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I-Total Coloring and VI-Total Coloring of mC_4 Vertex-Distinguished by Multiple Sets

□ **WANG Nana, CHEN Xiang'en**†

College of Mathematics and Statistics, Northwest Normal University, Lanzhou 730070, Gansu, China

Abstract: We give the optimal I-(VI-)total colorings of mC_4 which are vertex-distinguished by multiple sets by the use of the method of constructing a matrix whose entries are the suitable multiple sets or empty sets and the method of distributing color set in advance. Thereby we obtain I-(VI-)total chromatic numbers of mC_4 which are vertex-distinguished by multiple sets.

Key words: mC_4; I-total coloring; VI-total coloring; multiple sets; vertex-distinguished

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0 Introduction

All graphs discussed in this paper are simple, non-directed graphs. Many conclusions have been obtained regarding the vertex-distinguished proper edge coloring[1-3] and vertex-distinguished general edge coloring[4-7] of graphs. In 2008, Zhang *et al* [8] proposed vertex-distinguished total coloring and related conjectures of graphs. In 2014, Chen *et al*[9] introduced vertex-distinguished I-total coloring and related conjectures of graphs. Many studies have been made on vertex-distinguished I-(VI-)total colorings of graphs [10-12]. In this study, we consider vertex-distinguished I-(VI-)total colorings of mC_4 by multiple sets.

Let G be a simple graph. Suppose a mapping f: $V \cup E \rightarrow \{1, 2, \cdots, l\}$ is a general total coloring of G (not necessarily proper). If $\forall u, v \in V$, and u, v are adjacent vertices, we have $f(u) \neq f(v)$, and if $uv, vw \in E, uv \neq vw$, we have $f(uv) \neq f(vw)$, then f is called the I-total coloring of G. If any two adjacent edges of G receives different colors, then f is called VI-total coloring of G. Obviously, I-total coloring is VI-total coloring, and the reverse is uncertain. For an I- total coloring (resp. VI- total coloring) f of G, if l colors are used, then f is called l-I-total coloring of G (resp. l-VI-total coloring). Note that when we refer to the l-I-total coloring (resp. l-VI-total coloring) of graph, we always assume that the colors used are $1, 2, \cdots, l$.

Let f be a general total coloring of G. For any vertex x in G, $\tilde{C}_f(x)$ denotes the multiple set of colors of vertex x and edges that are incident of vertex x. $\tilde{C}_f(x)$ is said to be the color set of x under f. No confusion arises when using $\tilde{C}(x)$. Obviously, $|\tilde{C}_f(x)| = d_G(x) + 1$. If $\tilde{C}(u) \neq \tilde{C}(v)$ for any two distinct vertices u and v of G, then f is called

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Biography: WANG Nana, female, Master candidate, research direction: graph theory and its application. E-mail: wangnana202106@163.com

† To whom correspondence should be addressed. E-mail: chenxe@nwnu.edu.cn

vertex-distinguished by multiple sets. Let

$\tilde{\chi}^{\mathrm{i}}_{\mathrm{vt}}(G)=\min\{l|G$ has l-I-total coloring which is vertex-distinguished by multiple sets$\}$

and

$\tilde{\chi}^{\mathrm{vi}}_{\mathrm{vt}}(G)=\min\{l|G$ has l-VI-total coloring which is vertex-distinguished by multiple sets$\}$.

Then, $\tilde{\chi}^{\mathrm{i}}_{\mathrm{vt}}(G)$ is called the I-total chromatic number of G which is vertex-distinguished by multiple sets. Similarly, $\tilde{\chi}^{\mathrm{vi}}_{\mathrm{vt}}(G)$ is called the VI-total chromatic number of G which is vertex-distinguished by multiple sets. Let $n_i(G)$ represent the number of vertices of degree i. Suppose that

$$\tilde{\zeta}(G)=\min\{l|i\binom{l}{i}+\binom{l}{i+1}\geqslant n_i,\delta\leqslant i\leqslant\Delta\}.$$

Proposition 1 $\tilde{\chi}^{\mathrm{i}}_{\mathrm{vt}}(G)\geqslant\tilde{\chi}^{\mathrm{vi}}_{\mathrm{vt}}(G)\geqslant\tilde{\zeta}(G)$.

