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The Non-Convergence of Steiner Symmetrizations

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Abstract: In this paper, we demonstrate the existence of iterated Steiner symmetrizations of K that does not converge, even if the sequence of directions is dense in the unit sphere.

Key words: convex body; compact set; Steiner symmetrization; non-convergence

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0 Introduction

Over the past two centuries, Steiner symmetrization has frequently been employed as a tool for addressing various issues, such as solving isoperimetric problems related to convex bodies, establishing properties of volume and surface area, and proving certain convex geometric inequalities. In recent years, numerous papers have been published on Steiner symmetrizations, which are essential for examining these related problems, see e. g. Refs. [1-4]. Steiner symmetrization not only serves as the foundation for geometric analysis but also holds a significant role in other branches of mathematics. The authors of Refs. [2, 5] constructed a series of Steiner symmetrizations for a given convex body under a specific directional sequence, where the direction sequence is chosen from a finite set of directions on the unit sphere, and the series of Steiner symmetrizations converge to a ball.

There remain many fundamental unsolved problems concerning Steiner symmetrizations. One of the such questions is whether the sequence of Steiner symmetrizations converges to a ball when the direction sequence, which is used to construct Steiner symmetrizations, is dense on the unit sphere. In this paper, we identify suitable iterative sequences of Steiner symmetrizations that do not converge to a ball with the same volume as the initial convex body. Furthermore, we provide examples of specific cases that do not converge under Steiner symmetrizations. Convergent sequences employed to construct Steiner symmetrization typically depend on geometric functions that decrease monotonically along the sequence, such as volume and chord length. In this paper, our primary focus is on designing a sequence of directions that is non-convergent, meaning that the direction sequences diverge in certain cases.

Furthermore, is it possible for the Steiner symmetrization of a convex body to become non-convergent

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when the selected direction sequence is dense on the unit sphere S^{n-1}? Recently, some researchers have raised the same question and demonstrated that by rearranging the direction sequence, the outcomes of Steiner symmetrizations along any dense direction sequence can either converge or diverge [5]. Additionally, they reveal that any given Steiner symmetrization sequence, whether convergent or not, possesses a non-convergent subsequence [6]. In Refs. [5] and [6], the authors mainly focused on the case of convex bodies in the plane. In this paper, we consider the cases of convex bodies and compact sets in high-dimensional space. We construct a specific sequence of directions in high-dimensional space, which guarantees that the sequence of iterative Steiner symmetrizations of the convex body or compact set is non-convergent under this sequence of directions.

1 Preliminaries

Let K be an n-dimensional convex body in $\mathbb{R}^n$. The Steiner symmetrization $S_u K$ of K with respect to the hyperplane $u^{\perp}$ is the union of the line segments that are produced by translating the intersections of lines parallel to u and the convex body K, where the midpoints of all the line segments lie on the hyperplane $u^{\perp}$. For a given convex body $K \subset \mathbb{R}^n$, we randomly select a sequence of unit directions u_i on the unit sphere and perform iterative Steiner symmetrizations on K. The sequence of Steiner symmetrizations $S_{u_i} S_{u_{i-1}} \cdots S_{u_1} K$ can converge to a ball if the chosen sequence of the directions is good enough, see e.g. Ref. [7]. On the other hand, the direction set Λ must be dense and appropriately selected from the unit sphere to allow the sequence of Steiner symmetrizations $S_{u_i} S_{u_{i-1}} \cdots S_{u_1} K$ of convex body K to converge to a ball. Conversely, an arbitrarily countable dense sequence of directions might not necessarily result in convergence to the ball. Consequently, the order of directions could be crucial for generating the desired effect. In the subsequent analysis, we examine the non-convergence of Steiner symmetrizations applied to a specific convex body K. If we do not choose the appropriate sequence of directional u_i, then the limit of the Steiner symmetrization sequence $S_{u_i} S_{u_{i-1}} \cdots S_{u_1} K$ of the convex body K may not be convergent when $i \to \infty$.

Two essential lemmas and several basic properties of Steiner symmetrizations are given as the following. Standard references to the fundamental properties of Steiner symmetrization include Refs. [2,5,8].

Lemma 1[9] Let $p_1, p_2, \cdots$ be a sequence of positive prime integers. Then the sum

$$\sum_{i=1}^{\infty} \frac{1}{p_i} \tag{1}$$

diverges.

Lemma 2 (see Ref. [10], Theorem 2.2) Let $\{u_m\}$ be a sequence in S^{n-1} with $u_{m-1} \cdot u_m = \cos \alpha_m$, where $\{\alpha_m\}$ is a sequence in $(0, \pi/2)$ that satisfies

$$\sum_{m=1}^{\infty} \alpha_m^2 < \infty.$$

Then there exists a sequence of rotations $\{R_m\}$ such that for every non-empty compact set $K \subset \mathbb{R}^n$, the rotated symmetrization sequence

$$K_m = R_m S_{u_m} \cdots S_{u_1} K \tag{2}$$

converges in Hausdorff distance and in symmetric difference to a compact set L.

