma 1, we can get the number of perfect matchings in $G=(n+1,4,t)$.

Theorem 1 Let $G=(n+1,4,t)$ be a $(3,6)$-fullerene, where $0\leqslant t<4, t\equiv(n+1)\bmod 2$. When n is odd and $t=0$, denote by $N(n)$ the number of perfect matchings of G. Then

$$N(n)=1+2^{n+1}+2^{\frac{n+1}{2}}+\left(3+2\sqrt{2}\right)\left(2+\sqrt{2}\right)^{n-1}+\frac{1}{2}\left(2-\sqrt{2}\right)^{n+1} \quad (19)$$

Proof When n is odd and $t=0$, by Proposition 1, then $G\cong Q_n^1$ (see Fig. 1(a)). Denote by $C_1, C_2, \ldots, C_{n+1}$ the $n+1$ concentric rings of G from the inside to outside. Denote by $u_{i1}, u_{i2}, \cdots, u_{i8}$ the vertices of C_i along the clockwise direction of C_i such that u_{ij} and $u_{i,j+4}$ are arranged on the same line for $i=1,2,\cdots,n+1$, $j=1,2,3,4$. Let $E_1=\{u_{11}u_{21}, u_{13}u_{23}, u_{15}u_{25}, u_{17}u_{27}\}$ be a set of transversal edges (see Fig. 5).

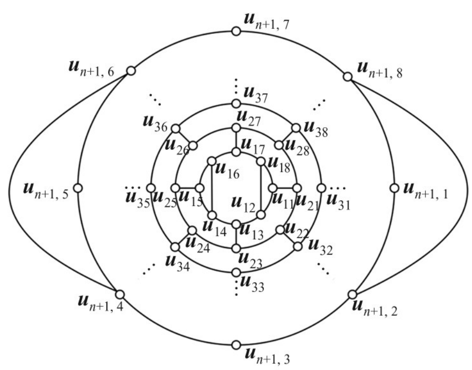


Fig. 5 The labeled graph Q_n^1

Let $\mathcal{M}$ be the perfect matchings set of G and $\mathcal{N}_i$ the perfect matchings set of G containing i edges in E_1. Thus $\mathcal{N}_i \cap \mathcal{N}_j = \varnothing\left(0 \leqslant i < j \leqslant 4\right)$, $\mathcal{M} = \bigcup_{i=0}^{4} \mathcal{N}_i$. That is, $|\mathcal{M}| = \sum_{i=0}^{4} |\mathcal{N}_i|$.

Since any perfect matching in $\mathcal{N}_0$ does not contain any edge of E_1, any perfect matching $M \in \mathcal{N}_0$, the edges of M must belong to $E\left(C_i\right)(1 \leqslant i \leqslant n+1)$. Thus there are two matching methods covering all vertices on $C_i(1 \leqslant i \leqslant n+1)$. According to the fractional multiplication, we can get $|\mathcal{N}_0| = 2^{n+1}$.

For $\mathcal{N}_1$, we select arbitrarily any traversal edge of E_1 as matched edge which can not be extended to a perfect matching of G as at least one vertex on C_1 is not covered, that is, there is no perfect matching in $\mathcal{N}_1$; Similarly, there is no perfect matching in $\mathcal{N}_3$. Thus $|\mathcal{N}_1| = |\mathcal{N}_3| = 0$.

For $\mathcal{N}_4$, there is exactly one perfect matching, that is, $|\mathcal{N}_4| = 1$.

For $\mathcal{N}_2$, $\mathcal{N}_2$ can be expressed as six types: denote by $\mathcal{N}_2^1$ the perfect matchings set containing edges $u_{11}u_{21}$, $u_{13}u_{23}$; denote by $\mathcal{N}_2^2$ the perfect matchings set containing edges $u_{11}u_{21}$, $u_{15}u_{25}$; denote by $\mathcal{N}_2^3$ the perfect matchings set containing edges $u_{11}u_{21}, u_{17}u_{27}$; denote by $\mathcal{N}_2^4$ the perfect matchings set containing edges $u_{13}u_{23}, u_{15}u_{25}$; denote by $\mathcal{N}_2^5$ the perfect matchings set containing edges $u_{13}u_{23}, u_{17}u_{27}$; denote by $\mathcal{N}_2^6$ the perfect matchings set containing edges $u_{15}u_{25}, u_{17}u_{27}$. Thus $\mathcal{N}_2^i \cap \mathcal{N}_2^j = \varnothing\left(1 \leqslant i < j \leqslant 6\right)$, $\mathcal{N}_2 = \bigcup_{i=1}^{6} \mathcal{N}_2^i$. That is, $|\mathcal{N}_2| = \sum_{i=1}^{6} |\mathcal{N}_2^i|$.

