I-Total Coloring and VI-Total Coloring of $mC_4$ Vertex-Distinguished by Multiple Sets

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Abstract: We give the optimal I-(VI-)total colorings of $mC_4$, which are vertex-distinguished by multiple sets by the use of the method of constructing a matrix whose entries are the suitable multiple sets or empty sets and the method of distributing color set in advance. Thereby we obtain I-(VI-)total chromatic numbers of $mC_4$ which are vertex-distinguished by multiple sets.

Key words: $mC_4$; I-total coloring; VI-total coloring; multiple sets; vertex-distinguished

CLC number: O157.5

0 Introduction

All graphs discussed in this paper are simple, non-directed graphs. Many conclusions have been obtained regarding the vertex-distinguished proper edge coloring[1-3] and vertex-distinguished general edge coloring[4-5] of graphs. In 2008, Zhang et al [6] proposed vertex-distinguished total coloring and related conjectures of graphs. In 2014, Chen et al[7] introduced vertex-distinguished I-total coloring and related conjectures of graphs. Many studies have been made on vertex-distinguished I-(VI-)total colorings of graphs [10-12]. In this study, we consider vertex-distinguished I-(VI-)total colorings of $mC_4$ by multiple sets.

Let $G$ be a simple graph. Suppose a mapping $f: V \cup E \rightarrow \{1,2,\cdots,l\}$ is a general total coloring of $G$ (not necessarily proper). If $\forall u,v \in V$, and $u,v$ are adjacent vertices, we have $f(u) \neq f(v)$, and if $uv,vw \in E, uv \neq vw$, we have $f(uv) \neq f(vw)$, then $f$ is called the I-total coloring of $G$. If any two adjacent edges of $G$ receives different colors, then $f$ is called VI-total coloring of $G$. Obviously, I-total coloring is VI-total coloring, and the reverse is uncertain. For an I-total coloring (resp.VI-total coloring) $f$ of $G$, if $l$ colors are used, then $f$ is called I-total coloring of $G$ (resp.l-VI-total coloring). Note that when we refer to the I-VI-total coloring (resp.l-VI-total coloring) of graph, we always assume that the colors used are 1,2,⋯,l.

Let $f$ be a general total coloring of $G$. For any vertex $x$ in $G$, $\tilde{C}_r(x)$ denotes the multiple set of colors of vertex $x$ and edges that are incident of vertex $x$. $\tilde{C}_r(x)$ is said to be the color set of $x$ under $f$. No confusion arises when using $\tilde{C}(x)$. Obviously, $|\tilde{C}_r(x)|=d_c(x)+1$. If $\tilde{C}(u) \neq \tilde{C}(v)$ for any two distinct vertices $u$ and $v$ of $G$, then $f$ is called...
vertex-distinguished by multiple sets. Let

\[ \chi_{\min}(G) = \min\{l | G \text{ has } l\text{-I-total coloring which is vertex-distinguished by multiple sets} \} \]

and

\[ \chi_{\max}(G) = \min\{l | G \text{ has } l\text{-VI-total coloring which is vertex-distinguished by multiple sets} \}. \]

Then, \( \chi_{\min}(G) \) is called the I-total chromatic number of \( G \) which is vertex-distinguished by multiple sets. Similarly, \( \chi_{\max}(G) \) is called the VI-total chromatic number of \( G \) which is vertex-distinguished by multiple sets. Let \( n_i(G) \) represent the number of vertices of degree \( i \). Suppose that

\[ \tilde{\chi}(G) = \min\{l\left[ l\binom{i}{1} + \binom{l}{i+1} \right] \geq n_i, \delta \leq i \leq \Delta \}. \]

**Proposition 1**

\[ \chi_{\min}(G) \geq \chi_{\max}(G) \geq \tilde{\chi}(G). \]

Let \( A_{i+j,0} = \begin{bmatrix} \{1,1,1\} & \{2,2,1\} & \{3,3,1\} \\ \{1,2,1\} & \{2,3,1\} & \{3,4,1\} \\ \vdots & \vdots & \vdots \\ \{1,1-1,1\} & \{2,1,1\} & \emptyset \\ \{1,1,1\} & \emptyset & \emptyset \end{bmatrix} \]

Submatrix \( A_{i+j,0} [i_1,i_2,\ldots,i_j,j_1,j_2,\ldots,j_l] \) is an \( r \times s \) matrix. It is comprised by all the elements which are only in \( i_1, i_2, \ldots, i_j, j_1, j_2, \ldots, j_l \)-th rows but also in \( j_1, j_2, \ldots, j_l \)-th columns of \( A_{i+j,0} \). The following six schemes are presented for the I-total coloring of \( C_4 \) which are vertex-distinguished by multiple sets. Note that all lowercase letters represent different colors.

