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Noether Theorem for Fractional Singular Systems

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Abstract: Noether theorems for two fractional singular systems are discussed. One system involves mixed integer and Caputo fractional derivatives, and the other involves only Caputo fractional derivatives. Firstly, the fractional primary constraints and the fractional constrained Hamilton equations are given. Then, the fractional Noether theorems of the two fractional singular systems are established, including the fractional Noether identities, the fractional Noether quasi-identities and the fractional conserved quantities. Finally, the results obtained are illustrated by two examples.

Key words: singular system; fractional primary constraint; fractional constrained Hamilton equation; Noether theorem; conserved quantity

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0 Introduction

Given a Lagrangian $L=L\left(t,\boldsymbol{q},\dot{\boldsymbol{q}}\right)$, $\boldsymbol{q}=\left(q_1,q_2,\cdots,q_n\right)$, $\dot{\boldsymbol{q}}=\left(\dot{q}_1,\dot{q}_2,\cdots,\dot{q}_n\right)$, we define

$$H_{ij}=\frac{\partial^2 L}{\partial\dot{q}_i\partial\dot{q}_j},\ i,j=1,2,\cdots,n \tag{1}$$

where the matrix $\left[H_{ij}\right]$ is called a Hessian matrix. If $\det\left[H_{ij}\right]\neq 0$, then the Lagrangian is called regular. If $\det\left[H_{ij}\right]=0$, then the Lagrangian is called singular. For example, the Parra's Lagrangian[1]

$$L=\frac{1}{2}m\left(\dot{q}_1^2+\dot{q}_2^2+l^2\dot{q}_3^2+2l\dot{q}_1\dot{q}_3\cos q_3+2l\dot{q}_2\dot{q}_3\sin q_3\right)+V\left(q_1,q_2,q_3\right) \tag{2}$$

the Deriglazov's Lagrangian[2]

$$L=q_2^2\dot{q}_1^2+q_1^2\dot{q}_2^2+2q_1q_2\dot{q}_1\dot{q}_2+V\left(q_1,q_2\right)\left(V=q_1^2+q_2^2\right) \tag{3}$$

the Cawley's Lagrangian[3]

$$L=\dot{q}_1\dot{q}_2+V\left(q_1,q_2,q_3\right)\left(V=\frac{1}{2}q_2q_3^2\right) \tag{4}$$

and the Mittelstaedt's Lagrangian[4]

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$$L=\frac{1}{2m}\left(\dot{q}_1+\dot{q}_2\right)^2+\frac{1}{2\mu}\dot{q}_3^2+V\left(q_1,q_2,q_3\right) \tag{5}$$

where l, m and μ are constants and V represents potential energy, they are all singular. In fact, singular systems have a close relationship with the condensed matter theories, the gauge field theories, the quantum field theories of anyons, the particle physics and so forth[5-7]. Regarding the singular systems, Dirac[8] was the first one to study their canonical equation. Then, singular systems were also investigated in several classical mechanics textbooks[5-7], including both the canonical equation and the Noether theorem.

The Noether theorem was introduced by the German female mathematician Noether[9]. The Noether symmetry method is one of the methods used to solve the differential equations of motion. There are many results on the Noether theorem[10-12]. Fractional calculus has been popular recently. The fractional derivatives that are used most often are the Riemann-Liouville, Caputo and Riesz fractional derivatives. In 2007, Frederico and Torres[13] initiated the study of the fractional Noether theorem. Based on a bilinear operator D ($D(I)=0$), they defined the fractional conserved quantity. Using this definition, fractional Noether theorems and their applications were discussed for several mechanics systems, such as the Lagrangian system[14-17], the Birkhoffian system[18,19], the Hamiltonian system[20], and the multidimensional Lagrangian system[21]. Two years later, with the idea of the definition of the classical conserved quantity, Atanacković *et al*[22] introduced another definition of the fractional conserved quantity ($\mathrm{d}I/\mathrm{d}t=0$). They held the point that this definition is more reasonable than the former definition. Then, fractional Noether theorems of the different mechanics systems, such as the Birkhoffian system[23-25], the Hamiltonian system[26,27] and the nonconservative system[28,29] were obtained.

At present, two fractional singular systems, one concerning the mixed integer and Caputo fractional derivatives and the other concerning the Caputo fractional derivatives, have been established, including the fractional primary constraints and the fractional constrained Hamilton equations[30]. The next task is to find the solutions to them. Therefore, in this paper, we intend to make use of the Noether symmetry method to complete this study.

1 Preliminaries

We give the definitions of the Riemann-Liouville and the Caputo fractional derivatives as follows. Given a function $f(t)$ and two constants α and β that satisfy $n-1\leqslant\alpha,\beta<n$, where n is an integer, the Riemann-Liouville fractional derivative and the Caputo fractional derivative have the forms[31]

$${}_{t_1}^{RL}D_t^{\alpha}f(t)=\frac{1}{\Gamma(n-\alpha)}\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^n\int_{t_1}^{t}(t-\xi)^{n-\alpha-1}f(\xi)\mathrm{d}\xi \tag{6}$$

$${}_{t}^{RL}D_{t_2}^{\beta}f(t)=\frac{1}{\Gamma(n-\beta)}\left(-\frac{\mathrm{d}}{\mathrm{d}t}\right)^n\int_{t}^{t_2}(\xi-t)^{n-\beta-1}f(\xi)\mathrm{d}\xi \tag{7}$$

$${}_{t_1}^{C}D_t^{\alpha}f(t)=\frac{1}{\Gamma(n-\alpha)}\int_{t_1}^{t}(t-\xi)^{n-\alpha-1}\left(\frac{\mathrm{d}}{\mathrm{d}\xi}\right)^n f(\xi)\mathrm{d}\xi \tag{8}$$

$${}_{t}^{C}D_{t_2}^{\beta}f(t)=\frac{1}{\Gamma(n-\beta)}\int_{t}^{t_2}(\xi-t)^{n-\beta-1}\left(-\frac{\mathrm{d}}{\mathrm{d}\xi}\right)^n f(\xi)\mathrm{d}\xi \tag{9}$$

here, α and β represent the orders of the fractional derivatives. When $\alpha,\beta\to1$, the fractional derivative operators reduce to the classical integer derivative operators, namely,

$${}_{t_1}^{RL}D_t^{1}={}_{t_1}^{C}D_t^{1}=\frac{\mathrm{d}}{\mathrm{d}t},\ {}_{t}^{RL}D_{t_2}^{1}={}_{t}^{C}D_{t_2}^{1}=-\frac{\mathrm{d}}{\mathrm{d}t} \tag{10}$$

Throughout this paper, we assume that $0<\alpha,\beta<1$.

For the Lagrangian $L_M=L_M\left(t,\boldsymbol{q_M},\dot{\boldsymbol{q}}_{\boldsymbol{M}},{}_{t_1}^{C}\boldsymbol{D}_t^{\alpha}\boldsymbol{q_M}\right)$, where $\boldsymbol{q_M}=\left(q_{M1},q_{M2},\cdots,q_{Mn}\right)$, $\dot{\boldsymbol{q}}_{\boldsymbol{M}}=\left(\dot{q}_{M1},\dot{q}_{M2},\cdots,\dot{q}_{Mn}\right)$, ${}_{t_1}^{C}\boldsymbol{D}_t^{\alpha}\boldsymbol{q_M}=\left({}_{t_1}^{C}D_t^{\alpha}q_{M1},{}_{t_1}^{C}D_t^{\alpha}q_{M2},\cdots,{}_{t_1}^{C}D_t^{\alpha}q_{Mn}\right)$, q_{Mj} are the generalized coordinates, $\dot{q}_{Mj}=\mathrm{d}q_{Mj}/\mathrm{d}t$ are the generalized velocities, ${}_{t_1}^{C}D_t^{\alpha}q_{Mj}$ are the Caputo derivatives of q_{Mj}, $q_{Mj}(\cdot)\in\mathbb{C}^2\left(\left[t_1,t_2\right];\mathbb{R}\right)$, $j=1,2,\cdots,n$, $L_M(\cdot,\cdot,\cdot,\cdot)\in\mathbb{C}^2\left(\left[t_1,t_2\right]\times\mathbb{R}^n\times\mathbb{R}^n\times\mathbb{R}^n;\mathbb{R}\right)$, and $0<\alpha<1$, we define the corresponding generalized momenta and the Hamiltonian as

$$p_{Mi}=\frac{\partial L_M\left(t,\boldsymbol{q_M},\dot{\boldsymbol{q}}_{\boldsymbol{M}},{}_{t_1}^{C}\boldsymbol{D}_{\boldsymbol{t}}^{\boldsymbol{\alpha}}\boldsymbol{q_M}\right)}{\partial\dot{q}_{Mi}},\ p_{Mi}^{\alpha}=\frac{\partial L_M\left(t,\boldsymbol{q_M},\dot{\boldsymbol{q}}_{\boldsymbol{M}},{}_{t_1}^{C}\boldsymbol{D}_{\boldsymbol{t}}^{\boldsymbol{\alpha}}\boldsymbol{q_M}\right)}{\partial\,{}_{t_1}^{C}D_t^{\alpha}q_{Mi}},$$

$$H_M=p_{Mi}\dot{q}_{Mi}+p_{Mi}^{\alpha}\cdot{}_{t_1}^{C}D_t^{\alpha}q_{Mi}-L_M\left(t,\boldsymbol{q_M},\dot{\boldsymbol{q}}_{\boldsymbol{M}},{}_{t_1}^{C}\boldsymbol{D}_{\boldsymbol{t}}^{\boldsymbol{\alpha}}\boldsymbol{q_M}\right),\ i=1,2,\cdots,n \tag{11}$$