Next we give the proof idea for finding the number of perfect matchings of $\mathcal{N}_2$. Let L_n be a subgraph of G formed by removing the inner cap of G and adding two pendant vertices. Since n is odd, $L_n \cong L_n^i(i=1,2,3)$. By Lemma 1, we know the number of perfect matchings in L_n. On the other hand, we can calculate the number of perfect matchings in $G-L_n$. Thus the number of perfect matchings of $\mathcal{N}_2^i$ can be obtained by using the fractional multiplication.

Let $M \in \mathcal{N}_2^1$, that is, M contains edges $u_{11}u_{21}, u_{13}u_{23}$. Then M can be divided into two parts: one part is the matching edges restricted in $L_n = G - C_1 + u_{11}u_{21} + u_{13}u_{23}$, and the other part is the matching edges in $G-L_n$. From the fractional multiplication, we can get $|\mathcal{N}_2^1| = |\mathcal{M}(L_n)| \times |\mathcal{M}(G-L_n)|$, where $|\mathcal{M}(L_n)|$ and $|\mathcal{M}(G-L_n)|$ represent the number of perfect matchings of L_n and $G-L_n$, respectively. By definition $L_n \cong L_n^3$, then $|\mathcal{M}(L_n)| = |\mathcal{M}(L_n^3)| = f_3(n)$, $|\mathcal{M}(G-L_n)| = 1$. Thus, $|\mathcal{N}_2^1| = f_3(n)$. Similarly, we can get $|\mathcal{N}_2^1| = |\mathcal{N}_2^3| = |\mathcal{N}_2^4| = |\mathcal{N}_2^6| = f_3(n)$. Let $M \in \mathcal{N}_2^2$, that is, M contains edges $u_{11}u_{21}$, $u_{15}u_{25}$. Then M can be divided into two parts: one part is the matching edges restricted in $L_n = G - C_1 + u_{11}u_{21} + u_{15}u_{25}$, and the other part is the matching edges in $G-L_n$. From the fractional multiplication, we can get $|\mathcal{N}_2^2| = |\mathcal{M}(L_n)| \times |\mathcal{M}(G-L_n)|$. By definition $L_n \cong L_n^2$, then $|\mathcal{M}(L_n)| = |\mathcal{M}(L_n^2)| = f_2(n)$, $|\mathcal{M}(G-L_n)| = 2$. Thus, $|\mathcal{N}_2^2| = 2f_2(n)$. Let $M \in \mathcal{N}_2^5$, that is, M contains edges $u_{13}u_{23}, u_{17}u_{27}$. Then M can be divided into two parts: one part is the matching edges restricted in $L_n = G - C_1 + u_{13}u_{23} + u_{17}u_{27}$, and the other part is the matching edges in $G-L_n$. By definition $L_n \cong L_n^1$, then $|\mathcal{M}(G-L_n)| = 0$. Thus, $|\mathcal{N}_2^5| = 0$.

Therefore, $|\mathcal{N}_2| = \sum_{i=1}^{6} |\mathcal{N}_2^i| = 2f_2(n) + 4f_3(n)$.

By Lemma 1, we can get the number of perfect matchings in G.

$$N(n) = |\mathcal{M}| = \sum_{i=0}^{4} |\mathcal{N}_i| = 1 + 2^{n+1} + 2f_2(n) + 4f_3(n)$$

$$= 1 + 2^{n+1} + 2^{\frac{n+1}{2}} + (3 + 2\sqrt{2})(2 + \sqrt{2})^{n-1} + \frac{1}{2}(2 - \sqrt{2})^{n+1}$$

Thus Theorem 1 is proved. On the other hand, we can verify that there are 13 perfect matchings in $G = (2,4,0)$ and 89 perfect matchings in $G = (4,4,0)$ by a simple calculation, which is consistent with the result calculated by formula (19).