In Fig. 1(a), the color set of each vertex of \( C_4 \) is \( \{a, b, c\} \). This coloring scheme is Co1(a; b; c).

In Fig. 1(b), the color set of each vertex of \( C_4 \) is \( \{a, b, c\} \). This coloring scheme is Co2(a; b; c; d; e; f).

In Fig. 1(c), the color set of each vertex of \( C_4 \) is \( \{a, b, c\} \). This coloring scheme is Co3(a; b; c; d; e; f).

In Fig. 2(a), the color set of each vertex of \( C_4 \) is

\[ \begin{align*}
(a) & \text{ Co1(a; b; c)} \\
(b) & \text{ Co2(a; b; c; d; e; f)} \\
(c) & \text{ Co3(a; b; c; d; e; f)}
\end{align*} \]

**Proof**

Obviously, I-total coloring is VI-total coloring. Thus \( \chi_{\min}(G) \leq \chi_{\max}(G) \).

Set \( t = \chi_{\max}(G) \). \( G \) has \( t \)-VI-total coloring which are vertex-distinguished by multiple sets. For \( \delta \leq i \leq \Delta \), considering the vertices of the degree \( i \), we obtain

[Equation]

Thus, \( t \leq \tilde{\chi}(G) \), namely \( \chi_{\max}(G) \geq \tilde{\chi}(G) \). This completes the proof.

### 1 Preliminaries

We first define a matrix \( A_{i+j,0} \) for any \( l \geq 4 \),

\[ \begin{bmatrix}
\{1,1,1\} & \{2,2,1\} & \{3,3,1\} \\
\{1,2,1\} & \{2,3,1\} & \{3,4,1\} \\
\vdots & \vdots & \vdots \\
\{1,1-1,1\} & \{2,1,1\} & \emptyset \\
\{1,1,1\} & \emptyset & \emptyset \\
\end{bmatrix} \]

\[ \{a,c,b\}, \{b,d,a\}, \{a,e,b\}, \{a,b,f\}, \{a,b,c,d,e,f\} \]

In Fig. 2(b), the color set of each vertex of \( C_4 \) is \( \{a,b,c,d\} \). This coloring scheme is Co5(a; b; c; d; e; f).

In Fig. 2(c), the color set of each vertex of \( C_4 \) is \( \{a,b,c,d\} \). This coloring scheme is Co6(a; b; c; d; e; f).

**Fig. 2** The coloring scheme Co4(a; b; c; d; e; f), Co5(a; b; c; d; e; f), and Co6(a; b; c; d; e; f).

**Lemma 1** When \( 1 \leq j \leq l-2 \) (\( j \) is an odd number), \( \{j,j,l\} \), \( \{j+1,j+1,l\} \), \( \{j,l,l\} \), \( \{j+1,l,l\} \) are the color sets of the vertices under I-total coloring of \( C_4 \) which are vertex-distinguished by multiple sets in Fig. 1(a).

**Lemma 2** When \( i = 0 \) (mod 2), \( j = 1 \) (mod 2), and \( A_{i+j-0} \) \((i+1,j+1)\) are not \( \emptyset \), \( \{j,j+j+1,l\} \), \( \{j+1,l+1,l\} \), \( \{j+l,j+1+l\} \) are the color sets of the vertices under I-total coloring of \( C_4 \) which are vertex-distinguished by multiple sets in Fig. 1(b).
Lemma 3 If \( l \equiv 1, 6 \pmod{8} \), \( l \geq 9 \), then for \( A_{e_i \cup \cdot} \), except \( \emptyset \) can be divided into \( \frac{1}{4} \left( \binom{l+1}{2} - 1 \right) \) groups, each group has four 3-subsets. These are the color sets of the vertices under 1-total coloring of \( C_4 \) which are vertex-distinguished by multiple sets.