Specifically, we assume that ${}_{t_1}^{C}D_t^{\alpha}q_{Mi}$ can always be described by a function that depends on the elements of t, q_{Mj}, $\dot{q}_{Mj}$ and p_{Mj}^{α}, namely, ${}_{t_1}^{C}D_t^{\alpha}q_{Mi}=h_{Mi}\left(t,q_{Mj},\dot{q}_{Mj},p_{Mj}^{\alpha}\right)$, $i,j=1,2,\cdots,n$.

In this case, we define the elements of the Hessian matrix as

$$H_{Mij}=\frac{\partial^2 L_M\left(t,\boldsymbol{q_M},\dot{\boldsymbol{q}}_{\boldsymbol{M}},{}_{t_1}^{C}\boldsymbol{D}_{\boldsymbol{t}}^{\boldsymbol{\alpha}}\boldsymbol{q_M}\right)}{\partial\dot{q}_{Mi}\partial\dot{q}_{Mj}},\ i,j=1,2,\cdots,n \tag{12}$$

if $\operatorname{rank}\left[H_{Mij}\right]=R$, $0\leqslant R<n$, then the fractional primary constraints with the mixed derivatives have the forms [30]

$$\phi_{Ma}\left(t,q_{Mj},p_{Mj},p_{Mj}^{\alpha}\right)=0 \tag{13}$$

where $a=1,2,\cdots,n-R$, $0\leqslant R<n$, $j=1,2,\cdots,n$. The fractional constrained Hamilton equations with the mixed derivatives have the forms [30]

$$\dot{q}_{Mi}=\frac{\partial H_M}{\partial p_{Mi}}+\lambda_{Ma}\frac{\partial\phi_{Ma}}{\partial p_{Mi}},\ \dot{p}_{Mi}=-\frac{\partial H_M}{\partial q_{Mi}}+{}_{t}^{RL}D_{t_2}^{\alpha}p_{Mi}^{\alpha}-\lambda_{Ma}\frac{\partial\phi_{Ma}}{\partial q_{Mi}},\ {}_{t_1}^{C}D_t^{\alpha}q_{Mi}=\frac{\partial H_M}{\partial p_{Mi}^{\alpha}}+\lambda_{Ma}\frac{\partial\phi_{Ma}}{\partial p_{Mi}^{\alpha}} \tag{14}$$

where $H_M=H_M\left(t,\boldsymbol{q_M},\boldsymbol{p_M},\boldsymbol{p_M^{\alpha}}\right)$, $\boldsymbol{q_M}=\left(q_{M1},q_{M2},\cdots,q_{Mn}\right)$, $\boldsymbol{p_M}=\left(p_{M1},p_{M2},\cdots,p_{Mn}\right)$, $\boldsymbol{p_M^{\alpha}}=\left(p_{M1}^{\alpha},p_{M2}^{\alpha},\cdots,p_{Mn}^{\alpha}\right)$, $\lambda_{Ma}(t)$ are the Lagrange multipliers, $a=1,2,\cdots,n-R$, $0\leqslant R<n$, and $i=1,2,\cdots,n$.

However, when the Lagrange multipliers cannot be solved, Eq. (14) is invalid, and the fractional constrained Hamilton equations with the mixed derivatives have another forms, which have been investigated in Ref. [30]. In this paper, we discuss only the case in which the Lagrange multipliers can be solved.

For the Lagrangian $L_C=L_C\left(t,\boldsymbol{q_C},{}_{t_1}^{C}\boldsymbol{D}_{\boldsymbol{t}}^{\boldsymbol{\alpha}}\boldsymbol{q_C}\right)$, where $\boldsymbol{q_C}=\left(q_{C1},q_{C2},\cdots,q_{Cn}\right)$, ${}_{t_1}^{C}\boldsymbol{D}_{\boldsymbol{t}}^{\boldsymbol{\alpha}}\boldsymbol{q_C}=\left({}_{t_1}^{C}D_t^{\alpha}q_{C1},{}_{t_1}^{C}D_t^{\alpha}q_{C2},\cdots,{}_{t_1}^{C}D_t^{\alpha}q_{Cn}\right)$, q_{Cj} are the generalized coordinates, ${}_{t_1}^{C}D_t^{\alpha}q_{Cj}$ are the Caputo derivatives of q_{Cj}, $q_{Cj}(\cdot)\in\mathbb{C}^2\left(\left[t_1,t_2\right];\mathbb{R}\right)$, $j=1,2,\cdots,n$, $L_C(\cdot,\cdot,\cdot)\in\mathbb{C}^2\left(\left[t_1,t_2\right]\times\mathbb{R}^n\times\mathbb{R}^n;\mathbb{R}\right)$, and $0<\alpha<1$, we define the corresponding generalized momenta and the Hamiltonian as

$$p_{Ci}=\frac{\partial L_C\left(t,\boldsymbol{q_C},{}_{t_1}^{C}\boldsymbol{D}_{\boldsymbol{t}}^{\boldsymbol{\alpha}}\boldsymbol{q_C}\right)}{\partial\,{}_{t_1}^{C}D_t^{\alpha}q_{Ci}},\ H_C=p_{Ci}\cdot{}_{t_1}^{C}D_t^{\alpha}q_{Ci}-L_C\left(t,\boldsymbol{q_C},{}_{t_1}^{C}\boldsymbol{D}_{\boldsymbol{t}}^{\boldsymbol{\alpha}}\boldsymbol{q_C}\right),\ i=1,2,\cdots,n \tag{15}$$

We assume that the Lagrangian $L_C\left(t,\boldsymbol{q_C},{}_{t_1}^{C}\boldsymbol{D}_{\boldsymbol{t}}^{\boldsymbol{\alpha}}\boldsymbol{q_C}\right)$ is singular, i.e., only R elements of ${}_{t_1}^{C}D_t^{\alpha}q_{Ci}$, $i=1,2,\cdots,n$, can be solved, where $0\leqslant R<n$.

In this case, the fractional primary constraints with the Caputo fractional derivatives have the forms [30]

$$\phi_{Ca}\left(t,q_{Cj},p_{Cj}\right)=0,\ a=1,2,\cdots,n-R,\ 0\leqslant R<n,\ j=1,2,\cdots,n \tag{16}$$

The fractional constrained Hamilton equations with the Caputo fractional derivatives have the forms [30]

$${}_{t_1}^{C}D_t^{\alpha}q_{Ci}=\frac{\partial H_C}{\partial p_{Ci}}+\lambda_{Ca}\frac{\partial\phi_{Ca}}{\partial p_{Ci}},\ {}_{t}^{RL}D_{t_2}^{\alpha}p_{Ci}=\frac{\partial H_C}{\partial q_{Ci}}+\lambda_{Ca}\frac{\partial\phi_{Ca}}{\partial q_{Ci}} \tag{17}$$

where $H_C=H_C\left(t,\boldsymbol{q_C},\boldsymbol{p_C}\right)$, $\boldsymbol{q_C}=\left(q_{C1},q_{C2},\cdots,q_{Cn}\right)$, $\boldsymbol{p_C}=\left(p_{C1},p_{C2},\cdots,p_{Cn}\right)$, λ_{Ca} are the Lagrange multipliers, $a=1,2,\cdots,n-R$, $0\leqslant R<n$, and $i=1,2,\cdots,n$.

Similarly, when the Lagrange multipliers cannot be solved, Eq. (17) is invalid, and the fractional constrained Hamilton equations with the Caputo fractional derivatives have another forms, which have been investigated in Ref. [30]. In this paper, we discuss only the case in which the Lagrange multipliers can be solved.

