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Note

On Packing Trees into Complete Bipartite Graphs

□ **PENG Yanling[1], WANG Hong[2]**

1. Department of Mathematics, Suzhou University of Science and Technology, Suzhou 215009, Jiangsu, China;

2. Department of Mathematics, University of Idaho, Moscow 83844-1103, Idaho, US

Abstract: Let $B_n(X,Y)$ denote the complete bipartite graph of order n with vertex partition sets X and Y. We prove that for each tree T of order n, there is a packing of k copies of T into a complete bipartite graph $B_{n+m}(X,Y)$. The ideal of the work comes from the "Tree packing conjecture" made by Gyráfás and Lehel. Bollobás confirmed the "Tree packing conjecture" for many small trees, who showed that one can pack $T_1, T_2, \cdots, T_{n/\sqrt{2}}$ into K_n and that a better bound would follow from a famous conjecture of Erdös. In a similar direction, Hobbs, Bourgeois and Kasiraj made the following conjecture: Any sequence of trees $T_2, \cdots, T_n$, with T_i having order i, can be packed into $K_{n-1,\lceil n/2\rceil}$. Further Hobbs, Bourgeois and Kasiraj proved that any two trees can be packed into a complete bipartite graph $K_{n-1,\lceil n/2\rceil}$. Motivated by these results, Wang Hong proposed the conjecture: For each tree T of order n, there is a k-packing of T in some complete bipartite graph $B_{n+k-1}(X,Y)$. In this paper, we prove a weak version of this conjecture.

Key words: packing of graphs; tree packing conjecture; embedding of graph

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0 Introduction

For graphs G and H, an embedding of G into H is an injection $\phi: V(G) \to V(H)$ such that $\phi(a)\phi(b) \in E(H)$ whenever $ab \in E(G)$. A packing of p graphs $G_1, G_2, \cdots, G_p$ into H is a p-tuple $\Phi = (\phi_1, \phi_2, \cdots, \phi_p)$ such that, for $i = 1, 2, \cdots, p$, ϕ_i is an embedding of G_i into H and the p sets $\phi_i(E(G_i))$ are mutually disjoint. When all G_i are isomorphic to G, we call it a k-parking of G. A bipartite graph G with the vertex partition $\mathbb{X} = (X_1, X_2)$ is denoted as $G(X_1, X_2)$ or $G(\mathbb{X})$. For a packing $\Phi = (\phi_1, \phi_2, \cdots, \phi_p)$ of $G_1(\mathbb{X}), G_2(\mathbb{X}), \cdots, G_p(\mathbb{X})$ into a bipartite graph $H(\mathbb{Y})$, we mean that Φ is a packing such that $\phi_i(X_j) \subset Y_j$, $i = 1, 2, \cdots, p, j = 1, 2$.

Packing problems are central to combinatorics. Many classical problems can be stated as packing problems, such as Mantel's Theorem which can be formulated by saying that if G is an n-vertex graph with less than $\binom{n}{2} - \frac{n^2}{4}$ edges, then the two graphs K_3 and G can be packed into K_n. The packing problem has received a lot of attention. Many interesting results and elegant proofs of these results were obtained. For a survey, see Refs. [1, 2]. Among the best known packing problems, the famous tree packing conjecture of Gyráfás and Lehel has driven a large amount of research in the area.

Conjecture 1 (Gyráfás and Lehel[3]) Given $n \in \mathbb{N}$ and trees $T_1, \cdots, T_n$ with T_i having order i, the graphs $T_1, \cdots, T_n$ can be packed into complete graph K_n.

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Biography: PENG Yanling, female, Professor, research direction: graph theory. E-mail: pengyanling@mail.usts.edu.cn

A packing of many of the small trees from Conjecture 1 was obtained by Bollobás[4], who showed that one can pack $T_1, \cdots, T_{n/\sqrt{2}}$ into K_n and that a better bound would follow from a famous conjecture of Erdös. In a similar direction, Hobbs, Bourgeois and Kasiraj made the following conjecture.