**Proof** We use Lemmas 1 and 2, and only consider the remaining entries of \( A_{e_i \cup \cdot} \).

Case 1: \( l \equiv 1 \pmod{8} \).

For each \( j \equiv 1 \pmod{8}, 1 \leq j \leq l - 8 \), considering the remaining entries of the \( j, j+2, j+4, j+6 \) columns: \( \{j, l-1, l\}, \{j+2, l-1, l\}, \{j+4, l-1, l\}, \{j+6, l-1, l\} \). These four 3-subsets are the color sets of the vertices in \( \text{Co4}(l; l-1; j+2; j+4; j+6) \).

Case 2: \( l \equiv 6 \pmod{8} \).

1. The remaining entries in \( 1, 2, l-3, l-2, l-1 \) columns can be divided into two groups, \( \{1, l-2, l\}, \{l-3, l-2, l\} \), \( \{l-1, l-1, l\} \), \( \{l-4, l-1, l\} \), \( \{l-6, l-1, l\} \), \( \{l-8, l-1, l\} \). The corresponding coloring schemes are \( \text{Co3}(l; l-1; l-3, 2, l) \) and \( \text{Co4}(l; l-1; 1, 2, l-3, l-2) \), respectively.

2. For each \( j \equiv 3 \pmod{8}, 3 \leq j \leq l-11 \), considering the remaining entries of the \( j, j+1, j+2, j+3, j+4, j+5, j+6, j+7 \) columns, which can be divided into three groups: \( \{j, l-1, l\}, \{j+1, l-1, l\}, \{j+2, l-1, l\} \), \( \{j+3, l-1, l\} \), \( \{j+4, l-1, l\} \), \( \{j+5, l-1, l\} \), \( \{j+6, l-1, l\} \), \( \{j+7, l-1, l\} \), \( \{j+8, l-1, l\} \), \( \{j+9, l-1, l\} \), \( \{j+10, l-1, l\} \). The corresponding coloring schemes are \( \text{Co4}(l; l-1; j+1; j+2; j+3) \), \( \text{Co4}(l; l-1; j+4; j+5; j+6; j+7) \), and \( \text{Co4}(l; l-2; j+2; j+4; j+6) \), respectively.

Lemma 4 If \( l \equiv 2, 5 \pmod{8} \), \( l \geq 10 \), then all non-empty sets in \( A_{e_i \cup \cdot} \) except for \( \{l-7, l-2, l\}, \{l-5, l-2, l\} \) (when \( l \equiv 2 \pmod{8} \)) or \( \{l-4, l-1, l\}, \{l-2, l-1, l\} \) (when \( l \equiv 5 \pmod{8} \)) can be divided into \( \frac{1}{4} \left( \binom{l+1}{2} - 3 \right) \) groups, and each group has four 3-subsets. These are the color sets of the vertices under 1-total coloring of \( C_4 \) which are vertex-distinguished by multiple sets.

**Proof** We use Lemmas 1 and 2, and only consider the remaining entries of \( A_{e_i \cup \cdot} \).

Case 1: \( l \equiv 2 \pmod{8} \).

For the remaining entries in \( 1, 2, l-3, l-2, l-1 \) columns, the grouping is obtained and the corresponding coloring scheme is determined using Lemma 3 Case 2.

For the six remaining entries in \( l-7, l-6, l-5, l-4 \) columns, there is a group \( \{l-7, l-1, l\}, \{l-6, l-1, l\}, \{l-5, l-1, l\}, \{l-4, l-1, l\} \), namely \( \text{Co4}(l; l-1; l-7; l-6; l-5; l-4) \).

This leaves the 3-subsets \( \{l-7, l-2, l\}, \{l-5, l-2, l\} \).

Case 2: \( l \equiv 5 \pmod{8} \).

For each \( j \equiv 1 \pmod{8}, 1 \leq j \leq l-5 \), considering the remaining entries in \( j, j+2, j+4, j+6 \) columns, the grouping is obtained and the corresponding coloring scheme is determined using Lemma 3 Case 1.