2 Fractional Noether Theorem with Mixed Derivatives

Noether symmetry with the mixed derivatives is determined by the Noether symmetric transformations, under which the fractional Hamilton action with the mixed derivatives

$$I_M=\int_{t_1}^{t_2}L_M\left(t,\boldsymbol{q_M},\dot{\boldsymbol{q}}_M,{}_{t_1}^{C}\boldsymbol{D}_t^{\alpha}\boldsymbol{q_M}\right)\mathrm{d}t=\int_{t_1}^{t_2}\left(p_{Mi}\dot{q}_{Mi}+p_{Mi}^{\alpha}\cdot{}_{t_1}^{C}D_t^{\alpha}q_{Mi}-H_M\left(t,\boldsymbol{q_M},\boldsymbol{p_M},\boldsymbol{p_M^{\alpha}}\right)\right)\mathrm{d}t \tag{18}$$

remains invariant. Therefore, if we want to study the Noether theorem, we first need to give the infinitesimal transformations with the mixed derivatives. Then, we discuss the change of the fractional Hamilton action (Eq. (18)) under the given infinitesimal transformations. Finally, the condition which is called the fractional Noether identity with the mixed derivatives is obtained.

Here, the infinitesimal transformations have the forms

$$\bar{t}=t+\Delta t,\ \bar{q}_{Mi}(\bar{t})=q_{Mi}(t)+\Delta q_{Mi},\ \bar{p}_{Mi}(\bar{t})=p_{Mi}(t)+\Delta p_{Mi},\ \bar{p}_{Mi}^{\alpha}(\bar{t})=p_{Mi}^{\alpha}(t)+\Delta p_{Mi}^{\alpha} \tag{19}$$

whose expansions are

$$\bar{t}=t+\theta_M\xi_{M0}\left(t,\boldsymbol{q_M},\boldsymbol{p_M},\boldsymbol{p_M^{\alpha}}\right)+o(\theta_M),\ \bar{q}_{Mi}(\bar{t})=q_{Mi}(t)+\theta_M\xi_{Mi}\left(t,\boldsymbol{q_M},\boldsymbol{p_M},\boldsymbol{p_M^{\alpha}}\right)+o(\theta_M),$$
$$\bar{p}_{Mi}(\bar{t})=p_{Mi}(t)+\theta_M\eta_{Mi}\left(t,\boldsymbol{q_M},\boldsymbol{p_M},\boldsymbol{p_M^{\alpha}}\right)+o(\theta_M),\ \bar{p}_{Mi}^{\alpha}(\bar{t})=p_{Mi}^{\alpha}(t)+\theta_M\eta_{Mi}^{\alpha}\left(t,\boldsymbol{q_M},\boldsymbol{p_M},\boldsymbol{p_M^{\alpha}}\right)+o(\theta_M) \tag{20}$$

where ξ_{M0}, ξ_{Mi}, η_{Mi} and η_{Mi}^{α} are called the infinitesimal generators with the mixed derivatives, θ_M is a small parameter, and $i=1,2,\cdots,n$.

We denote the change of the fractional Hamilton action as ΔI_M, namely, $\Delta I_M=\bar{I}_M-I_M$. If we consider only the linear part of θ_M, then we have

$$\begin{aligned}
\Delta I_M=&\int_{\bar{t}_1}^{\bar{t}_2}\left(\bar{p}_{Mi}\dot{\bar{q}}_{Mi}+\bar{p}_{Mi}^{\alpha}{}_{\bar{t}_1}^{C}D_{\bar{t}}^{\alpha}\bar{q}_{Mi}-H_M\left(t,\bar{\boldsymbol{q}}_M,\bar{\boldsymbol{p}}_M,\bar{\boldsymbol{p}}_M^{\alpha}\right)\right)\mathrm{d}\bar{t}-\int_{t_1}^{t_2}\left(p_{Mi}\dot{q}_{Mi}+p_{Mi}^{\alpha}{}_{t_1}^{C}D_t^{\alpha}q_{Mi}-H_M\left(t,\boldsymbol{q_M},\boldsymbol{p_M},\boldsymbol{p_M^{\alpha}}\right)\right)\mathrm{d}t\\
=&\int_{t_1}^{t_2}\Bigg[\left(p_{Mi}+\Delta p_{Mi}\right)\left(\dot{q}_{Mi}+\Delta\dot{q}_{Mi}\right)+\left(p_{Mi}^{\alpha}+\Delta p_{Mi}^{\alpha}\right)\cdot\left({}_{t_1}^{C}D_t^{\alpha}q_{Mi}+{}_{t_1}^{C}D_t^{\alpha}\delta q_{Mi}+\Delta t\frac{\mathrm{d}}{\mathrm{d}t}{}_{t_1}^{C}D_t^{\alpha}q_{Mi}-\frac{1}{\Gamma(1-\alpha)}\left((t-t_1)^{-\alpha}\dot{q}_{Mi}(t_1)\Delta t_1\right)\right)\\
&-H_M\left(t+\Delta t,q_{Mj}+\Delta q_{Mj},p_{Mj}+\Delta p_{Mj},p_{Mj}^{\alpha}+\Delta p_{Mj}^{\alpha}\right)\Bigg]\left(1+\frac{\mathrm{d}}{\mathrm{d}t}\Delta t\right)\mathrm{d}t-\int_{t_1}^{t_2}\left(p_{Mi}\dot{q}_{Mi}+p_{Mi}^{\alpha}{}_{t_1}^{C}D_t^{\alpha}q_{Mi}-H_M\left(t,\boldsymbol{q_M},\boldsymbol{p_M},\boldsymbol{p_M^{\alpha}}\right)\right)\mathrm{d}t\\
=&\int_{t_1}^{t_2}\Bigg[p_{Mi}\dot{q}_{Mi}+p_{Mi}\Delta\dot{q}_{Mi}+\Delta p_{Mi}\dot{q}_{Mi}+p_{Mi}^{\alpha}{}_{t_1}^{C}D_t^{\alpha}q_{Mi}+p_{Mi}^{\alpha}{}_{t_1}^{C}D_t^{\alpha}\delta q_{Mi}+p_{Mi}^{\alpha}\Delta t\frac{\mathrm{d}}{\mathrm{d}t}{}_{t_1}^{C}D_t^{\alpha}q_{Mi}+\Delta p_{Mi}^{\alpha}{}_{t_1}^{C}D_t^{\alpha}q_{Mi}-\frac{p_{Mi}^{\alpha}}{\Gamma(1-\alpha)}(t-t_1)^{-\alpha}\dot{q}_{Mi}(t_1)\Delta t_1\\
&-H_M\left(t,\boldsymbol{q_M},\boldsymbol{p_M},\boldsymbol{p_M^{\alpha}}\right)-\frac{\partial H_M}{\partial t}\Delta t-\frac{\partial H_M}{\partial q_{Mi}}\Delta q_{Mi}-\frac{\partial H_M}{\partial p_{Mi}}\Delta p_{Mi}-\frac{\partial H_M}{\partial p_{Mi}^{\alpha}}\Delta p_{Mi}^{\alpha}+\left(p_{Mi}\dot{q}_{Mi}+p_{Mi}^{\alpha}{}_{t_1}^{C}D_t^{\alpha}q_{Mi}-H_M\right)\frac{\mathrm{d}}{\mathrm{d}t}\Delta t\Bigg]\mathrm{d}t\\
&-\int_{t_1}^{t_2}\left(p_{Mi}\dot{q}_{Mi}+p_{Mi}^{\alpha}{}_{t_1}^{C}D_t^{\alpha}q_{Mi}-H_M\left(t,\boldsymbol{q_M},\boldsymbol{p_M},\boldsymbol{p_M^{\alpha}}\right)\right)\mathrm{d}t\\
=&\theta_M\int_{t_1}^{t_2}\Bigg[p_{Mi}\dot{\xi}_{Mi}+p_{Mi}^{\alpha}{}_{t_1}^{C}D_t^{\alpha}\left(\xi_{Mi}-\dot{q}_{Mi}\xi_{M0}\right)+\left(p_{Mi}^{\alpha}\frac{\mathrm{d}}{\mathrm{d}t}{}_{t_1}^{C}D_t^{\alpha}q_{Mi}-\frac{\partial H_M}{\partial t}\right)\xi_{M0}+\lambda_{Ma}\frac{\partial\phi_{Ma}}{\partial p_{Mi}^{\alpha}}\eta_{Mi}^{\alpha}\\
&-\frac{p_{Mi}^{\alpha}}{\Gamma(1-\alpha)}(t-t_1)^{-\alpha}\dot{q}_{Mi}(t_1)\xi_{M0}(t_1)+\left(p_{Mi}^{\alpha}{}_{t_1}^{C}D_t^{\alpha}q_{Mi}-H_M\right)\dot{\xi}_{M0}-\frac{\partial H_M}{\partial q_{Mi}}\xi_{Mi}+\lambda_{Ma}\frac{\partial\phi_{Ma}}{\partial p_{Mi}}\eta_{Mi}\Bigg]\mathrm{d}t
\end{aligned} \tag{21}$$

where

$${}_{\bar{t}_1}^{C}D_{\bar{t}}^{\alpha}\bar{q}_{Mi}={}_{t_1}^{C}D_t^{\alpha}q_{Mi}+{}_{t_1}^{C}D_t^{\alpha}\delta q_{Mi}+\Delta t\frac{\mathrm{d}}{\mathrm{d}t}{}_{t_1}^{C}D_t^{\alpha}q_{Mi}-\frac{1}{\Gamma(1-\alpha)}(t-t_1)^{-\alpha}\dot{q}_{Mi}(t_1)\Delta t_1,$$
$$\xi_{M0}(t_1)=\xi_{M0}\left(t_1,\boldsymbol{q_M}(t_1),\boldsymbol{p_M}(t_1),\boldsymbol{p_M^{\alpha}}(t_1)\right),\ \Delta\dot{q}_{Mi}=\theta_M\left(\dot{\xi}_{Mi}-\dot{q}_{Mi}\dot{\xi}_{M0}\right).$$