Proof Obviously, I-total coloring is VI-total coloring. Thus $\tilde{\chi}^{\mathrm{vi}}_{\mathrm{vt}}(G)\leqslant\tilde{\chi}^{\mathrm{i}}_{\mathrm{vt}}(G)$.

Set $t=\tilde{\chi}^{\mathrm{vi}}_{\mathrm{vt}}(G)$. G has t-VI-total coloring which are vertex-distinguished by multiple sets. For $\delta\leqslant i\leqslant\Delta$. Considering the vertices of the degree i, we obtain

$$i\binom{t}{i}+\binom{t}{i+1}\geqslant n_i.$$

Thus, $t\in\{l|i\binom{l}{i}+\binom{l}{i+1}\geqslant n_i,\delta\leqslant i\leqslant\Delta\}$. Therefore, $t\geqslant\tilde{\zeta}(G)$, namely $\tilde{\chi}^{\mathrm{vi}}_{\mathrm{vt}}(G)\geqslant\tilde{\zeta}(G)$. This completes the proof.

1 Preliminaries

We first define a matrix $\boldsymbol{A}_{l\times(l-1)}$, for any $l\geqslant 4$,

$$\boldsymbol{A}_{l\times(l-1)}=\begin{pmatrix}\{1,1,l\} & \{2,2,l\} & \{3,3,l\} & \cdots & \{l-2,l-2,l\} & \{l-1,l-1,l\}\\ \{1,2,l\} & \{2,3,l\} & \{3,4,l\} & \cdots & \{l-2,l-1,l\} & \{l-1,l,l\}\\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots\\ \{1,l-1,l\} & \{2,l,l\} & \varnothing & \cdots & \varnothing & \varnothing\\ \{1,l,l\} & \varnothing & \varnothing & \cdots & \varnothing & \varnothing\end{pmatrix}.$$

Let $1\leqslant i_1<i_2<\cdots<i_r\leqslant l$, $1\leqslant j_1<j_2<\cdots<j_s\leqslant l-1$. Submatrix $\boldsymbol{A}_{l\times(l-1)}[i_1,i_2,\cdots,i_r|j_1,j_2,\cdots,j_s]$ is an $r\times s$ matrix. It is comprised by all the elements which are only in i_1-, i_2-,$\cdots$, or i_r-th rows but also in j_1-, j_2-, $\cdots$, or j_s-th columns of $\boldsymbol{A}_{l\times(l-1)}$. The following six schemes are presented for the I-total coloring of C_4 which are vertex-distinguished by multiple sets. Note that all lowercase letters represent different colors.

In Fig. 1(a), the color set of each vertex of C_4 is $\{a,a,b\},\{b,b,a\},\{a,a,c\},\{c,c,a\}$. This coloring scheme is Co1($a;b;c$).

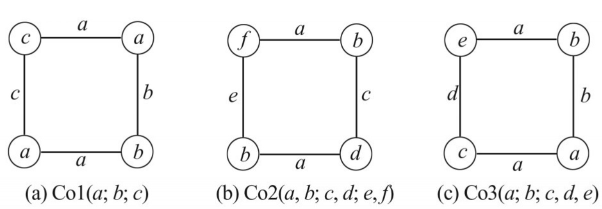
In Fig. 1(b), the color set of each vertex of C_4 is $\{a,b,c\},\{c,d,a\},\{a,b,e\},\{e,f,a\}$. This coloring scheme is Co2($a,b;c,d;e,f$).

In Fig. 1(c), the color set of each vertex of C_4 is $\{a,b,b\},\{b,a,a\},\{a,c,d\},\{d,e,a\}$. This coloring scheme is Co3($a;b;c,d,e$).

In Fig. 2(a), the color set of each vertex of C_4 is $\{a,c,b\},\{b,d,a\},\{a,e,b\},\{b,f,a\}$. This coloring scheme is Co4($a;b;c;d;e;f$).