This leaves the 3-subsets \( \{l-4, l-1, l\}, \{l-2, l-1, l\} \).

Lemma 5 If \( l \equiv 7, 0 \pmod{8} \), \( l \geq 15 \), then all non-empty sets in \( A_{e_i \cup \cdot} \) except for \( \{l-6, l-1, l\}, \{l-4, l-1, l\}, \{l-2, l-1, l\} \) (when \( l \equiv 7 \pmod{8} \)) or \( \{l-5, l-1, l\}, \{l-4, l-1, l\} \) (when \( l \equiv 0 \pmod{8} \)) can be divided into \( \frac{1}{4} \left( \binom{l+1}{2} - 4 \right) \) groups, and each group has four 3-subsets. These are the color sets of the vertices under 1-total coloring of \( C_4 \) which are vertex-distinguished by multiple sets.

**Proof** We use Lemmas 1 and 2, and only consider the entries of \( A_{e_i \cup \cdot} \).

Case 1: \( l \equiv 7 \pmod{8} \).

For each \( j \equiv 1 \pmod{8}, 1 \leq j \leq l-7 \), considering the remaining entries in columns \( j, j+2, j+4, j+6 \), the grouping is obtained and the corresponding coloring scheme is determined using Lemma 3 Case 1.

This leaves the 3-subsets \( \{l-6, l-1, l\}, \{l-4, l-1, l\}, \{l-2, l-1, l\} \).

Case 2: \( l \equiv 0 \pmod{8} \).

For the remaining entries in \( 1, 2, l-3, l-2, l-1 \) columns, the grouping is obtained and the corresponding coloring scheme is determined using Lemma 3 Case 2 .

For \( j \equiv 3 \pmod{8}, 3 \leq j \leq l-6 \), considering the remaining entries of the \( j, j+1, j+2, j+3, j+4, j+5, j+6, j+7 \) columns, the grouping is obtained and the corresponding coloring scheme is determined using Lemma 3 Case 2.

This leaves the 3-subsets \( \{l-5, l-2, l\}, \{l-4, l-1, l\} \).

Lemma 6 If \( l \equiv 3 \pmod{8} \), \( l \geq 11 \), then all non-empty sets in \( A_{e_i \cup \cdot} \) except \( \{l-2, l-1, l\} \) can be divided into \( \frac{1}{4} \left( \binom{l+1}{2} - 2 \right) \) groups, and each group has four 3-subsets. These are the color sets of the vertices under 1-total coloring of \( C_4 \) which are vertex-distinguished by multiple sets.
Proof We use Lemmas 1 and 2, and only consider the remaining entries of $A_{j,n-l-\eta}^\ast$.

For each $j \equiv 1 \pmod{8}$, $l \leq j \leq l-3$, considering the remaining entries of the $j$, $j+2$, $j+4$, $j+6$ columns, the grouping is obtained and the corresponding coloring scheme is determined using Lemma 3 Case 1.

This leaves 3-subset $\{l-2, l-1, l\}$.

Lemma 7 If $l \equiv 4 \pmod{8}$, $l \geq 12$, then all non-empty sets in $A_{j,n-l-\eta}$ except for $\{l-9, l-2, l\}, \{l-7, l-2, l\}, \{l-5, l-2, l\}, \{l-5, l-1, l\}, \{l-4, l-1, l\}$ can be divided into $\left\lfloor \frac{l+1}{2} \right\rfloor - 6$ groups, and each group has four 3-subsets. These are the color sets of the vertices under l-total colorings of $C_n$ which are vertex-distinguished by multiple sets.

Proof We use Lemmas 1 and 2, and only consider the remaining entries of $A_{j,n-l-\eta}^\ast$.