That fractional Hamilton action remains invariant implies $\Delta I_M=0$, therefore, from Eq. (21), we have

$$\begin{aligned}
&p_{Mi}\dot{\xi}_{Mi}+p_{Mi}^{\alpha}{}_{t_1}^{C}D_t^{\alpha}\left(\xi_{Mi}-\dot{q}_{Mi}\xi_{M0}\right)+\left(p_{Mi}^{\alpha}\frac{\mathrm{d}}{\mathrm{d}t}{}_{t_1}^{C}D_t^{\alpha}q_{Mi}-\frac{\partial H_M}{\partial t}\right)\xi_{M0}-\frac{\partial H_M}{\partial q_{Mi}}\xi_{Mi}+\lambda_{Ma}\frac{\partial\phi_{Ma}}{\partial p_{Mi}^{\alpha}}\eta_{Mi}^{\alpha}\\
&+\lambda_{Ma}\frac{\partial\phi_{Ma}}{\partial p_{Mi}}\eta_{Mi}-\frac{p_{Mi}^{\alpha}}{\Gamma(1-\alpha)}(t-t_1)^{-\alpha}\dot{q}_{Mi}(t_1)\xi_{M0}(t_1)+\left(p_{Mi}^{\alpha}{}_{t_1}^{C}D_t^{\alpha}q_{Mi}-H_M\right)\dot{\xi}_{M0}=0,\ i=1,2,\cdots,n
\end{aligned} \tag{22}$$

Equation (22) is called the fractional Noether identity with the mixed derivatives for the fractional constrained Hamiltonian system (Eq. (14)). The infinitesimal transformations in this case are called the Noether symmetric transformations, which determine the Noether symmetry.

In this paper, we adopt Atanacković's definition of the fractional conserved quantity. We review it first.

Definition 1 [22] A quantity C is called a fractional conserved quantity if and only if $\mathrm{d}C/\mathrm{d}t=0$ holds.

Theorem 1 If the infinitesimal generators ξ_{M0}, ξ_{Mi}, η_{Mi} and η_{Mi}^{α} satisfy the fractional Noether identity (Eq. (22)), then a fractional conserved quantity with the mixed derivatives exists for the fractional constrained Hamiltonian system (Eq. (14)), as follows:

$$C_M = \left(p^{\alpha}_{Mi\,t_1}{}^{C}D^{\alpha}_{t} q_{Mi} - H_M\right)\xi_{M0} + \int_{t_1}^{t}\left[p^{\alpha}_{Mi\,t_1}{}^{C}D^{\alpha}_{\tau}\left(\xi_{Mi} - \dot{q}_{Mi}\xi_{M0}\right) - \left(\xi_{Mi} - \dot{q}_{Mi}\xi_{M0}\right){}^{RL}_{\tau}D^{\alpha}_{t_2} p^{\alpha}_{Mi}\right]\mathrm{d}\tau$$
$$+p_{Mi}\xi_{Mi} - \int_{t_1}^{t}\frac{p^{\alpha}_{Mi}}{\Gamma(1-\alpha)}\left(\tau - t_1\right)^{-\alpha}\dot{q}_{Mi}\left(t_1\right)\xi_{M0}\left(t_1\right)\mathrm{d}\tau = \mathrm{const} \tag{23}$$

Proof It is obtained from Eqs. (14), (23) that

$$\frac{\mathrm{d}}{\mathrm{d}t}C_M = \left(\dot{p}^{\alpha}_{Mi\,t_1}{}^{C}D^{\alpha}_{t} q_{Mi} + p^{\alpha}_{Mi}\frac{\mathrm{d}}{\mathrm{d}t}{}^{C}_{t_1}D^{\alpha}_{t} q_{Mi} - \frac{\partial H_M}{\partial t} - \frac{\partial H_M}{\partial p_{Mi}}\dot{p}_{Mi} - \frac{\partial H_M}{\partial q_{Mi}}\dot{q}_{Mi} - \frac{\partial H_M}{\partial p^{\alpha}_{Mi}}\dot{p}^{\alpha}_{Mi}\right)\xi_{M0} + \left(p^{\alpha}_{Mi\,t_1}{}^{C}D^{\alpha}_{t} q_{Mi} - H_M\right)\dot{\xi}_{M0}$$
$$+p^{\alpha}_{Mi\,t_1}{}^{C}D^{\alpha}_{t}\left(\xi_{Mi} - \dot{q}_{Mi}\xi_{M0}\right) - \left(\xi_{Mi} - \dot{q}_{Mi}\xi_{M0}\right){}^{RL}_{t}D^{\alpha}_{t_2} p^{\alpha}_{Mi} + p_{Mi}\dot{\xi}_{Mi} - \frac{p^{\alpha}_{Mi}}{\Gamma(1-\alpha)}\left(t - t_1\right)^{-\alpha}\dot{q}_{Mi}\left(t_1\right)\xi_{M0}\left(t_1\right) + \dot{p}_{Mi}\xi_{Mi}$$
$$= \frac{\partial H_M}{\partial q_{Mi}}\xi_{Mi} + \dot{p}_{Mi}\xi_{Mi} + \left(\dot{p}^{\alpha}_{Mi\,t_1}{}^{C}D^{\alpha}_{t} q_{Mi} - \frac{\partial H_M}{\partial p_{Mi}}\dot{p}_{Mi} - \frac{\partial H_M}{\partial q_{Mi}}\dot{q}_{Mi} - \frac{\partial H_M}{\partial p^{\alpha}_{Mi}}\dot{p}^{\alpha}_{Mi}\right)\xi_{M0}$$
$$-\left(\xi_{Mi} - \dot{q}_{Mi}\xi_{M0}\right){}^{RL}_{t}D^{\alpha}_{t_2} p^{\alpha}_{Mi} - \lambda_{Ma}\frac{\partial \phi_{Ma}}{\partial p^{\alpha}_{Mi}}\eta^{\alpha}_{Mi} - \lambda_{Ma}\frac{\partial \phi_{Ma}}{\partial p_{Mi}}\eta_{Mi}$$
$$= \left(\dot{p}_{Mi} + \frac{\partial H_M}{\partial q_{Mi}} - {}^{RL}_{t}D^{\alpha}_{t_2} p^{\alpha}_{Mi} + \lambda_{Ma}\frac{\partial \phi_{Ma}}{\partial q_{Mi}}\right)\left(\xi_{Mi} - \dot{q}_{Mi}\xi_{M0}\right) - \lambda_{Ma}\frac{\partial \phi_{Ma}}{\partial p^{\alpha}_{Mi}}\eta^{\alpha}_{Mi} - \lambda_{Ma}\frac{\partial \phi_{Ma}}{\partial q_{Mi}}\left(\xi_{Mi} - \dot{q}_{Mi}\xi_{M0}\right)$$
$$+\left(\dot{p}^{\alpha}_{Mi\,t_1}{}^{C}D^{\alpha}_{t} q_{Mi} - \dot{p}^{\alpha}_{Mi}\frac{\partial H_M}{\partial p^{\alpha}_{Mi}} - \frac{\partial H_M}{\partial p_{Mi}}\dot{p}_{Mi}\right)\xi_{M0} - \lambda_{Ma}\frac{\partial \phi_{Ma}}{\partial p_{Mi}}\eta_{Mi} + \dot{p}_{Mi}\xi_{M0}\left(\frac{\partial H_M}{\partial p_{Mi}} + \lambda_{Ma}\frac{\partial \phi_{Ma}}{\partial p_{Mi}}\right)$$
$$= -\lambda_{Ma}\frac{\partial \phi_{Ma}}{\partial p^{\alpha}_{Mi}}\eta^{\alpha}_{Mi} - \lambda_{Ma}\frac{\partial \phi_{Ma}}{\partial p_{Mi}}\eta_{Mi} + \dot{p}^{\alpha}_{Mi}\xi_{M0}\lambda_{Ma}\frac{\partial \phi_{Ma}}{\partial p^{\alpha}_{Mi}} - \lambda_{Ma}\frac{\partial \phi_{Ma}}{\partial q_{Mi}}\left(\xi_{Mi} - \dot{q}_{Mi}\xi_{M0}\right) + \dot{p}_{Mi}\xi_{M0}\lambda_{Ma}\frac{\partial \phi_{Ma}}{\partial p_{Mi}}$$
$$= -\lambda_{Ma}\frac{\partial \phi_{Ma}}{\partial p^{\alpha}_{Mi}}\left(\eta^{\alpha}_{Mi} - \dot{p}^{\alpha}_{Mi}\xi_{M0}\right) - \lambda_{Ma}\frac{\partial \phi_{Ma}}{\partial p_{Mi}}\left(\eta_{Mi} - \dot{p}_{Mi}\xi_{M0}\right) - \lambda_{Ma}\frac{\partial \phi_{Ma}}{\partial q_{Mi}}\left(\xi_{Mi} - \dot{q}_{Mi}\xi_{M0}\right) = 0,$$