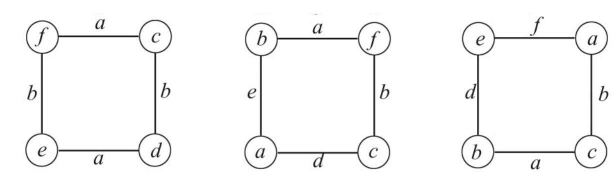
In Fig. 2(b), the color set of each vertex of C_4 is $\{a,f,b\},\{b,c,d\},\{d,a,e\},\{e,b,a\}$. This coloring scheme is Co5($a;b;c,d;e;f$).

In Fig. 2(c), the color set of each vertex of C_4 is $\{f,a,b\},\{b,c,a\},\{a,b,d\},\{d,e,f\}$. This coloring scheme is Co6($a,b;c;d,e,f$).



(a) Co1($a;b;c$) (b) Co2($a,b;c,d;e,f$) (c) Co3($a;b;c,d,e$)

Fig.1 The coloring scheme Co1($a;b;c$), Co2($a,b;c,d;e,f$), and Co3($a;b;c,d,e$)



(a) Co4($a;b;c;d;e;f$) (b) Co5($a;b;c,d;e;f$) (c) Co6($a,b;c;d,e,f$)

Fig.2 The coloring scheme Co4($a;b;c;d;e;f$), Co5($a;b;c,d;e;f$), and Co6($a,b;c;d,e,f$)

Lemma 1 When $1\leqslant j\leqslant l-2$ (j is an odd number), $\{j,j,l\}$, $\{j+1,j+1,l\}$, $\{j,l,l\}$, $\{j+1,l,l\}$ are the color sets of the vertices under I-total coloring of C_4 which are vertex-distinguished by multiple sets in Fig. 1(a).

Lemma 2 When $i\equiv 0\pmod 2$, $j\equiv 1\pmod 2$, and $\boldsymbol{A}_{l\times(l-1)}(i,i+1|j,j+1)$ are not $\varnothing$, $\{j,j+i-1,l\}$, $\{j,j+i,l\},\{j+1,j+i,l\},\{j+1,j+i+1,l\}$ are the color sets of the vertices under I-total coloring of C_4 which are vertex-distinguished by multiple sets in Fig. 1(b).

Lemma 3 If $l \equiv 1, 6 \pmod 8$, $l \geqslant 9$, then for $\boldsymbol{A}_{l\times(l-1)}$ except $\varnothing$ can be divided into $\frac{1}{4}\left[\binom{l+1}{2}-1\right]$ groups, each group has four 3-subsets. These are the color sets of the vertices under I-total coloring of C_4 which are vertex-distinguished by multiple sets.

Proof We use Lemmas 1 and 2, and only consider the remaining entries of $\boldsymbol{A}_{l\times(l-1)}$.

Case 1: $l \equiv 1 \pmod 8$.

For each $j \equiv 1 \pmod 8), 1 \leqslant j \leqslant l-8$, considering the remaining entries of the $j, j+2, j+4, j+6$ columns: $\{j, l-1, l\}$, $\{j+2, l-1, l\}$, $\{j+4, l-1, l\}$, $\{j+6, l-1, l\}$. These four 3-subsets are the color sets of the vertices in Co4$(l; l-1; j; j+2; j+4; j+6)$.

Case 2: $l \equiv 6 \pmod 8$.

Ⅰ. The remaining entries in 1, 2, $l-3$, $l-2$, $l-1$ columns can be divided into two groups, $\{\{1, l-2, l\}$, $\{l-3, l-2, l\}$, $\{l-1, l-1, l\}, \{l-1, l, l\}\}$; $\{\{1, l-1, l\}, \{2, l-1, l\}, \{l-3, l-1, l\}, \{l-2, l-1, l\}\}$. The corresponding coloring schemes are Co3$(l; l-1; l-3, l-2, 1)$ and Co4$(l; l-1; 1; 2; l-3; l-2)$, respectively.