For the remaining entries in $1, 2, l-3, l-2, l-1$ columns, the grouping is obtained and the corresponding coloring scheme is determined using Lemma 3 Case 2. For $j \equiv 3 \pmod{8}$, $3 \leq l \leq 10(l \geq 20)$, considering the remaining entries in columns $j, j+1, j+2, j+3, j+4, j+5, j+6, j+7$, the grouping is obtained and the corresponding coloring scheme is determined using Lemma 3 Case 2. For the remaining in $l-9, l-8, l-7, l-6$ columns, there is a group $\{\{l-9, l-1, l\}, \{l-8, l-1, l\}, \{l-7, l-1, l\}, \{l-6, l-1, l\}\}$ except for $\{l-9, l-2, l\}, \{l-7, l-2, l\}$, namely $\mathcal{C}_4(l; l-1; l-6; l-7; l-8; l-9)$.

This leaves the 3-subsets $\{l-9, l-2, l\}, \{l-7, l-2, l\}, \{l-5, l-2, l\}, \{l-5, l-1, l\}, \{l-4, l-1, l\}$.

2 Main Results and Their Proofs

Theorem 1 If $2\left(\frac{l-1}{2}\right) + \left(\frac{l-1}{3}\right) < 4m \leq 2\left(\frac{l}{2}\right)$ with $m \geq 1, l \geq 3$, then $\mathcal{C}_n(mC_l) = l$.

Proof Obviously, there is $l = \mathcal{C}_n(mC_l) \leq \tilde{\mathcal{C}}_n(G)$.

Therefore, we can directly give the l-total coloring of $mC_l$ which are vertex-distinguished by multiple sets.

(1) When $m = 1, l = 3$. Use $\{1, 2\}, \{1, 2, 3\}, \{1, 3, 3\}$, that is, $\mathcal{C}_4(1; 3; 3)$ to color the first $C_n$. Thus, the multiple 3-subsets $\{2, 2, 3\}, \{2, 3, 3\}, \{1, 2, 3\}$ remain.

(2) When $2 \leq m \leq 4, l = 4$. Based on the l-total coloring of the first $C_n$, we start coloring from the second $C_n$. According to Lemma 1, one $C_n$ can be colored with $\mathcal{C}_1(4; 1; 2)$. Subsequently, the multiple 3-subsets $\{3, 3, 3\}, \{3, 3, 4\}, \{1, 2, 4\}, \{1, 3, 4\}, \{2, 3, 4\}$ remain. The third $C_n$ is colored with $\mathcal{C}_2(3; 4; 2)$, under which the color sets of four vertices are $\{3, 3, 4\}, \{3, 3, 4\}$ and (1) remaining $\{2, 2, 3\}, \{2, 3, 3\}$. The fourth $C_n$ is colored with (1) remaining $\{1, 2, 3\}$ and $\{2, 4\}, \{1, 3, 4\}, \{2, 3, 4\}$, as illustrated in Fig. 3. Thus far, all multiple 3-subsets of $\{1, 2, 3, 4\}$ have been used.

Fig. 3 Vertex-distinguished l-total coloring of $C_n$
6; 1, 4) and Co2(8, 5; 4, 6; 3, 4). The remaining 3-subsets are \{7, 7, 8\}, \{7, 8, 8\}, \{1, 6, 8\}, \{1, 7, 8\}, \{2, 7, 8\}, \{3, 6, 8\}, \{3, 7, 8\}, \{4, 7, 8\}, \{5, 6, 8\}, \{5, 7, 8\}, \{6, 7, 8\}. The 3-subsets \{1, 6, 8\}, \{5, 6, 8\}, \{7, 7, 8\}, \{7, 8, 8\} are used to color the 26-th \(C_i\) with Co3(8; 7; 1, 6, 5). The 27-th \(C_i\) is colored with Co4(8; 7; 1; 2, 5; 6), under which the color sets of four vertices are \{1, 7, 8\}, \{2, 7, 8\}, \{5, 7, 8\}, \{6, 7, 8\}. The 3-subsets \{3, 6, 8\}, \{3, 7, 8\}, \{4, 7, 8\} and \(\Box\) remaining \(3, 4, 5\) are used to color the 28-th \(C_i\) with Co5(8; 3; 5, 4; 7; 6). Consequently, all multiple 3-subsets of \{1, 2, 3, 4, 5, 6, 7, 8\} have been used.