Theorem 2 Let $G = (n+1,4,t)$ be a $(3,6)$-fullerene, where $0 \leqslant t < 4, t \equiv (n+1) \bmod 2$. When n is odd and $t=2$, denote by $N(n)$ the number of perfect matchings of G. Then

$$N(n) = 1 + 2^{n+1} - 2^{\frac{n+1}{2}} + \left(3 + 2\sqrt{2}\right)\left(2 + \sqrt{2}\right)^{n-1} + \frac{1}{2}\left(2 - \sqrt{2}\right)^{n+1} \tag{20}$$

Proof When n is odd and $t=2$, by Proposition 1, then $G \cong Q_n^2$ (see Fig. 1(b)). Denote by $C_1, C_2, \cdots, C_{n+1}$ the $n+1$ concentric rings of G from the inside to outside. Denote by $u_{i1}, u_{i2}, \cdots, u_{i8}$ the vertices of C_i along the clockwise direction of C_i such that u_{ij} and $u_{i,j+4}$ are arranged on the same line for $i = 1, 2, \cdots, n+1$, $j = 1, 2, 3, 4$. Let $E_1 = \{u_{11}u_{21}, u_{13}u_{23}, u_{15}u_{25}, u_{17}u_{27}\}$ be a set of transversal edges.

Let $\mathcal{M}$ be the perfect matchings set of G and $\mathcal{N}_i$ the perfect matchings set of G containing i edges in E_1. Thus $\mathcal{N}_i \cap \mathcal{N}_j = \varnothing (0 \leqslant i < j \leqslant 4)$, $\mathcal{M} = \bigcup_{i=0}^{4} \mathcal{N}_i$. That is, $|\mathcal{M}| = \sum_{i=0}^{4} |\mathcal{N}_i|$.

Since any perfect matching in $\mathcal{N}_0$ does not contain any edge of E_1, any perfect matching $M \in \mathcal{N}_0$, the edges of M must belong to $E(C_i)(1 \leqslant i \leqslant n+1)$. Thus there are two matching methods covering all vertices on $C_i(1 \leqslant i \leqslant n+1)$. According to the fractional multiplication, we can get $|\mathcal{N}_0| = 2^{n+1}$.

For $\mathcal{N}_1$, we select arbitrarily any traversal edge of E_1 as matched edge which cannot be extended to a perfect matching of G as at least one vertex on C_1 is not covered, that is, there is no perfect matching in $\mathcal{N}_1$; Similarly, there is no perfect matching in $\mathcal{N}_3$. Thus $|\mathcal{N}_1| = |\mathcal{N}_3| = 0$.

For $\mathcal{N}_4$, there is exactly one perfect matching, that is, $|\mathcal{N}_4| = 1$.

For $\mathcal{N}_2$, $\mathcal{N}_2$ can be expressed as six types: denote by $\mathcal{N}_2^1$ the perfect matchings set containing edges $u_{11}u_{21}$, $u_{13}u_{23}$; denote by $\mathcal{N}_2^2$ the perfect matchings set containing edges $u_{11}u_{21}, u_{15}u_{25}$; denote by $\mathcal{N}_2^3$ the perfect matchings set containing edges $u_{11}u_{21}, u_{17}u_{27}$; denote by $\mathcal{N}_2^4$ the perfect matchings set containing edges $u_{13}u_{23}, u_{15}u_{25}$; denote by $\mathcal{N}_2^5$ the perfect matchings set containing edges $u_{13}u_{23}, u_{17}u_{27}$; denote by $\mathcal{N}_2^6$ the perfect matchings set containing edges $u_{15}u_{25}$, $u_{17}u_{27}$. Thus $\mathcal{N}_2^i \cap \mathcal{N}_2^j = \varnothing (1 \leqslant i < j \leqslant 6)$, $\mathcal{N}_2 = \bigcup_{i=1}^{6} \mathcal{N}_2^i$. That is, $|\mathcal{N}_2| = \sum_{i=1}^{6} |\mathcal{N}_2^i|$.

Next we give the proof idea for finding the number of perfect matchings of $\mathcal{N}_2$. Let L_n be a subgraph of G formed by removing the inner cap of G and adding two pendant vertices. Since n is odd, $L_n \cong L_n^i (i = 1, 2, 3)$. By Lemma 1, we know the number of perfect matchings in L_n. On the other hand, we can calculate the number of perfect matchings in $G - L_n$. Thus the number of perfect matchings of $\mathcal{N}_2^i$ can be obtained by using the fractional multiplication.