Ⅱ. For each $j \equiv 3 \pmod 8), 3 \leqslant j \leqslant l-11$, considering the remaining entries of the $j, j+1, j+2, j+3, j+4, j+5, j+6, j+7$ columns, which can be divided into three groups: $\{\{j, l-1, l\}, \{j+1, l-1, l\}, \{j+2, l-1, l\}, \{j+3, l-1, l\}\}$, $\{\{j+4, l-1, l\}, \{j+5, l-1, l\}, \{j+6, l-1, l\}, \{j+7, l-1, l\}\}$, $\{\{j, l-2, l\}, \{j+2, l-2, l\}, \{j+4, l-2, l\}, \{j+6, l-2, l\}\}$. The corresponding coloring schemes are Co4$(l; l-1; j; j+1; j+2; j+3)$, Co4$(l; l-1; j+4; j+5; j+6; j+7)$, and Co4$(l; l-2; j; j+2; j+4; j+6)$, respectively.

Lemma 4 If $l \equiv 2, 5 \pmod 8$, $l \geqslant 10$, then all nonempty sets in $\boldsymbol{A}_{l\times(l-1)}$ except for $\{l-7, l-2, l\}, \{l-5, l-2, l\}$ (when $l \equiv 2 \pmod 8$) or $\{l-4, l-1, l\}$, $\{l-2, l-1, l\}$(when $l \equiv 5 \pmod 8$) can be divided into $\frac{1}{4}\left[\binom{l+1}{2}-3\right]$ groups, and each group has four 3-subsets. These are the color sets of the vertices under I-total coloring of C_4 which are vertex-distinguished by multiple sets.

Proof We use Lemmas 1 and 2, and only consider the remaining entries of $\boldsymbol{A}_{l\times(l-1)}$.

Case 1: $l \equiv 2 \pmod 8$.

For the remaining entries in $1, 2, l-3, l-2, l-1$ columns, the grouping is obtained and the corresponding coloring scheme is determined using Lemma 3 Case 2 Ⅰ. For $j \equiv 3 \pmod 8), 3 \leqslant j \leqslant l-8$, considering the remaining entries of the $j, j+1, j+2, j+3, j+4, j+5, j+6, j+7$ columns, the grouping is obtained and the corresponding coloring scheme is determined using Lemma 3 Case 2 Ⅱ. For the six remaining entries in $l-7$, $l-6, l-5, l-4$ columns, there is a group $\{\{l-7, l-1, l\}, \{l-6, l-1, l\}, \{l-5, l-1, l\}, \{l-4, l-1, l\}\}$, namely Co4$(l; l-1; l-7; l-6; l-5; l-4)$.

This leaves the 3-subsets $\{l-7, l-2, l\}, \{l-5, l-2, l\}$.

Case 2: $l \equiv 5 \pmod 8$.

For each $j \equiv 1 \pmod 8), 1 \leqslant j \leqslant l-5$, considering the remaining entries in $j, j+2, j+4, j+6$ columns, the grouping is obtained and the corresponding coloring scheme is determined using Lemma 3 Case 1.

This leaves the 3-subsets $\{l-4, l-1, l\}, \{l-2, l-1, l\}$.

Lemma 5 If $l \equiv 7, 0 \pmod 8), l \geqslant 15$, then all nonempty sets in $\boldsymbol{A}_{l\times(l-1)}$ except for $\{l-6, l-1, l\}, \{l-4, l-1, l\}, \{l-2, l-1, l\}$ (when $l \equiv 7 \pmod 8$) or $\{l-5, l-1, l\}, \{l-5, l-1, l\}, \{l-4, l-1, l\}$ (when $l \equiv 0 \pmod 8$) can be divided into $\frac{1}{4}\left[\binom{l+1}{2}-4\right]$ groups, and each group has four 3-subsets. These are the color sets of the vertices under I-total coloring of C_4 which are vertex-distinguished by multiple sets.

Proof We use Lemmas 1 and 2, and only consider the entries of $\boldsymbol{A}_{l\times(l-1)}$.

Case 1: $l \equiv 7 \pmod 8$.

For each $j \equiv 1 \pmod 8), 1 \leqslant j \leqslant l-7$, considering the remaining entries in columns $j, j+2, j+4, j+6$, the grouping is obtained and the corresponding coloring scheme is determined using Lemma 3 Case 1.

This leaves the 3-subsets $\{l-6, l-1, l\}, \{l-4, l-1, l\}, \{l-2, l-1, l\}$.

Case 2: $l \equiv 0 \pmod 8$.