7. Let \(I \geq 9\), we recursively proceed as following process.

We have obtained \((I - 1)\)-total coloring of \(\left[\frac{1}{4} \left(2 \binom{l - 2}{2} + \binom{l - 1}{3}\right)\right]C_i\) which are vertex-distinguished by multiple sets. On this basis, we will construct the I-total coloring from the \(\left[\frac{1}{4} \binom{l - 1}{2} + \binom{l - 1}{3}\right] + 1\)-th \(C_i\) to \(\left[\frac{1}{4} \binom{l - 1}{2} + \binom{l - 1}{3}\right] + 1\)-th \(C_i\) which are vertex-distinguished by multiple sets.

When \(I \equiv 1 \pmod{8}, I \geq 9\). Using Lemma 3, we can obtain the \(I\)-I-total coloring of \(\left[\frac{1}{4} \binom{l - 1}{2} + \binom{l - 1}{3}\right]C_i\) which are vertex-distinguished by multiple sets, and we have used all 3-subsets of \{1, 2, \cdots, I\}.

When \(I \equiv 2 \pmod{8}, I \geq 10\). Using Lemma 4, we can obtain the \(I\)-I-total coloring of \(\left[\frac{1}{4} \binom{l - 1}{2} + \binom{l - 1}{3}\right] - 2\) \(C_i\), which are vertex-distinguished by multiple sets, and we have used all 3-subsets of \{1, 2, \cdots, I\} except for \{\{l - 7, l - 2, l\}, \{l - 5, l - 2, l\}\}.

When \(I \equiv 3 \pmod{8}, I \geq 11\). Using Lemma 6, we can obtain \(I\)-I-total coloring of \(\left[\frac{1}{4} \binom{l - 1}{2} + \binom{l - 1}{3}\right] - 1\) \(C_i\), which are vertex-distinguished by multiple sets. We have used all 3-subsets of \{1, 2, \cdots, I\} except for \{\{l - 2, l - 1, l\}\} and the above mentioned \{\{l - 8, l - 3, l - 1\}, \{l - 6, l - 3, l - 1\}\}.

When \(I \equiv 4 \pmod{8}, I \geq 12\). Using Lemma 7, we can obtain the \(I\)-I-total coloring of \(\left[\frac{1}{4} \binom{l - 1}{2} + \binom{l - 1}{3}\right] - 8\) \(C_i\), which are vertex-distinguished by multiple sets. We have used all 3-subsets of \{1, 2, \cdots, I\} except for \{\{l - 9, l - 2, l\}, \{l - 7, l - 2, l\}, \{l - 5, l - 2, l\}, \{l - 5, l - 1, l\}, \{l - 5, l - 1, l\}\} and the above mentioned \{\{l - 3, l - 2, l - 1\}, \{l - 9, l - 4, l - 2\}, \{l - 7, l - 4, l - 2\}\}. The \(\left[\frac{1}{4} \binom{l - 1}{2} + \binom{l - 1}{3}\right] - 4\) th \(C_i\) is colored with Co2(l - 2, l - 4; 1, l - 7, l - 9, l - 1)\(\cdots\), under which the color sets of four vertices are \{\{l - 9, l - 2, l\}, \{l - 7, l - 2, l\}, \{l - 9, l - 4, l - 2\}, \{l - 7, l - 4, l - 2\}\). The 3-subsets \{\{l - 5, l - 2, l\}, \{l - 5, l - 1, l\}, \{l - 4, l - 1, l\}, \{l - 3, l - 2, l\} are used to color the \(\left[\frac{1}{4} \binom{l - 1}{2} + \binom{l - 1}{3}\right] - 2\) th \(C_i\) with Co5(l - 1, l - 3, l - 2, l - 5, l - 4). At this time, we obtained the \(I\)-I-total coloring of \(\left[\frac{1}{4} \binom{l - 1}{2} + \binom{l - 1}{3}\right] - 2\) \(C_i\) which are vertex-distinguished by multiple sets. Moreover, all multiple 3-subsets of \{1, 2, \cdots, I\} have been used.

When \(I \equiv 5 \pmod{8}, I \equiv 13\). Using Lemma 4, we can obtain the \(I\)-I-total coloring of \(\left[\frac{1}{4} \binom{l - 1}{2} + \binom{l - 1}{3}\right] - 5\) \(C_i\), which are vertex-distinguished by multiple sets. We have used all 3-subsets of \{1, 2, \cdots, I\} except for \{\{l - 4, l - 1, l\}, \{l - 2, l - 1, l\}\}.