Let $M \in \mathcal{N}_2^1$, that is, M contains edges $u_{11}u_{21}, u_{13}u_{23}$. Then M can be divided into two parts: one part is the matching edges restricted in $L_n = G - C_1 + u_{11}u_{21} + u_{13}u_{23}$, and the other part is the matching edges in $G - L_n$. From the fractional multiplication, we can get $|\mathcal{N}_2^1| = |\mathcal{M}(L_n)| \times |\mathcal{M}(G - L_n)|$, where $|\mathcal{M}(L_n)|$ and $|\mathcal{M}(G - L_n)|$ represent the number of perfect matchings of L_n and $G - L_n$, respectively. By definition $L_n \cong L_n^3$, then $|\mathcal{M}(L_n)| = |\mathcal{M}(L_n^3)| = f_3(n)$, $|\mathcal{M}(G - L_n)| = 1$. Thus, $|\mathcal{N}_2^1| = f_3(n)$. Similarly, we can get $|\mathcal{N}_2^1| = |\mathcal{N}_2^3| = |\mathcal{N}_2^4| = |\mathcal{N}_2^6| = f_3(n)$. Let $M \in \mathcal{N}_2^2$, that is, M contains edges $u_{11}u_{21}, u_{15}u_{25}$. Then M can be divided into two parts: one part is the matching edges restricted in $L_n = G - C_1 + u_{11}u_{21} + u_{15}u_{25}$, and the other part is the matching edges in $G - L_n$. From the fractional multiplication, we can get $|\mathcal{N}_2^2| = |\mathcal{M}(L_n)| \times |\mathcal{M}(G - L_n)|$. By definition $L_n \cong L_n^1$, then $|\mathcal{M}(L_n)| = |\mathcal{M}(L_n^1)| = f_1(n)$, $|\mathcal{M}(G - L_n)| = 2$. Thus, $|\mathcal{N}_2^2| = 2f_1(n)$. Let $M \in \mathcal{N}_2^5$, that is, M contains edges $u_{13}u_{23}$, $u_{17}u_{27}$. Then M can be divided into two parts: one part is the matching edges restricted in $L_n = G - C_1 + u_{13}u_{23} + u_{17}u_{27}$, and the other part is the matching edges in $G - L_n$. By definition $L_n \cong L_n^2$, then $|\mathcal{M}(G - L_n)| = 0$. Thus, $|\mathcal{N}_2^5| = 0$.

Therefore, $|\mathcal{N}_2| = \sum_{i=1}^{6} |\mathcal{N}_2^i| = 2f_1(n) + 4f_3(n)$.

By Lemma 1, we can get the number of perfect matchings in G.

$$N(n) = |\mathcal{M}| = \sum_{i=0}^{4} |\mathcal{N}_i| = 1 + 2^{n+1} + 2f_1(n) + 4f_3(n)$$
$$= 1 + 2^{n+1} - 2^{\frac{n+1}{2}} + (3 + 2\sqrt{2})(2 + \sqrt{2})^{n-1} + \frac{1}{2}(2 - \sqrt{2})^{n+1}$$

Thus Theorem 2 is proved. On the other hand, we can verify that there are 9 perfect matchings in $G = (2, 4, 2)$ by a simple calculation, which is consistent with the result calculated by formula (20).

Theorem 3 Let $G = (n+1, 4, t)$ be a $(3,6)$-fullerene, where $0 \leqslant t < 4, t \equiv (n+1) \bmod 2$. When n is even, denote by $N(n)$ the number of perfect matchings of G. Then

$$N(n) = 1 + 2^{n+1} + \left(3 - 2\sqrt{2}\right)\left(2 - \sqrt{2}\right)^{n-1} + \left(10 + 7\sqrt{2}\right)\left(2 + \sqrt{2}\right)^{n-2} \quad (21)$$

Proof When n is even, by Proposition 1, then $G \cong Q_n^3$ (see Fig. 1(c)). Denote by $C_1, C_2, \cdots, C_{n+1}$ the $n+1$ concentric rings of G from the inside to outside. Denote by $u_{i1}, u_{i2}, \cdots, u_{i8}$ the vertices of C_i along the clockwise direction of C_i such that u_{ij} and $u_{i,j+4}$ are arranged on the same line for $i = 1, 2, \cdots, n+1$, $j = 1, 2, 3, 4$. Let $E_1 = \{u_{11}u_{21}, u_{13}u_{23}, u_{15}u_{25}, u_{17}u_{27}\}$ be a set of transversal edges.