For the remaining entries of $1, 2, l-3, l-2, l-1$ columns, the grouping is obtained and the corresponding coloring scheme is determined using Lemma 3 Case 2 Ⅰ. For $j \equiv 3 \pmod 8), 3 \leqslant j \leqslant l-6$, considering the remaining entries of the $j, j+1, j+2, j+3, j+4, j+5, j+6, j+7$ columns, the grouping is obtained and the corresponding coloring scheme is determined using Lemma 3 Case 2.

This leaves the 3-subsets $\{l-5, l-2, l\}, \{l-5, l-1, l\}, \{l-4, l-1, l\}$.

Lemma 6 If $l \equiv 3 \pmod 8$, $l \geqslant 11$, then all nonempty sets in $\boldsymbol{A}_{l\times(l-1)}$ except $\{l-2, l-1, l\}$ can be divided into $\frac{1}{4}\left[\binom{l+1}{2}-2\right]$ groups, and each group has four 3-subsets. These are the color sets of the vertices under I-total coloring of C_4 which are vertex-distinguished by multiple sets.

Proof We use Lemmas 1 and 2, and only consider the remaining entries of $\boldsymbol{A}_{l\times(l-1)}$.

For each $j\equiv 1\ (\mathrm{mod}\ 8), 1\leqslant j\leqslant l-3$, considering the remaining entries of the $j, j+2, j+4, j+6$ columns, the grouping is obtained and the corresponding coloring scheme is determined using Lemma 3 Case 1.

This leaves 3-subset $\{l-2,\ l-1, l\}$.

Lemma 7 If $l\equiv 4\ (\mathrm{mod}\ 8),\ l\geqslant 12$, then all nonempty sets in $\boldsymbol{A}_{l\times(l-1)}$ except for $\{l-9, l-2, l\},\{l-7,\ l-2, l\},\{l-5, l-2, l\},\{l-5, l-1, l\},\{l-4, l-1, l\}$ can be divided into $\frac{1}{4}\left[\binom{l+1}{2}-6\right]$ groups, and each group has four 3-subsets. These are the color sets of the vertices under I-total colorings of C_4 which are vertex-distinguished by multiple sets.

Proof We use Lemmas 1 and 2, and only consider the remaining entries of $\boldsymbol{A}_{l\times(l-1)}$.

For the remaining entries in $1, 2, l-3, l-2, l-1$ columns, the grouping is obtained and the corresponding coloring scheme is determined using Lemma 3 Case 2 Ⅰ. For $j\equiv 3\ (\mathrm{mod}\ 8),\ 3\leqslant j\leqslant l-10 (l\geqslant 20)$, considering the remaining entries in columns $j, j+1, j+2, j+3, j+4, j+5, j+6, j+7$, the grouping is obtained and the corresponding coloring scheme is determined using Lemma 3 Case 2. For the remaining in $l-9, l-8, l-7,\ l-6$ columns, there is a group $\{\{l-9, l-1, l\},\{l-8, l-1, l\},\{l-7, l-1, l\},\{l-6, l-1, l\}\}$ except for $\{l-9, l-2, l\}$, $\{l-7, l-2, l\}$, namely Co4$(l; l-1; l-6; l-7; l-8; l-9)$.

This leaves the 3-subsets $\{l-9, l-2, l\},\{l-7, l-2, l\},\{l-5, l-2, l\},\{l-5, l-1, l\},\{l-4, l-1, l\}$.

2 Main Results and Their Proofs

Theorem 1 If $2\binom{l-1}{2}+\binom{l-1}{3}<4m\leqslant 2\binom{l}{2}+\binom{l}{3}, m\geqslant 1, l\geqslant 3$, then $\tilde{\chi}_{\mathrm{vt}}^{\mathrm{i}}(mC_4)=l$.

Proof Obviously, there is $l=\tilde{\zeta}(mC_4)\leqslant\tilde{\chi}_{\mathrm{vt}}^{\mathrm{i}}(G)$. Therefore, we can directly give the l-I-total coloring of mC_4 which are vertex-distinguished by multiple sets.

① When $m=1, l=3$. Use $\{1,1,2\},\{1,2,2\},\{1,1,3\},\{1,3,3\}$, that is, Co1(1; 3; 2) to color the first C_4. Thus, the multiple 3-subsets $\{2,2,3\},\{2,3,3\},\{1,2,3\}$ remain.