When \(I \equiv 6 \pmod{8}, I \equiv 14\). Using Lemma 3, we can obtain the \(I\)-I-total coloring of \(\left[\frac{1}{4} \binom{l - 1}{2} + \binom{l - 1}{3}\right] - 2\) \(C_i\), which are vertex-distinguished by multiple sets. We have used all 3-subsets of \{1, 2, \cdots, I\} except for the above mentioned \{\{l - 5, l - 2, l - 1\}, \{l - 3, l - 2, l - 1\}\}.

When \(I \equiv 7 \pmod{8}, I \equiv 15\). Using Lemma 5, we can obtain the \(I\)-I-total coloring of \(\left[\frac{1}{4} \binom{l - 1}{2} + \binom{l - 1}{3}\right] - 5\) \(C_i\), which are vertex-distinguished by multiple sets. We have used all the 3-subsets of \{1, 2, \cdots, I\} except for \{\{l - 6, l - 1, l\}, \{l - 4, l - 1, l\}, \{l - 2, l - 1, l\}\} and the above mentioned \{\{l - 6, l - 3, l - 2\}, \{l - 4, l - 3, l - 2\}\}. The 3-subsets \{\{l - 6, l - 1, l\}, \{l - 4, l - 1, l\}, \{l - 2, l - 1, l\}, \{l - 6, l - 3, l - 2\}\} are used to color the \(\left[\frac{1}{4} \binom{l - 1}{2} + \binom{l - 1}{3}\right] - 3\) th \(C_i\) with Co6(l - 1, l - 4; l - 6, l - 3, l - 2). Then the 3-subset \{\{l - 4, l - 3, l - 2\}\} remains.

When \(I \equiv 0 \pmod{8}, I \equiv 16\). Using Lemma 5, we can obtain the \(I\)-I-total coloring of \(\left[\frac{1}{4} \binom{l - 1}{2} + \binom{l - 1}{3}\right] - 4\) \(C_i\), which are vertex-distinguished by multiple sets. We have used all the 3-subsets of \{1, 2, \cdots, I\} except for \{\{l - 5, l - 2, l\}, \{l - 5, l - 1, l\}, \{l - 4, l - 1, l\}\} and the above mentioned \{\{l - 5, l - 4, l - 3\}\}. The \(\left[\frac{1}{4} \binom{l - 1}{2} + \binom{l - 1}{3}\right] - 5\) th \(C_i\) is colored with the above four 3-subsets, that is, Co5(l - 1, l - 4, l - 3).
5; \ l - 3, \ l - 4; \ l - 1; \ l - 2). Thus far, all multiple 3-subsets of \{1, 2, \ldots, \l\} have been used.

The theorem is proven.

**Theorem 2** If \(2\left(\binom{l-1}{2} + \binom{l-1}{3}\right) < 4m \leq 2\left(\binom{l}{2}\right) + \binom{l}{3}\), \(m \geq 1, l \geq 3\), then \(\chi_v(mC_4) = l\).

**Proof** This conclusion can be obtained by the proof of Proposition 1 and Theorem 1.

3 Conclusion

In this study, the I-(VI-)total chromatic numbers of \(mC_4\) have been obtained, which are vertex-distinguished by multiple sets. According to the characteristics of the cycles and multiple sets, the \(mC_4\) (even number) of the I-(VI-)total chromatic numbers and VI-total of the multiple sets can be similarly obtained according to the above methods. That is, if \(2\left(\binom{l-1}{2} + \binom{l-1}{3}\right) < nm \leq 2\left(\binom{l}{2}\right) + \binom{l}{3}\), \(m \geq 1, l \geq 3\) is satisfied, then \(\chi_v(mC_4) = l\) and \(\chi_v(mC_4) = l\), and two cases of recursive boundary conditions can be inferred in the proof process: if \(3 | n\), then \(2n\); if \(3 | n\), then \(6n\). The I-(VI-)total colorings of odd order cycles which are vertex-distinguished by multiple sets will be studied at a later stage.

References