Let $\mathcal{M}$ be the perfect matchings set of G and $\mathcal{N}_i$ the perfect matchings set of G containing i edges in E_1. Thus $\mathcal{N}_i \cap \mathcal{N}_j = \varnothing\left(0 \leqslant i < j \leqslant 4\right)$, $\mathcal{M} = \bigcup_{i=0}^{4} \mathcal{N}_i$. That is, $|\mathcal{M}| = \sum_{i=0}^{4} |\mathcal{N}_i|$.

Since any perfect matching in $\mathcal{N}_0$ does not contain any edge of E_1, any perfect matching $M \in \mathcal{N}_0$, the edges of M must belong to $E\left(C_i\right)(1 \leqslant i \leqslant n+1)$. Thus there are two matching methods covering all vertices on $C_i(1 \leqslant i \leqslant n+1)$. According to the fractional multiplication, we can get $|\mathcal{N}_0| = 2^{n+1}$.

For $\mathcal{N}_1$, we select arbitrarily any traversal edge of E_1 as matched edge which can not be extended to a perfect matching of G as at least one vertex on C_1 is not covered, that is, there is no perfect matching in $\mathcal{N}_1$; Similarly, there is no perfect matching in $\mathcal{N}_3$. Thus $|\mathcal{N}_1| = |\mathcal{N}_3| = 0$.

For $\mathcal{N}_4$, there is exactly one perfect matching, that is, $|\mathcal{N}_4| = 1$.

For $\mathcal{N}_2$, $\mathcal{N}_2$ can be expressed as six types: denote by $\mathcal{N}_2^1$ the perfect matchings set containing edges $u_{11}u_{21}$, $u_{13}u_{23}$; denote by $\mathcal{N}_2^2$ the perfect matchings set containing edges $u_{11}u_{21}, u_{15}u_{25}$; denote by $\mathcal{N}_2^3$ the perfect matchings set containing edges $u_{11}u_{21}, u_{17}u_{27}$; denote by $\mathcal{N}_2^4$ the perfect matchings set containing edges $u_{13}u_{23}$, $u_{15}u_{25}$; denote by $\mathcal{N}_2^5$ the perfect matchings set containing edges $u_{13}u_{23}, u_{17}u_{27}$; denote by $\mathcal{N}_2^6$ the perfect matchings set containing edges $u_{15}u_{25}, u_{17}u_{27}$. Thus $\mathcal{N}_2^i \cap \mathcal{N}_2^j = \varnothing\left(1 \leqslant i < j \leqslant 6\right)$, $\mathcal{N}_2 = \bigcup_{i=1}^{6} \mathcal{N}_2^i$. That is, $|\mathcal{N}_2| = \sum_{i=1}^{6} |\mathcal{N}_2^i|$.

Next we give the proof idea for finding the number of perfect matchings of $\mathcal{N}_2$. Let L_n be a subgraph of G formed by removing the inner cap of G and adding two pendant vertices. Since n is even, $L_n \cong L_n^i(i=4,5,6)$. By Lemma 1, we know the number of perfect matchings in L_n. On the other hand, we can calculate the number of perfect matchings in $G-L_n$. Thus the number of perfect matchings of $\mathcal{N}_2^i$ can be obtained by using the fractional multiplication.