② When $2\leqslant m\leqslant 4, l=4$. Based on I-total coloring of the first C_4, we start coloring from the second C_4. According to Lemma 1, one C_4 can be colored with Co1(4; 1; 2). Subsequently, the multiple 3-subsets $\{3,3,4\},\{3,4,4\},\{1,2,4\},\{1,3,4\},\{2,3,4\}$ remain. The third C_4 is colored with Co1(3; 4; 2), under which the color sets of four vertices are $\{3,3,4\},\{3,4,4\}$ and ① remaining $\{2,2,3\},\{2,3,3\}$. The fourth C_4 is colored with ① remaining $\{1,2,3\}$ and $\{1,2,4\},\{1,3,4\},\{2,3,4\}$, as illustrated in Fig. 3. Thus far, all multiple 3-subsets of $\{1,2,3,4\}$ have been used.

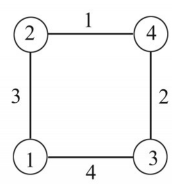


Fig.3 Vertex-distinguished I-total coloring of C_4

③ When $5\leqslant m\leqslant 7,\ l=5$. Based on the previous step, we start coloring from the 5-th C_4. According to Lemmas 1 and 2, $3C_4$ can be colored with Co1(5; 1; 2), Co1(5; 3; 4), and Co2(5, 3; 2, 4; 1, 2). Subsequently, the multiple 3-subset $\{1,4,5\}$, $\{3,4,5\}$ remain.

④ When $8\leqslant m\leqslant 12, l=6$. Based on the I-total coloring of $7C_4$ which are vertex-distinguished by multiple sets, we start coloring from the 8-th C_4. According to Lemmas 1 and 2, $3C_4$ can be colored with Co1(6; 1; 2), Co1(6; 3; 4), and Co2(6, 3; 2, 4; 1, 2). The remaining entries are $\{5,5,6\},\{5,6,6\},\{1,4,6\},\{1,5,6\},\{2,5,6\},\{3,4,6\},\{3,5,6\},\{4,5,6\}$, which can be divided into two groups that can color $2C_4$ with Co3(6; 5; 1, 4, 3) and Co4(6; 5; 1; 2; 3; 4). As all multiple 3-subsets $\{1,2,3,4,5,6\}$ containing 6 are used up in the above coloring, the remaining 3-subsets are still $\{1,4,5\},\{3,4,5\}$.

⑤ When $13\leqslant m\leqslant 19, l=7$. Based on the I-total coloring of $12C_4$ which are vertex-distinguished by multiple sets, we color from the 13-th to 19-th C_4. According to Lemmas 1 and 2, $6C_4$ can be colored with the Co1(7; 1; 2), Co1(7; 3; 4), Co1(7; 5; 6), Co2(7, 3; 2, 4; 1, 2), Co2(7, 5; 2, 6; 1, 4), and Co2(7, 5; 4, 6; 3, 4). Now, the 3-subsets $\{1,6,7\},\{3,6,7\},\{5,6,7\}$ and ③ remaining $\{1,4,5\}$ are used to color the 19-th C_4 with Co6(7, 6; 3; 1, 4, 5). Due to the above coloring, all multiple 3-subsets containing 7 are used. Thus, the remaining 3-subset $\{3,4,5\}$ is still left.

⑥ When $20\leqslant m\leqslant 28, l=8$. We start from the 20-th C_4 based on the preceding coloring. According to Lemmas 1 and 2, $6C_4$ can be colored with Co1(8; 1; 2), Co1(8; 3; 4), Co1(8; 5; 6), Co2(8, 3; 2, 4; 1, 2), Co2(8, 5; 2,

$6;1,4)$ and Co2$(8,5;4,6;3,4)$. The remaining 3-subsets are $\{7,7,8\},\{7,8,8\},\{1,6,8\},\{1,7,8\},\{2,7,8\},\{3,6,8\},\{3,7,8\},\{4,7,8\},\{5,6,8\},\{5,7,8\},\{6,7,8\}$. The 3-subsets $\{1,6,8\},\{5,6,8\},\{7,7,8\},\{7,8,8\}$ are used to color the 26-th C_4 with Co3$(8;7;1,6,5)$. The 27-th C_4 is colored with Co4$(8;7;1;2;5;6)$, under which the color sets of four vertices are $\{1,7,8\},\{2,7,8\},\{5,7,8\},\{6,7,8\}$. The 3-subsets $\{3,6,8\},\{3,7,8\},\{4,7,8\}$ and ③ remaining $\{3,4,5\}$ are used to color the 28-th C_4 with Co5$(8;3;5,4;7;6)$. Consequently, all multiple 3-subsets of $\{1,2,3,4,5,6,7,8\}$ have been used.