2 The Non-Convergence of Steiner Symmetrizations

In this section we will construct several cases where the Steiner symmetrizations do not converge. First, we need to find a column of suitable dense direction sequence on the unit sphere. We perform iterate Steiner symmetrizations of the given convex body in these directions, and finally summarize the results of the symmetrizations. Moreover, we prove the limits of some directional sequences under Steiner symmetrizations that either do not exist, or do not converge to an ellipsoid or to a non-convex body (non-compact set) for some unique convex bodies. Using the examples given in the plane, we can quickly get the high dimensional case through low dimensional recurrence.

2.1 The Non-Convergence of Convex Bodies

2.1.1 Construction of directional sequences

Let $p_1, p_2, \cdots$ be a sequence of positive prime integers. By Lemma 1, we have the sequence sum

$$\sum_{i=1}^{\infty} \frac{1}{p_i}$$

diverge. Now for any $m \geqslant 1$, denote the counter clockwise angle

$$\theta_m = \sum_{i=1}^{m} \frac{\sqrt{2}}{p_i} \tag{3}$$

in $\mathbb{R}^2$, and θ_m is represented by radian, let $\boldsymbol{u}_m$ be the unit vector in $\mathbb{R}^2$. When $\theta_m \to \infty$, we have each continuous incremental angle $\sqrt{2}/p_m \to 0$, where the unit vector $\boldsymbol{u}_m$ is an element in a countable dense subset on the unit circle.

Now we look at the following formula

$$\prod_{m=1}^{\infty}\cos\left(\frac{\sqrt{2}}{p_m}\right)=\prod_{m=1}^{\infty}\left(1-2\sin^2\frac{\sqrt{2}}{2p_m}\right)$$
$$\geqslant\prod_{m=1}^{\infty}\left(1-2\left(\frac{\sqrt{2}}{2p_m}\right)^2\right)\geqslant\prod_{m=1}^{\infty}\left(1-\frac{1}{p_m^2}\right)\tag{4}$$

Applying the Euler product formula [11] to the above formula, we obtain

$$\left(\prod_{m=1}^{\infty}\cos\left(\frac{\sqrt{2}}{p_m}\right)\right)^{-1}\leqslant\prod_{m=1}^{\infty}\left(\frac{1}{1-\frac{1}{p_m^2}}\right)$$
$$=\prod_{m=1}^{\infty}\left(1+\frac{1}{p_m^2}+\frac{1}{p_m^4}+\cdots\right)=\sum_{k=1}^{\infty}\frac{1}{k^2}=\frac{\pi^2}{6}\tag{5}$$

Thus,

$$\prod_{m=1}^{\infty}\cos\frac{\sqrt{2}}{p_m}\geqslant\frac{6}{\pi^2}\tag{6}$$

2.1.2 Non-convergence of line segments

Let ℓ be a line segment perpendicular to the horizontal axis and centered at the origin with a length of 1. Now make the line segment ℓ Steiner symmetrization under the direction sequence $\{u_m\}$ to get S_{u_m}. As a result, for every symmetrization done, the previous segment is projected onto a line perpendicular to the $\{u_m\}$. Thus the length of the subsequent segment is equal to that of the previous segment multiplied by the incremental cosine $\cos\left(\sqrt{2}/p_i\right)$. Moreover, since the limit of $\prod_{i=1}^{\infty}\cos\left(\sqrt{2}/p_i\right)$ in the formula (6) is constant and greater than zero, and the direction angle θ_m always circles around the circle in a certain order, the Steiner symmetrization of the line segment ℓ always circulates in the circle, and its length is close to a value greater than zero in a limit.

Therefore, it was concluded that the sequence of line segments

$$\ell_m=S_{u_m}\cdots S_{u_2}S_{u_1}\ell\tag{7}$$

has no limit.

2.1.3 Non-convergence of special convex bodies

Let K be a cylindrical convex body of area ε and contain a summation axis. According to the monotonicity of the Steiner symmetrizations on convex bodies, in the Steiner symmetrization sequence

$$K_m=S_{u_m}\cdots S_{u_2}S_{u_1}K\tag{8}$$

each element has a corresponding axis of symmetrization ℓ_m, so that the diameter of each K_m is greater than $6/\pi^2$. The Steiner symmetrization shows that each K_m has the same area ε as the original convex body K, where ε can be arbitrarily small, so that the sequence K_m can never converge to a ball. Actually, when $\varepsilon<9/\pi^3$, the sequence K_m does not converge, because the symmetry axis of K_m keeps consistently rotating, but does not converge to a very small area of ε.