Conjecture 2 (Hobbs, Bourgeois and Kasiraj[5]) Any sequence of trees $T_2, \cdots, T_n$, with T_i having order i, can be packed into $K_{n-1,\lceil n/2 \rceil}$.

The conjecture has been verified for several very special classes of trees. Hobbs, Bourgeois and Kasiraj[5] proved that any two trees of order m and n with $m<n$ can be packed into a complete bipartite graph $K_{n-1,\lceil n/2 \rceil}$. Yuster[6] proved that any sequence of trees $T_2, \cdots, T_s$, $s < \sqrt{5/8}\, n$ can be packed into $K_{n-1,\lceil n/2 \rceil}$. Motivated by these results, Wang proposed the following conjecture.

Conjecture 3 (Wang[7]) For each tree T of order n, there is a k-packing of T in some complete bipartite graph $B_{n+k-1}(X, Y)$.

This conjecture is true for $k=2$ and $k=3$ (see Theorem 1 and Theorem 2).

Theorem 1[8] Let $S(U_0, U_1)$ and $T(V_0, V_1)$ be two trees of order n with $|U_i| = |V_i|$ $(i=0, 1)$. Then there exists a complete bipartite graph $B_{n+1}(X_0, X_1)$ such that there is a packing of $S(U_0, U_1)$ and $T(V_0, V_1)$ in $B_{n+1}(X_0, X_1)$.

Theorem 2[7] For each tree T of order n, there is a 3-packing of T in some complete bipartite graph $B_{n+2}(X, Y)$.

In this paper we prove the following theorem.

Theorem 3 For each tree T of order n, whose bipartite vertex classes are of size k_1 and k_2, there is a k-packing of T in some complete bipartite graph $B_{n+m}(X, Y)$, where $m=(h_1-k_1)+(h_2-k_2)$, $h_1=|X|$ and $h_2=|Y|$.

1 Proof of Theorem 3

We recall the following lemma due to Yuster[6].

Lemma 1[6] Let H be a bipartite graph with vertex classes H_1 and H_2 of sizes h_1 and h_2 , respectively, $h_1 \leqslant h_2$. Let T be a tree whose bipartite vertex classes are of size k_1 and k_2. If $k_1 \leqslant h_1$ and $k_2 \leqslant h_2$ and $e(H) \geqslant k_2 h_1 + k_1 h_2 + k_1 + k_2 - h_1 - h_2 - k_1 k_2$, then H contains a subgraph isomorphic to T.

Proof of Theorem 3 Let T be a tree of order n, whose bipartite vertex classes are of size k_1 and k_2, where $k_1+k_2=n$. Let $B_n(X, Y)$ be a complete bipartite graph of order n with vertex partition sets X and Y of sizes k_1 and k_2, respectively. Now we add some vertices into X and Y such that $|X|=h_1$, $|Y|=h_2$, $h_1 \leqslant h_2$ and $(h_1-k_1)(h_2-k_2)+(h_1-k_1)+(h_2-k_2) \geqslant (k-1)(n-1)$. So we get a complete bipartite graph $B_{n+m}(X, Y)$ of order $n+m$, where $m=(h_1-k_1)+(h_2-k_2)$. Clearly, $B_{n+m}(X, Y)$ contains a copy of T. Suppose that we have already packed $k-1$ copies of T in $B_{n+m}(X, Y)$. Let H be the spanning subgraph of $B_{n+m}(X, Y)$ which contains all the edges that do not appear in the packing. It is easy to see that $e(H) = h_1 h_2 - (k_1+k_2-1)(k-1)$. Since $(h_1-k_1)(h_2-k_2)+(h_1-k_1)+(h_2-k_2) \geqslant (k-1)(n-1)$, we have $e(H) \geqslant k_2 h_1 + k_1 h_2 + k_1 + k_2 - h_1 - h_2 - k_1 k_2$. By Lemma 1, we find a copy of T in H, and add T to the packing. So there is a k-packing of T in the complete bipartite graph $B_{n+m}(X, Y)$.

The proof is completed. □

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