⑦ Let $l \geqslant 9$, we recursively proceed as following process.

We have obtained $(l-1)$-I-total coloring of $\left\lfloor \frac{1}{4}\left[2\binom{l-1}{2}+\binom{l-1}{3}\right]\right\rfloor C_4$ which are vertex-distinguished by multiple sets. On this basis, we will construct the I-total coloring from the $\left\lfloor \frac{1}{4}\left[2\binom{l-1}{2}+\binom{l-1}{3}\right]\right\rfloor+1$-th C_4 to $\left\lfloor \frac{1}{4}\left[2\binom{l}{2}+\binom{l}{3}\right]\right\rfloor$-th C_4 which are vertex-distinguished by multiple sets.

When $l \equiv 1 \pmod 8, l \geqslant 9$. Using Lemma 3, we can obtain the l-I-total coloring of $\frac{1}{4}\left[2\binom{l}{2}+\binom{l}{3}\right] C_4$, which are vertex-distinguished by multiple sets, and we have used all 3-subsets of $\{1,2,\cdots,l\}$.

When $l \equiv 2 \pmod 8, l \geqslant 10$. Using Lemma 4, we can obtain the l-I-total coloring of $\frac{1}{4}\left[2\binom{l}{2}+\binom{l}{3}-2\right] C_4$, which are vertex-distinguished by multiple sets, and we have used all 3-subsets of $\{1,2,\cdots,l\}$ except for $\{l-7,l-2,l\},\{l-5,l-2,l\}$.

When $l \equiv 3 \pmod 8, l \geqslant 11$. Using Lemma 6, we can obtain l-I-total coloring of $\frac{1}{4}\left[2\binom{l}{2}+\binom{l}{3}-1\right] C_4$, which are vertex-distinguished by multiple sets. We have used all 3-subsets of $\{1,2,\cdots,l\}$ except for $\{l-2,l-1,l\}$ and the above mentioned $\{l-8,l-3,l-1\},\{l-6,l-3,l-1\}$.

When $l \equiv 4 \pmod 8, l \geqslant 12$. Using Lemma 7, we can obtain the l-I-total coloring of $\frac{1}{4}\left[2\binom{l}{2}+\binom{l}{3}-8\right] C_4$, which are vertex-distinguished by multiple sets. We have used all 3-subsets of $\{1,2,\cdots,l\}$ except for $\{l-9,l-2,l\},\{l-7,l-2,l\},\{l-5,l-2,l\},\{l-5,l-1,l\},\{l-4,l-1,l\}$ and the above mentioned $\{l-3,l-2,l-1\},\{l-9,l-4,l-2\},\{l-7,l-4,l-2\}$. The $\frac{1}{4}\left[2\binom{l}{2}+\binom{l}{3}-4\right]$-th C_4 is colored with Co2$(l-2,l-4;l-7,l;l-9,l)$, under which the color sets of four vertices are $\{l-9,l-2,l\},\{l-7,l-2,l\},\{l-9,l-4,l-2\},\{l-7,l-4,l-2\}$. The 3-subsets $\{l-5,l-2,l\},\{l-5,l-1,l\},\{l-4,l-1,l\},\{l-3,l-2,l-1\}$ are used to color the $\frac{1}{4}\left[2\binom{l}{2}+\binom{l}{3}\right]$-th C_4 with Co5$(l;l-1;l-3,l-2;l-5;l-4)$. At this time, we obtained the l-I-total coloring of $\frac{1}{4}\left[2\binom{l}{2}+\binom{l}{3}\right] C_4$, which are vertex-distinguished by multiple sets. Moreover, all multiple 3-subsets of $\{1,2,\cdots,l\}$ have been used.