where

$$\delta\phi_{Ma}\left(t, q_{Mj}, p_{Mj}, p^{\alpha}_{Mj}\right) = \frac{\partial \phi_{Ma}}{\partial q_{Mi}}\delta q_{Mi} + \frac{\partial \phi_{Ma}}{\partial p_{Mi}}\delta p_{Mi} + \frac{\partial \phi_{Ma}}{\partial p^{\alpha}_{Mi}}\delta p^{\alpha}_{Mi}$$
$$= \frac{\partial \phi_{Ma}}{\partial q_{Mi}}\left(\Delta q_{Mi} - \dot{q}_{Mi}\Delta t\right) + \frac{\partial \phi_{Ma}}{\partial p_{Mi}}\left(\Delta p_{Mi} - \dot{p}_{Mi}\Delta t\right) + \frac{\partial \phi_{Ma}}{\partial p^{\alpha}_{Mi}}\left(\Delta p^{\alpha}_{Mi} - \dot{p}^{\alpha}_{Mi}\Delta t\right)$$
$$= \theta_M\left[\frac{\partial \phi_{Ma}}{\partial q_{Mi}}\left(\xi_{Mi} - \dot{q}_{Mi}\xi_{M0}\right) + \frac{\partial \phi_{Ma}}{\partial p_{Mi}}\left(\eta_{Mi} - \dot{p}_{Mi}\xi_{M0}\right) + \frac{\partial \phi_{Ma}}{\partial p^{\alpha}_{Mi}}\left(\eta^{\alpha}_{Mi} - \dot{p}^{\alpha}_{Mi}\xi_{M0}\right)\right] = 0$$

The proof is completed.

Furthermore, if the fractional Hamilton action (Eq. (18)) does not remain invariant under the infinitesimal transformations with the mixed derivatives, for instance, if $\Delta I_M = \bar{I}_M - I_M = -\int_{t_1}^{t_2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\Delta G_M\right)\mathrm{d}t$, where $\Delta G_M = \theta_M G_M$ and $G_M = G_M\left(t, q_{Mj}, p^{\alpha}_{Mj}, p_{Mj}\right)$ is called a gauge function with the mixed derivatives, then from Eq. (21), we obtain

$$p_{Mi}\dot{\xi}_{Mi} + p^{\alpha}_{Mi\,t_1}{}^{C}D^{\alpha}_{t}\left(\xi_{Mi} - \dot{q}_{Mi}\xi_{M0}\right) + \left(p^{\alpha}_{Mi}\frac{\mathrm{d}}{\mathrm{d}t}{}^{C}_{t_1}D^{\alpha}_{t} q_{Mi} - \frac{\partial H_M}{\partial t}\right)\xi_{M0} - \frac{\partial H_M}{\partial q_{Mi}}\xi_{Mi} + \lambda_{Ma}\frac{\partial \phi_{Ma}}{\partial p^{\alpha}_{Mi}}\eta^{\alpha}_{Mi}$$
$$+\lambda_{Ma}\frac{\partial \phi_{Ma}}{\partial p_{Mi}}\eta_{Mi} - \frac{p^{\alpha}_{Mi}}{\Gamma(1-\alpha)}\left(t - t_1\right)^{-\alpha}\dot{q}_{Mi}\left(t_1\right)\xi_{M0}\left(t_1\right) + \left(p^{\alpha}_{Mi\,t_1}{}^{C}D^{\alpha}_{t} q_{Mi} - H_M\right)\dot{\xi}_{M0} + \dot{G}_M = 0 \tag{24}$$

Equation (24) is called the fractional Noether-quasi identity with the mixed derivatives for the fractional constrained Hamiltonian system (Eq. (14)). The infinitesimal transformations in this case are called the Noether-quasi symmetric transformations with the mixed derivatives, which determine the Noether-quasi symmetry with the mixed derivatives. Then, a fractional conserved quantity can also be obtained from the Noether-quasi symmetry.

Theorem 2 If the infinitesimal generators ξ_{M0}, ξ_{Mi}, η_{Mi}, η^{α}_{Mi} and a gauge function G_M satisfy the fractional Noether-quasi identity (Eq. (24)), then a fractional conserved quantity with the mixed derivatives exists for the fractional constrained Hamiltonian system (Eq. (14))

$$C_{GM}=\left(p_{Mi\,t_1}^{\alpha}{}^{C}D_{t}^{\alpha}q_{Mi}-H_{M}\right)\xi_{M0}+\int_{t_1}^{t}\left[p_{Mi\,t_1}^{\alpha}{}^{C}D_{\tau}^{\alpha}\left(\xi_{Mi}-\dot{q}_{Mi}\xi_{M0}\right)-\left(\xi_{Mi}-\dot{q}_{Mi}\xi_{M0}\right){}_{\tau}^{RL}D_{t_2}^{\alpha}p_{Mi}^{\alpha}\right]\mathrm{d}\tau$$
$$+p_{Mi}\xi_{Mi}-\int_{t_1}^{t}\frac{p_{Mi}^{\alpha}}{\Gamma(1-\alpha)}\left(\tau-t_1\right)^{-\alpha}\dot{q}_{Mi}\left(t_1\right)\xi_{M0}\left(t_1\right)\mathrm{d}\tau+G_{M}=\text{const} \tag{25}$$

Proof The intended result can be obtained from Eqs. (14), (24) and (25).

Remark 1 The Noether-quasi symmetry with the mixed derivatives is more general than the Noether symmetry with the mixed derivatives. In fact, by setting $G_M=0$, Theorem 2 reduces to Theorem 1.

An example is presented to illustrate the results and methods above.

Example 1 For the Lagrangian

$$L_M=\dot{q}_{M1}q_{M2}-q_{M1}\dot{q}_{M2}+q_{M1}^2+q_{M2}^2+\frac{1}{2}\left[\left({}_{t_1}^{C}D_t^{\alpha}q_{M1}\right)^2+\left({}_{t_1}^{C}D_t^{\alpha}q_{M2}\right)^2\right] \tag{26}$$

find its conserved quantity.