Let $M \in \mathcal{N}_2^1$, that is, M contains edges $u_{11}u_{21}, u_{13}u_{23}$. Then M can be divided into two parts: one part is the matching edges restricted in $L_n = G - C_1 + u_{11}u_{21} + u_{13}u_{23}$, and the other part is the matching edges in $G-L_n$. From the fractional multiplication, we can get $|\mathcal{N}_2^1| = \left|\mathcal{M}\left(L_n\right)\right| \times \left|\mathcal{M}\left(G-L_n\right)\right|$, where $\left|\mathcal{M}\left(L_n\right)\right|$ and $\left|\mathcal{M}\left(G-L_n\right)\right|$ represent the number of perfect matchings of L_n and $G-L_n$, respectively. By definition $L_n \cong L_n^5$, then $\left|\mathcal{M}\left(L_n\right)\right| = \left|\mathcal{M}\left(L_n^5\right)\right| = f_5(n)$, $\left|\mathcal{M}\left(G-L_n\right)\right| = 1$. Thus, $|\mathcal{N}_2^1| = f_5(n)$. Similarly, we can get $|\mathcal{N}_2^1| = |\mathcal{N}_2^6| = f_5(n)$. Let $M \in \mathcal{N}_2^2$, that is, M contains edges $u_{11}u_{21}, u_{15}u_{25}$. Then M can be divided into two parts: one part is the matching edges restricted in $L_n = G - C_1 + u_{11}u_{21} + u_{15}u_{25}$, and the other part is the matching edges in $G-L_n$. From the fractional multiplication, we can get $|\mathcal{N}_2^2| = \left|\mathcal{M}\left(L_n\right)\right| \times \left|\mathcal{M}\left(G-L_n\right)\right|$. By definition $L_n \cong L_n^4$, then $\left|\mathcal{M}\left(L_n\right)\right| = \left|\mathcal{M}\left(L_n^4\right)\right| = f_4(n)$, $\left|\mathcal{M}\left(G-L_n\right)\right| = 2$. Thus, $|\mathcal{N}_2^2| = 2f_4(n)$. Let $M \in \mathcal{N}_2^3$, that is, M contains edges $u_{11}u_{21}$, $u_{17}u_{27}$. Then M can be divided into two parts: one part is the matching edges restricted in $L_n = G - C_1 + u_{11}u_{21} + u_{17}u_{27}$, and the other part is the matching edges in $G-L_n$. From the fractional multiplication, we can get $|\mathcal{N}_2^3| = \left|\mathcal{M}\left(L_n\right)\right| \times \left|\mathcal{M}\left(G-L_n\right)\right|$. By definition $L_n \cong L_n^6$, then $\left|\mathcal{M}\left(L_n\right)\right| = \left|\mathcal{M}\left(L_n^6\right)\right| = f_6(n)$, $\left|\mathcal{M}\left(G-L_n\right)\right| = 1$. Thus, $|\mathcal{N}_2^3| = f_6(n)$. Similarly, we can get $|\mathcal{N}_2^3| = |\mathcal{N}_2^4| = f_6(n)$. Let $M \in \mathcal{N}_2^5$, that is, M contains edges $u_{13}u_{23}, u_{17}u_{27}$. Then M can be divided into two parts: one part is the matching edges restricted in graph $L_n = G - C_1 + u_{13}u_{23} + u_{17}u_{27}$, and the other part is the matching edges in $G-L_n$. By definition $L_n \cong L_n^4$, then $\left|\mathcal{M}\left(G-L_n\right)\right| = 0$. Thus, $|\mathcal{N}_2^5| = 0$.

Therefore, $|\mathcal{N}_2| = \sum_{i=1}^{6} |\mathcal{N}_2^i| = 2f_4(n) + 2f_5(n) + 2f_6(n)$.

By Lemma 1, we can get the number of perfect matchings in G.

$$
\begin{aligned}
N(n) = |\mathcal{M}| = \sum_{i=0}^{4} |\mathcal{N}_i| &= 1 + 2^{n+1} + 2f_4(n) + 2f_5(n) + 2f_6(n) \\
&= 1 + 2^{n+1} + \left(3 - 2\sqrt{2}\right)(2 - \sqrt{2})^{n-1} \\
&\quad + (10 + 7\sqrt{2})(2 + \sqrt{2})^{n-2}.
\end{aligned}
$$

Thus Theorem 3 is proved. On the other hand, we can verify that there are 29 perfect matchings in $G = (3,4,t)(t=1,3)$ by a simple calculation, which is consistent with the result calculated by formula (21).

Let $G = (n+1, 4, t)$ be a $(3,6)$-fullerene with p vertices. Since G has $n+1$ concentric rings, 8 vertices on each ring, and all vertices of G are on these concentric rings, then $8(n+1) = p$, that is, $n = \frac{p}{8} - 1$. By combining Theorems 1-3, we can get the perfect matchings number of $G = (n+1, 4, t)$.