We have shown that some special convex body or compact set do not necessarily converge after Steiner symmetrizations under a countably dense sequence of orientations. In this section, we use specific examples to show that convex bodies are not valid for Steiner symmetrizations under a specific directional sequence when the divergent series (1) are used as the basis to facilitate the calculation. Refs. [10, 12] have shown that a more general family of examples can be constructed starting with any decreasing sequence of incremental angles θ_i provided that $\sum_{i=1}^{\infty}\theta_i^2$ converges and $\sum_{i=1}^{\infty}\theta_i$ diverges. By applying the iterated Steiner symmetrizations in the resulting sequence of directions to a sufficiently eccentric ellipsoid, we will obtain a sequence of ellipsoids whose principal axes rotate forever without converging to a ball.

2.1.4 Another way of expressing non-convergence

First construct a column of sequence θ_m in $(0,\pi/2)$ with

$$\sum_{m=1}^{\infty}\theta_m=\infty,\sum_{m=1}^{\infty}\theta_m^2<\infty\tag{9}$$

and set $\gamma=\prod_{m=1}^{\infty}\cos\theta_m$.

For every $m\in\mathbb{N}_+$, define $\beta_m=\sum_{k=1}^{\infty}\theta_k$ and $u_m=(\cos\beta_m,\sin\beta_m)$. Here we need to pay attention to $\beta\in(0,1)$. Indeed, if m is greater than a specific N, then (9) indicates that $\theta_m\in(0,1)$. The above reasoning leads to

$$\cos\theta_m=1-2\sin^2\frac{\theta_m}{2}\geqslant1-\frac{\theta_m^2}{2},$$

and when $x\in(0,1/2)$, we can get

$$\ln(1-x)\geqslant-\left(1+o(1)\right)x.$$

Thus

$$\sum_{m=1}^{\infty}\ln\cos\theta_m\geqslant\sum_{m=1}^{N}\ln\cos\theta_m-\left(1+o(1)\right)\sum_{m=N+1}^{\infty}\frac{\theta_m^2}{2}>-\infty.$$

By the above inequality and $\gamma=e^{\sum_{m=1}^{\infty}\ln\cos\theta_m}$, we have $\gamma>0$.

Let K be a disk of diameter less than γ, and it contains a line segment ℓ perpendicular to the horizontal axis and of length 1. Now the convex body K and the line segment ℓ are made Steiner symmetrical simultaneously, which gives the convex bodies sequence K_m and line segments sequence ℓ_m. If each Steiner symmetrization S_{u_m} is projected from the former line segment ℓ_{m-1} onto the $\boldsymbol{u}_m^{\perp}$, then the length of ℓ_m is equal to the length of ℓ_{m-1} multiplied by the $\cos\theta_m$. Since β_m is diverged, the line segment ℓ_m always rotates around the circle, and thus the length of the line segment ℓ_m is monotonically reduced to γ. Because $K_m \supset \ell_m$, for each m, there is a K_m diameter larger than γ. When the sequence K_m is convergent, its limiting value must contain a disk of diameter γ. In addition, the area of K_m is equal to the area of the convex body K, so this is a contradiction of the above situation. In conclusion, the sequence K_m does not converge.

2.2 The Non-Convergence of Compact set

Define a compact set K which consists of a line segment ℓ and a ball B_r with radius r, select the orientation sequence $\{u_m\}$ in S^1 and $u_{m-1}\cdot u_m = \cos\theta_m$, $\gamma = \prod_{m=1}^{\infty}\cos\theta_m$, where the angle of rotation direction sequence θ_m belongs to $(0, \pi/2)$ and satisfies $\sum_{m=1}^{\infty}\theta_m^2 < \infty$. For each nonempty compact set $K \subset \mathbb{R}^n$, we can find a rotating sequence $\{R_m\}$ such that the Steiner symmetric sequence of K with respect to $\{R_m\}$ is

$$K_m = R_m S_{u_m} \cdots S_{u_1} K \tag{10}$$

By Lemma 2, we know that the limit of the sequence (10) must contain a ball B_r of radius r and a line segment ℓ of length γ. Meanwhile, if we want to make $\sum_{m=1}^{\infty}\theta_m^2$ arbitrarily small, we can make γ approach to 1 infinitely by removing several initial terms. Specifically, we can assume that $\gamma > 2/\pi$. Let K be the convex hull of the line segment ℓ in the non-convergence convex bodies and the ball B_r with center at the origin, where $r > 0$. If the rotational symmetrization sequence (10) converges, then its limit value must also contain both a ball B_r of radius r and a line segment ℓ of length γ. So we conclude that the area of any ellipsoid containing the above set is no less than $\pi\gamma r/2$. On the other hand, the area of K_m is equal to the area of K, and the upper bound of the area of K is $r/\left(\sqrt{1-4r^2}\right)$, then we can obtain a diamond equal to the K area converging to a circle centered on the origin, where the longer diamond is a line segment ℓ of length 1. Because if r is small enough, then its area will be smaller than $\pi\gamma r/2$, by the choice of γ , the limit is not an ellipsoid.

So the above example shows that any compact convex set containing a ball B_r and a line segment ℓ has an area greater than or equal to $\gamma r/2$. Since the area of K is equal to the area of a ball of radius r, namely πr^2, the limiting value of K_m is not a compact convex set when $\pi r < \gamma/2$.

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