For this Lagrangian L_M, there exist two fractional primary constraints [30]

$$\phi_{M1}=p_{M1}-q_{M2}=0,\ \phi_{M2}=p_{M2}+q_{M1}=0 \tag{27}$$

In addition, all the Lagrange multipliers can be obtained [30]:

$$\lambda_{M1}=-q_{M2}-\frac{1}{2}{}_{t}^{RL}D_{t_2}^{\alpha}p_{M2}^{\alpha},\ \lambda_{M2}=q_{M1}+\frac{1}{2}{}_{t}^{RL}D_{t_2}^{\alpha}p_{M1}^{\alpha} \tag{28}$$

The fractional constrained Hamilton equations can also be established [30]:

$$\dot{q}_{M1}=-q_{M2}-\frac{1}{2}{}_{t}^{RL}D_{t_2}^{\alpha}p_{M2}^{\alpha},\ \dot{q}_{M2}=q_{M1}+\frac{1}{2}{}_{t}^{RL}D_{t_2}^{\alpha}p_{M1}^{\alpha},\ \dot{p}_{M1}=q_{M1}+\frac{1}{2}{}_{t}^{RL}D_{t_2}^{\alpha}p_{M1}^{\alpha},$$
$$\dot{p}_{M2}=q_2+\frac{1}{2}{}_{t}^{RL}D_{t_2}^{\alpha}p_{M2}^{\alpha},\ {}_{t_1}^{C}D_t^{\alpha}q_{M1}=p_{M1}^{\alpha},\ {}_{t_1}^{C}D_t^{\alpha}q_{M2}=p_{M2}^{\alpha} \tag{29}$$

The fractional Noether-quasi identity (Eq. (24)) gives

$$p_{M1}\dot{\xi}_{M1}+p_{M1\,t_1}^{\alpha}{}^{C}D_t^{\alpha}\left(\xi_{M1}-\dot{q}_{M1}\xi_{M0}\right)+\left(p_{M1}^{\alpha}\frac{\mathrm{d}}{\mathrm{d}t}{}_{t_1}^{C}D_t^{\alpha}q_{M1}+p_{M2}^{\alpha}\frac{\mathrm{d}}{\mathrm{d}t}{}_{t_1}^{C}D_t^{\alpha}q_{M2}\right)\xi_{M0}+\lambda_{M1}\eta_{M1}+\lambda_{M2}\eta_{M2}$$
$$+2q_{M1}\xi_{M1}+2q_{M2}\xi_{M2}-\frac{p_{M1}^{\alpha}}{\Gamma(1-\alpha)}\left(t-t_1\right)^{-\alpha}\dot{q}_{M1}\left(t_1\right)\xi_{M0}\left(t_1\right)+p_{M2}\dot{\xi}_{M2}-\frac{p_{M2}^{\alpha}}{\Gamma(1-\alpha)}\left(t-t_1\right)^{-\alpha}\dot{q}_{M2}\left(t_1\right)\xi_{M0}\left(t_1\right)$$
$$+p_{M2\,t_1}^{\alpha}{}^{C}D_t^{\alpha}\left(\xi_{M2}-\dot{q}_{M2}\xi_{M0}\right)+\left(p_{M1\,t_1}^{\alpha}{}^{C}D_t^{\alpha}q_{M1}+p_{M2\,t_1}^{\alpha}{}^{C}D_t^{\alpha}q_{M2}-H_M\right)\dot{\xi}_{M0}+\dot{G}_M=0 \tag{30}$$

Through computation, we can verify that

$$\xi_{M0}=-1,\ \xi_{M1}=\xi_{M2}=0,\ \eta_{M1}=\eta_{M2}=0,\ \eta_{M1}^{\alpha}=\eta_{M2}^{\alpha}=0,\ G_M=0 \tag{31}$$

is a solution to Eq. (30). Finally, Theorem 2 gives the fractional conserved quantity

$$C_{GM}=\int_{t_1}^{t}\left(p_{M1}^{\alpha}\frac{\mathrm{d}}{\mathrm{d}\tau}{}_{t_1}^{C}D_{\tau}^{\alpha}q_{M1}+p_{M2}^{\alpha}\frac{\mathrm{d}}{\mathrm{d}\tau}{}_{t_1}^{C}D_{\tau}^{\alpha}q_{M2}-\dot{q}_{M1}{}_{\tau}^{RL}D_{t_2}^{\alpha}p_{M1}^{\alpha}-\dot{q}_{M2}{}_{\tau}^{RL}D_{t_2}^{\alpha}p_{M2}^{\alpha}\right)\mathrm{d}\tau-\left[\frac{1}{2}\left(p_{M1}^{\alpha}\right)^2+\frac{1}{2}\left(p_{M2}^{\alpha}\right)^2+q_{M1}^2+q_{M2}^2\right]=\text{const} \tag{32}$$

3 Fractional Noether Theorem with only Caputo Fractional Derivatives

Noether symmetry with the Caputo fractional derivative is determined by the Noether symmetric transformations under which the fractional Hamilton action with the Caputo fractional derivatives

$$I_C=\int_{t_1}^{t_2}L_C\left(t,\boldsymbol{q_C},{}_{t_1}^{C}\boldsymbol{D}_t^{\alpha}\boldsymbol{q_C}\right)\mathrm{d}t=\int_{t_1}^{t_2}\left(p_{Ci}\cdot{}_{t_1}^{C}D_t^{\alpha}q_{Ci}-H_C\left(t,\boldsymbol{q_C},\boldsymbol{p_C}\right)\right)\mathrm{d}t \tag{33}$$

remains invariant. Similarly, if we want to study the Noether theorem, we first need to give the infinitesimal transformations with the Caputo fractional derivative; then, we discuss the change of the fractional Hamilton action (Eq. (33)) under the given infinitesimal transformations. Finally, the condition called the fractional Noether identity with the Caputo fractional derivatives is obtained.

Here, the infinitesimal transformations have the forms

$$\bar{t}=t+\Delta t,\ \bar{q}_{Ci}\left(\bar{t}\right)=q_{Ci}(t)+\Delta q_{Ci},\ \bar{p}_{Ci}\left(\bar{t}\right)=p_{Ci}(t)+\Delta p_{Ci} \tag{34}$$

whose expansions are

$$\bar{t}=t+\theta_C\xi_{C0}(t,\boldsymbol{q_C},\boldsymbol{p_C})+o(\theta_C),\ \bar{q}_{Ci}(\bar{t})=q_{Ci}(t)+\theta_C\xi_{Ci}(t,\boldsymbol{q_C},\boldsymbol{p_C})+o(\theta_C),\ \bar{p}_{Ci}(\bar{t})=p_{Ci}(t)+\theta_C\eta_{Ci}(t,\boldsymbol{q_C},\boldsymbol{p_C})+o(\theta_C) \tag{35}$$

where ξ_{C0}, ξ_{Ci} and η_{Ci} are called the infinitesimal generators with the Caputo fractional derivatives, θ_C is a small parameter, and $i=1,2,\cdots,n$.

We denote the change of the fractional Hamilton action (Eq. (33)) as ΔI_C; namely, $\Delta I_C=\bar{I}_C-I_C$. If we consider only the linear part of θ_C, then we have

$$\begin{aligned}
\Delta I_C=&\int_{\bar{t}_1}^{\bar{t}_2}\left(\bar{p}_{Ci}\,{}_{\bar{t}_1}^{C}D_{\bar{t}}^{\alpha}\bar{q}_{Ci}-H_C(t,\bar{\boldsymbol{q}}_{\boldsymbol{C}},\bar{\boldsymbol{p}}_{\boldsymbol{C}})\right)\mathrm{d}\bar{t}-\int_{t_1}^{t_2}\left(p_{Ci}\cdot{}_{t_1}^{C}D_t^{\alpha}q_{Ci}-H_C(t,\boldsymbol{q_C},\boldsymbol{p_C})\right)\mathrm{d}t\\
=&\int_{t_1}^{t_2}\Bigg[\left(p_{Ci}+\Delta p_{Ci}\right)\cdot\left({}_{t_1}^{C}D_t^{\alpha}q_{Ci}+{}_{t_1}^{C}D_t^{\alpha}\delta q_{Ci}+\Delta t\frac{\mathrm{d}}{\mathrm{d}t}{}_{t_1}^{C}D_t^{\alpha}q_{Ci}-\frac{1}{\Gamma(1-\alpha)}(t-t_1)^{-\alpha}\dot{q}_{Ci}(t_1)\Delta t_1\right)\\
&-H_C\left(t+\Delta t,q_{Cj}+\Delta q_{Cj},p_{Cj}+\Delta p_{Cj}\right)\Bigg]\cdot\left(1+\frac{\mathrm{d}}{\mathrm{d}t}\Delta t\right)\mathrm{d}t-\int_{t_1}^{t_2}\left(p_{Ci}\cdot{}_{t_1}^{C}D_t^{\alpha}q_{Ci}-H_C(t,\boldsymbol{q_C},\boldsymbol{p_C})\right)\mathrm{d}t\\
=&\int_{t_1}^{t_2}\Bigg[p_{Ci}\,{}_{t_1}^{C}D_t^{\alpha}q_{Ci}+p_{Ci}\,{}_{t_1}^{C}D_t^{\alpha}\delta q_{Ci}+p_{Ci}\Delta t\frac{\mathrm{d}}{\mathrm{d}t}{}_{t_1}^{C}D_t^{\alpha}q_{Ci}+\Delta p_{Ci}\,{}_{t_1}^{C}D_t^{\alpha}q_{Ci}-H_C(t,\boldsymbol{q_C},\boldsymbol{p_C})-\frac{p_{Ci}}{\Gamma(1-\alpha)}(t-t_1)^{-\alpha}\dot{q}_{Ci}(t_1)\Delta t_1\\
&-\frac{\partial H_C}{\partial t}\Delta t-\frac{\partial H_C}{\partial p_{Ci}}\Delta p_{Ci}+\left(p_{Ci}\,{}_{t_1}^{C}D_t^{\alpha}q_{Ci}-H_C\right)\frac{\mathrm{d}}{\mathrm{d}t}\Delta t-\frac{\partial H_C}{\partial q_{Ci}}\Delta q_{Ci}\Bigg]\mathrm{d}t-\int_{t_1}^{t_2}\left(p_{Ci}\cdot{}_{t_1}^{C}D_t^{\alpha}q_{Ci}-H_C(t,\boldsymbol{q_C},\boldsymbol{p_C})\right)\mathrm{d}t\\
=&\theta_C\int_{t_1}^{t_2}\Bigg[p_{Ci}\,{}_{t_1}^{C}D_t^{\alpha}\left(\xi_{Ci}-\dot{q}_{Ci}\xi_{C0}\right)+\left(p_{Ci}\frac{\mathrm{d}}{\mathrm{d}t}{}_{t_1}^{C}D_t^{\alpha}q_{Ci}-\frac{\partial H_M}{\partial t}\right)\xi_{C0}-\frac{\partial H_C}{\partial q_{Ci}}\xi_{Ci}\\
&-\frac{p_{Ci}}{\Gamma(1-\alpha)}(t-t_1)^{-\alpha}\dot{q}_{Ci}(t_1)\xi_{C0}(t_1)+\left(p_{Ci}\,{}_{t_1}^{C}D_t^{\alpha}q_{Ci}-H_C\right)\dot{\xi}_{C0}+\lambda_{Ca}\frac{\partial\phi_{Ca}}{\partial p_{Ci}}\eta_{Ci}\Bigg]\mathrm{d}t
\end{aligned}\tag{36}$$