Corollary 1 Let $G = (n+1, 4, t)$ be a $(3,6)$-fullerene with p vertices, and denote by $|\mathcal{M}(G)|$ the number of perfect matchings of G. Then

$$|\mathcal{M}(G)| = \begin{cases} 1+2^{\frac{p}{8}}+2^{\frac{p}{16}}+\left(3+2\sqrt{2}\right)\left(2+\sqrt{2}\right)^{\frac{p}{8}-2}+\frac{1}{2}\left(2-\sqrt{2}\right)^{\frac{p}{8}}, \text{when } n \text{ is odd}, t=0. \\ 1+2^{\frac{p}{8}}-2^{\frac{p}{16}}+\left(3+2\sqrt{2}\right)\left(2+\sqrt{2}\right)^{\frac{p}{8}-2}+\frac{1}{2}\left(2-\sqrt{2}\right)^{\frac{p}{8}}, \text{ when } n \text{ is odd}, t=2. \\ 1+2^{\frac{p}{8}}+\left(3-2\sqrt{2}\right)\left(2-\sqrt{2}\right)^{\frac{p}{8}-2}+\left(10+7\sqrt{2}\right)\left(2+\sqrt{2}\right)^{\frac{p}{8}-3}, \text{ when } n \text{ is even.} \end{cases} \quad (22)$$

References

[1] Trinajstić N. *Chemical Graph Theory*[M]. Boca Ratan: CRC Press, 2018.

[2] Randić M. On the characterization of local aromatic properties in benzenoid hydrocarbons[J]. *Tetrahedron*, 1974, **30** (14): 2067-2074.

[3] Swinborne R, Herndon W C, Gutman I. Kekulé structures and resonance energies of benzenoid hydrocarbons[J]. *Tetrahedron Letters*, 1975, **16**(10): 755-758.

[4] Valiant L G. The complexity of computing the permanent[J]. *Theoretical Compute Science*, 1979, **8**(2): 189-201.

[5] Došlić T. Cyclical edge-connectivity of fullerene graphs and (*k*, 6)-cages[J]. *Journal of Mathematical Chemistry*, 2003, **33** (2): 103-112.

[6] Kardoš F, Král' D, Miškuf J, *et al*. Fullerene graphs have exponentially many perfect matchings[J]. *Journal of Mathematical Chemistry*, 2009, **46**(2): 443-447.

[7] Lovász L, Plummer M D. *Matching Theory*[M]. New York: North-Holland Press, 1986.

[8] Esperet L, Kardoš F, King A D, *et al*. Exponentially many perfect matchings in cubic graphs[J]. *Advances in Mathematics*, 2011, **227**(4): 1646-1664.

[9] Feng X, Zhang L, Zhang M. Enumeration of perfect matchings of lattice graphs by Pfaffians[J]. *Applied Mathematics and Computation*, 2018, **338**: 412-420.

[10] Zhang L Z, Wei S L, Lu F L. The number of Kekulé structures of polyominos on the torus[J]. *Journal of Mathematical Chemistry*, 2013, **51**(1): 354-368.

[11] Yang R, Zhang H P. Hexagonal resonance of (3,6)-fullerens [J]. *Journal of Mathematical Chemistry*, 2012, **50**(1): 261-273.

[12] Sun C H, Zhang H P. On bicriticality of (3, 6) -fullerene graphs[J]. *Journal of Mathematical Chemistry*, 2018, **56**(9): 2785-2793.

[13] Shi L J, Zhang H P. Forcing and anti-forcing numbers of (3, 6) -fullerenes[J]. *MATCH Commun Math Comput Chem*, 2016, **76**(3): 597-614.

[14] John P E, Sachs H. Spectra of toroidal graphs[J]. *Discrete Mathematics*, 2009, **309**(9): 2663-2681.

[15] Goodey P R. A class of Hamiltonian polytopes[J]. *Journal of Graph Theory*, 1977, **1**(2): 181-185.

[16] Grünbaum B, Motzkin T S. The number of hexagons and the simplicity of geodesics on certain polyhedra[J]. *Canadian Journal of Mathematics*, 1963, **15**: 744-751.

[17] Bondy J A, Murty U S R. *Graph Theory*[M]. Berlin: Springer-Verlag, 2008.

[18] Brualdi R A. *Introductory Combinatorics*[M]. New York: North-Holland Press, 2009.

[19] Diestel R. *Graph Theory*[M]. New York: Springer-Verlag, 2005.

□