When $l \equiv 5 \pmod 8, l \geqslant 13$. Using Lemma 4, we can obtain the l-I-total coloring of $\frac{1}{4}\left[2\binom{l}{2}+\binom{l}{3}-2\right] C_4$, which are vertex-distinguished by multiple sets. We have used all 3-subsets of $\{1,2,\cdots,l\}$ except for $\{l-4,l-1,l\},\{l-2,l-1,l\}$.

When $l \equiv 6 \pmod 8, l \geqslant 14$. Using lemma 3, we can obtain the l-I-total coloring of $\frac{1}{4}\left[2\binom{l}{2}+\binom{l}{3}-2\right] C_4$, which are vertex-distinguished by multiple sets. We have used all 3-subsets of $\{1,2,\cdots,l\}$ except for the above mentioned $\{l-5,l-2,l-1\},\{l-3,l-2,l-1\}$.

When $l \equiv 7 \pmod 8, l \geqslant 15$. Using lemma 5, we can obtain the l-I-total coloring of $\frac{1}{4}\left[2\binom{l}{2}+\binom{l}{3}-5\right] C_4$, which are vertex-distinguished by multiple sets. We have used all the 3-subsets of $\{1,2,\cdots,l\}$ except for $\{l-6,l-1,l\},\{l-4,l-1,l\},\{l-2,l-1,l\}$ and the above mentioned $\{l-6,l-3,l-2\},\{l-4,l-3,l-2\}$. The 3-subsets $\{l-6,l-1,l\},\{l-4,l-1,l\},\{l-2,l-1,l\},\{l-6,l-3,l-2\}$ are used to color the $\frac{1}{4}\left[2\binom{l}{2}+\binom{l}{3}-1\right]$-th C_4 with Co6$(l,l-1;l-4;l-6,l-3,l-2)$. Then the 3-subset $\{l-4,l-3,l-2\}$ remains.

When $l \equiv 0 \pmod 8, l \geqslant 16$. Using Lemma 5, we can obtain the l-I-total coloring of $\frac{1}{4}\left[2\binom{l}{2}+\binom{l}{3}-4\right] C_4$, which are vertex-distinguished by multiple sets. We have used all the 3-subsets of $\{1,2,\cdots,l\}$ except for $\{l-5,l-2,l\},\{l-5,l-1,l\},\{l-4,l-1,l\}$ and the above mentioned $\{l-5,l-4,l-3\}$. The $\frac{1}{4}\left[2\binom{l}{2}+\binom{l}{3}\right]$-th C_4 is colored with the above four 3-subsets, that is, Co5$(l;l-$

$5; l-3, l-4; l-1; l-2$). Thus far, all multiple 3-subsets of $\{1, 2, \cdots, l\}$ have been used.

The theorem is proven.

Theorem 2 If $2\binom{l-1}{2}+\binom{l-1}{3}<4m\leqslant 2\binom{l}{2}+\binom{l}{3}, m\geqslant 1, l\geqslant 3, \tilde{\chi}_{\mathrm{vt}}^{\mathrm{vi}}(mC_4)=l$.

Proof This conclusion can be obtained by the proof of Proposition 1 and Theorem 1.

3 Conclusion

In this study, the I-(VI-)total chromatic numbers of mC_4 have been obtained, which are vertex-distinguished by multiple sets. According to the characteristics of the cycles and multiple sets, the mC_n (even number) of the I-(VI-)total chromatic numbers and VI-total of the multiple sets can be similarly obtained according to the above methods. That is, if $2\binom{l-1}{2}+\binom{l-1}{3}<nm\leqslant 2\binom{l}{2}+\binom{l}{3}, m\geqslant 1, l\geqslant 3$ is satisfied, then $\tilde{\chi}_{\mathrm{vt}}^{\mathrm{i}}(mC_n)=l$ and $\tilde{\chi}_{\mathrm{vt}}^{\mathrm{vi}}(mC_n)=l$, and two cases of recursive boundary conditions can be inferred in the proof process: if $3\nmid n$, then $2n$; if $3|n$, then $6n$. The I-(VI-)total colorings of odd order cycles which are vertex-distinguished by multiple sets will be studied at a later stage.

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