where

$${}_{t_1}^{C}D_{\bar{t}}^{\alpha}\bar{q}_{Ci}={}_{t_1}^{C}D_t^{\alpha}q_{Ci}+{}_{t_1}^{C}D_t^{\alpha}\delta q_{Ci}+\Delta t\frac{\mathrm{d}}{\mathrm{d}t}{}_{t_1}^{C}D_t^{\alpha}q_{Ci}-\frac{1}{\Gamma(1-\alpha)}(t-t_1)^{-\alpha}\dot{q}_{Ci}(t_1)\Delta t_1,\ \xi_{C0}(t_1)=\xi_{C0}\left(t_1,\boldsymbol{q_C}(t_1),\boldsymbol{p_C}(t_1)\right).$$

That fractional Hamilton action (Eq. (33)) remains invariant implies $\Delta I_C=0$; therefore, from Eq. (36), we have

$$\begin{aligned}
&p_{Ci}\,{}_{t_1}^{C}D_t^{\alpha}\left(\xi_{Ci}-\dot{q}_{Ci}\xi_{C0}\right)+\left(p_{Ci}\frac{\mathrm{d}}{\mathrm{d}t}{}_{t_1}^{C}D_t^{\alpha}q_{Ci}-\frac{\partial H_C}{\partial t}\right)\xi_{C0}-\frac{\partial H_C}{\partial q_{Ci}}\xi_{Ci}+\lambda_{Ca}\frac{\partial\phi_{Ca}}{\partial p_{Ci}}\eta_{Ci}\\
&-\frac{p_{Ci}}{\Gamma(1-\alpha)}(t-t_1)^{-\alpha}\dot{q}_{Ci}(t_1)\xi_{C0}(t_1)+\left(p_{Ci}\,{}_{t_1}^{C}D_t^{\alpha}q_{Ci}-H_C\right)\dot{\xi}_{C0}=0,\ i=1,2,\cdots,n
\end{aligned}\tag{37}$$

Equation (37) is called the fractional Noether identity with the Caputo fractional derivatives for the fractional constrained Hamiltonian system (Eq. (17)). The infinitesimal transformations in this case are called the Noether symmetric transformations with the Caputo fractional derivatives, which determine the Noether symmetry.

Theorem 3 If the infinitesimal generators ξ_{C0}, ξ_{Ci}, η_{Ci} and η_{Ci}^{α} satisfy the fractional Noether identity (Eq. (37)), then a fractional conserved quantity with the Caputo fractional derivatives exists for the fractional constrained Hamiltonian system (Eq. (17)) as follows:

$$\begin{aligned}
C_C=&\left(p_{Ci}\,{}_{t_1}^{C}D_t^{\alpha}q_{Ci}-H_C\right)\xi_{C0}+\int_{t_1}^{t}\left[p_{Ci}\,{}_{t_1}^{C}D_{\tau}^{\alpha}\left(\xi_{Ci}-\dot{q}_{Ci}\xi_{C0}\right)-\left(\xi_{Ci}-\dot{q}_{Ci}\xi_{C0}\right){}_{\tau}^{RL}D_{t_2}^{\alpha}p_{Ci}\right]\mathrm{d}\tau\\
&-\int_{t_1}^{t}\frac{p_{Ci}}{\Gamma(1-\alpha)}(\tau-t_1)^{-\alpha}\dot{q}_{Ci}(t_1)\xi_{C0}(t_1)\mathrm{d}\tau=\mathrm{const}
\end{aligned}\tag{38}$$

Proof It is obtained from Eqs. (17), (37) and (38) that

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}C_C=&\left(\dot{p}_{Ci}\,{}_{t_1}^{C}D_t^{\alpha}q_{Ci}+p_{Ci}\frac{\mathrm{d}}{\mathrm{d}t}{}_{t_1}^{C}D_t^{\alpha}q_{Ci}-\frac{\partial H_C}{\partial t}-\frac{\partial H_C}{\partial q_{Ci}}\dot{q}_{Ci}-\frac{\partial H_C}{\partial p_{Ci}}\dot{p}_{Ci}\right)\xi_{C0}+\left(p_{Ci}\,{}_{t_1}^{C}D_t^{\alpha}q_{Ci}-H_C\right)\dot{\xi}_{C0}\\
&+p_{Ci}\,{}_{t_1}^{C}D_t^{\alpha}\left(\xi_{Ci}-\dot{q}_{Ci}\xi_{C0}\right)-\left(\xi_{Ci}-\dot{q}_{Ci}\xi_{C0}\right){}_{t}^{RL}D_{t_2}^{\alpha}p_{Ci}-\frac{p_{Ci}}{\Gamma(1-\alpha)}(t-t_1)^{-\alpha}\dot{q}_{Ci}(t_1)\xi_{C0}(t_1)\\
=&\frac{\partial H_C}{\partial q_{Ci}}\xi_{Ci}+\left(\dot{p}_{Ci}\,{}_{t_1}^{C}D_t^{\alpha}q_{Ci}-\frac{\partial H_C}{\partial p_{Ci}}\dot{p}_{Ci}-\frac{\partial H_C}{\partial q_{Ci}}\dot{q}_{Ci}\right)\xi_{C0}-\left(\xi_{Ci}-\dot{q}_{Ci}\xi_{C0}\right){}_{t}^{RL}D_{t_2}^{\alpha}p_{Ci}^{\alpha}-\lambda_{Ca}\frac{\partial\phi_{Ca}}{\partial p_{Ci}}\eta_{Ci}
\end{aligned}$$

$$=\left(\frac{\partial H_C}{\partial q_{Ci}}-{}_t^{RL}D_{t_2}^{\alpha}p_{Ci}+\lambda_{Ca}\frac{\partial\phi_{Ca}}{\partial q_{Ci}}\right)\left(\xi_{Ci}-\dot q_{Ci}\xi_{C0}\right)-\lambda_{Ca}\frac{\partial\phi_{Ca}}{\partial p_{Ci}}\eta_{Ci}-\lambda_{Ca}\frac{\partial\phi_{Ca}}{\partial q_{Ci}}\left(\xi_{Ci}-\dot q_{Ci}\xi_{C0}\right)+\left(\dot p_{Ci}{}_{t_1}^{C}D_t^{\alpha}q_{Ci}-\frac{\partial H_C}{\partial p_{Ci}}\dot p_{Ci}\right)\xi_{C0}$$

$$=-\lambda_{Ca}\frac{\partial\phi_{Ca}}{\partial p_{Ci}}\eta_{Ci}+\dot p_{Ci}\xi_{C0}\lambda_{Ca}\frac{\partial\phi_{Ca}}{\partial p_{Ci}}-\lambda_{Ca}\frac{\partial\phi_{Ca}}{\partial q_{Ci}}\left(\xi_{Ci}-\dot q_{Ci}\xi_{C0}\right)=-\lambda_{Ca}\frac{\partial\phi_{Ca}}{\partial p_{Ci}}\left(\eta_{Ci}-\dot p_{Ci}\xi_{C0}\right)-\lambda_{Ca}\frac{\partial\phi_{Ca}}{\partial q_{Ci}}\left(\xi_{Ci}-\dot q_{Ci}\xi_{C0}\right)=0,$$

where

$$\delta\phi_{Ca}\left(t,q_{Cj},p_{Cj}\right)=\frac{\partial\phi_{Ca}}{\partial q_{Ci}}\delta q_{Ci}+\frac{\partial\phi_{Ca}}{\partial p_{Ci}}\delta p_{Ci}=\frac{\partial\phi_{Ca}}{\partial q_{Ci}}\left(\Delta q_{Ci}-\dot q_{Ci}\Delta t\right)+\frac{\partial\phi_{Ca}}{\partial p_{Ci}}\left(\Delta p_{Ci}-\dot p_{Ci}\Delta t\right)$$
$$=\theta_C\left[\frac{\partial\phi_{Ca}}{\partial q_{Ci}}\left(\xi_{Ci}-\dot q_{Ci}\xi_{C0}\right)+\frac{\partial\phi_{Ca}}{\partial p_{Ci}}\left(\eta_{Ci}-\dot p_{Ci}\xi_{C0}\right)\right]=0.$$

The proof is completed.

Let $\Delta I_C=\bar I_C-I_C=-\int_{t_1}^{t_2}\frac{\mathrm d}{\mathrm dt}\left(\Delta G_C\right)\mathrm dt$, where $\Delta G_C=\theta_C G_C$ and $G_C=G_C\left(t,q_{Cj},p_{Cj}\right)$ is a gauge function with the Caputo fractional derivatives; then, from Eq. (37), we obtain

$$p_{Ci}{}_{t_1}^{C}D_t^{\alpha}\left(\xi_{Ci}-\dot q_{Ci}\xi_{C0}\right)+\left(p_{Ci}\frac{\mathrm d}{\mathrm dt}{}_{t_1}^{C}D_t^{\alpha}q_{Ci}-\frac{\partial H_C}{\partial t}\right)\xi_{C0}-\frac{\partial H_C}{\partial q_{Ci}}\xi_{Ci}+\lambda_{Ca}\frac{\partial\phi_{Ca}}{\partial p_{Ci}}\eta_{Ci}$$
$$-\frac{p_{Ci}}{\Gamma(1-\alpha)}\left(t-t_1\right)^{-\alpha}\dot q_{Ci}\left(t_1\right)\xi_{C0}\left(t_1\right)+\left(p_{Ci}{}_{t_1}^{C}D_t^{\alpha}q_{Ci}-H_C\right)\dot\xi_{C0}+\dot G_C=0,\ i=1,2,\cdots,n \tag{39}$$

Equation (39) is called the fractional Noether-quasi identity with the Caputo fractional derivatives for the fractional constrained Hamiltonian system (Eq. (17)). The infinitesimal transformations in this case are called Noether-quasi symmetric transformations, which determine the Noether-quasi symmetry. Then, a fractional conserved quantity with the Caputo fractional derivatives can be obtained.

Theorem 4 If the infinitesimal generators ξ_{C0}, ξ_{Ci}, η_{Ci}, and a gauge function G_C satisfy the fractional Noether-quasi identity (Eq. (39)), then a fractional conserved quantity with the Caputo fractional derivatives exists for the fractional constrained Hamiltonian system (Eq. (17))

$$C_{GC}=\left(p_{Ci}{}_{t_1}^{C}D_t^{\alpha}q_{Ci}-H_C\right)\xi_{C0}+\int_{t_1}^{t}\left[p_{Ci}{}_{t_1}^{C}D_\tau^{\alpha}\left(\xi_{Ci}-\dot q_{Ci}\xi_{C0}\right)-\left(\xi_{Ci}-\dot q_{Ci}\xi_{C0}\right){}_\tau^{RL}D_{t_2}^{\alpha}p_{Ci}\right]\mathrm d\tau$$
$$-\int_{t_1}^{t}\frac{p_{Ci}}{\Gamma(1-\alpha)}\left(\tau-t_1\right)^{-\alpha}\dot q_{Ci}\left(t_1\right)\xi_{C0}\left(t_1\right)\mathrm d\tau+G_C=\text{const.} \tag{40}$$

Proof The intended result can be obtained from Eqs. (17), (39) and (40).

Remark 2 Based on the Caputo fractional derivatives, the Noether-quasi symmetry is more general than the Noether symmetry. In fact, by setting $G_C=0$, Theorem 4 reduces to Theorem 3.

Remark 3 Based on the Caputo fractional derivatives, if let $\alpha\to 1$, then the fractional primary constraint (Eq. (16)), the fractional constrained Hamilton equation (Eq. (17)) and the Noether theorem (Theorem 3) reduce to the corresponding classical integer-order cases, which are consistent with the results in Ref. [6].

An example is presented to illustrate the results and methods above.

Example 2 For the Lagrangian

$$L_C=q_{C2}\cdot{}_{t_1}^{C}D_t^{\alpha}q_{C1}-q_{C1}\cdot{}_{t_1}^{C}D_t^{\alpha}q_{C2}+\left(q_{C1}\right)^2+\left(q_{C2}\right)^2 \tag{41}$$

find its conserved quantity.

For this Lagrangian L_C, there exist two fractional primary constraints [30]:

$$\phi_{C1}=p_{C1}-q_{C2}=0,\ \phi_{C2}=p_{C2}+q_{C1}=0. \tag{42}$$

The fractional constrained Hamilton equations can also be established [30]:

$${}_t^{RL}D_{t_2}^{\alpha}p_{C1}=-2q_{C1}+{}_{t_1}^{C}D_t^{\alpha}q_{C2},\ {}_t^{RL}D_{t_2}^{\alpha}p_{C2}=-2q_{C2}-{}_{t_1}^{C}D_t^{\alpha}q_{C1} \tag{43}$$

The fractional Noether-quasi identity (Eq. (39)) gives

$$p_{C1}{}_{t_1}^{C}D_t^{\alpha}\left(\xi_{C1}-\dot q_{C1}\xi_{C0}\right)+p_{C2}{}_{t_1}^{C}D_t^{\alpha}\left(\xi_{C2}-\dot q_{C2}\xi_{C0}\right)+2q_{C1}\xi_{C1}+2q_{C2}\xi_{C2}+\lambda_{C1}\eta_{C1}+\lambda_{C2}\eta_{C2}+\left(p_{C1}\frac{\mathrm d}{\mathrm dt}{}_{t_1}^{C}D_t^{\alpha}q_{C1}+p_{C2}\frac{\mathrm d}{\mathrm dt}{}_{t_1}^{C}D_t^{\alpha}q_{C2}\right)\xi_{C0}$$
$$-\frac{p_{C1}}{\Gamma(1-\alpha)}\left(t-t_1\right)^{-\alpha}\dot q_{C1}\left(t_1\right)\xi_{C0}\left(t_1\right)-\frac{p_{C2}}{\Gamma(1-\alpha)}\left(t-t_1\right)^{-\alpha}\dot q_{C2}\left(t_1\right)\xi_{C0}\left(t_1\right)+\left(p_{C1}{}_{t_1}^{C}D_t^{\alpha}q_{C1}+p_{C2}{}_{t_1}^{C}D_t^{\alpha}q_{C2}-H_C\right)\dot\xi_{C0}+\dot G_C=0 \tag{44}$$

Through computation, we can verify that

$$\xi_{C0}=-1,\ \xi_{C1}=\xi_{C2}=0,\ \eta_{C1}=\eta_{C2}=0,\ G_C=0 \tag{45}$$

is a solution to Eq. (44). Finally, Theorem 4 gives the fractional conserved quantity

$$C_{GC}=\int_{t_1}^{t}\left(p_{C1}\frac{\mathrm{d}}{\mathrm{d}\tau}{}_{t_1}^{C}D_{\tau}^{\alpha}q_{C1}+p_{C2}\frac{\mathrm{d}}{\mathrm{d}\tau}{}_{t_1}^{C}D_{\tau}^{\alpha}q_{C2}\right)\mathrm{d}\tau-\left[p_{C1}{}_{t_1}^{C}D_{t}^{\alpha}q_{C1}+p_{C2}{}_{t_1}^{C}D_{t}^{\alpha}q_{C2}+q_{C1}^{2}+q_{C2}^{2}\right] \tag{46}$$

4 Conclusion

Noether theorems for the singular systems with the mixed derivatives and with only Caputo fractional derivatives are studied for the first time. Theorems 1-4 are all new work. Besides, the constrained Hamiltonian system on time scales is another topic deserved to be done.

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