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Existence of Global Attractor for a 3D Brinkman-Forchheimer Equation in Some Poincaré Unbounded Domains

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Abstract: In this paper, we study the existence of global attractor of a class of three-dimensional Brinkman-Forchheimer equation in some unbounded domains which satisfies Poincaré inequality. We use the tail estimation method to establish the asymptotic compactness of the solution operator and then prove the existence of the global attractor in $(H_0^1(\Omega))^3$.

Key words: Brinkman-Forchheimer equation; global attractor; asymptotic compactness; tail estimation method

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0 Introduction

In this paper, we study a class of three-dimensional Brinkman-Forchheimer equation as follows:

$$\begin{cases} u_t - \gamma\Delta u + au + b|u|u + c|u|^2 u + \nabla p = f(x), & (x,t)\in\Omega\times\mathbb{R}^+ \\ \operatorname{div} u = 0, & (x,t)\in\Omega\times\mathbb{R}^+ \\ u(x,t)\big|_{\partial\Omega} = 0, & t\in\mathbb{R}^+ \\ u(x,0) = u_0(x), & x\in\Omega \end{cases} \tag{1}$$

where $\Omega\subseteq\mathbb{R}^3$ is an open set, not necessarily bounded, which is sufficiently regular and satisfies Poincaré's inequality. Here $u=u(x,t)=(u_1(x,t),u_2(x,t),u_3(x,t))$ is the fluid velocity vector, γ is the Brinkman coefficient, $a>0$ is the Darcy coefficient, $b>0,\ c>0$ are the Forchheimer coefficients, p is the pressure and $u_0=u_0(x)$ is the initial data.

Brinkman-Forchheimer equation describes the motion of fluid flow in a saturated porous medium[1,2], and has been studied by many researchers[3-6]. From the physical viewpoint, Gilver and Altobelli[7] obtained a determination of effective viscosity for the Brinkman-Forchheimer flows model. Nield[8] dealt with the momentum equation in a porous medium, involving the fluid mechanics of the interface region between a porous medium and a fluid layer. Vafai and Kim[9,10] obtained an exact solution to Brinkman-Forchheimer equation by using a generalized momentum equation.

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From the mathematical viewpoint, the discussion of three-dimensional Brinkman-Forchheimer equations is mainly concerned with the well-posedness, regularity and long-time behavior of solutions. The global attractor is the core concept of infinite dimensional dynamical systems. If a system has a global attractor, the attractor will contain all possible limit states of the solution of the system. For the autonomous Brinkman-Forchheimer equation, Uğurlu[11] showed the existence of global attractor of the system (1) in $(H_0^1(\Omega))^3$. If the term "$c|u|^2u$" in system (1) is replaced by "$c|u|^\beta u$", Ouyang and Yang[12] proved the existence of global attractor in $(H_0^1(\Omega))^3$ when $1<\beta<\frac{4}{3}$ by condition-(C) method. In Ref. [13], the existence of *D*-pullback attractors for the three-dimensional non-autonomous Brinkman-Forchheimer equation is deduced by establishing the *D*-pullback asymptotical compactness of θ-cocycle. In Ref.[14], Song *et al* discussed the L^2-decay of the weak solution of the Brinkman-Forchheimer equation in three-dimensional full space. In Ref.[15], Qiao *et al* proved the existence of the global attractor for the strong solution of the Brinkman-Forchheimer equation in a three-dimensional bounded domain. In Ref.[16], Song and Wu discussed a non-autonomous Brinkman-Forchheimer equation with singularly oscillating external force in 3D bounded domains. To the best of our knowledge, there is no discussion of the existence of global attractors in three-dimensional unbounded domains for Brinkman-Forchheimer equation. In this paper, we will discuss the existence of global attractor of system (1) in three-dimensional unbounded domains that satisfy the Poincaré's inequality. We will use the method of uniformly estimating the tail of the solution to obtain the asymptotic compactness of the corresponding solution operator of the equation. This method was first proposed by Wang in Ref.[17].

The structure of this paper is arranged as follows: In Section 1, we give some function space symbols and some inequalities that will be used later. Meanwhile, we provide some uniform estimates of the solution of equation (1), which will be used in the following two sections. In Section 2, we estimate the boundedness of the tail of the solution in $(H_0^1(\Omega))^3$. In Section 3 we prove the asymptotic compactness of the solution in $(H_0^1(\Omega))^3$ and then obtain the existence of global attractor.

1 Preliminaries

Let $\mathbf{L}^p(\Omega)=(L^p(\Omega))^3$, $\mathbf{H}^2(\Omega)=(H^2(\Omega))^3$. Throughout this paper, we use $\|\cdot\|_p$ to denote the norm in $\mathbf{L}^p(\Omega)$. C stands for a generic positive constant, depending on Ω and some constants, but independent of time t.

The Hausdorff semidistance in X from set B_1 to set B_2 is defined as

$$\mathrm{dist}_X(B_1,B_2)=\sup_{b_1\in B_1}\inf_{b_2\in B_2}\|b_1-b_2\|_X.$$

We set $E=\{u|u\in(C_0^\infty(\Omega))^3, \mathrm{div}\, u=0\}$, H is the closure of the set E in $(L^2(\Omega))^3$ topology, and V is the closure of the set E in $(H_0^1(\Omega))^3$ topology. H' and V' are the dual spaces of H and V, clearly, $V\hookrightarrow H\equiv H'\hookrightarrow V'$, where the injection is dense and continuous. We denote by $(\cdot,\cdot)$ and $\|\cdot\|$ the inner product and norm in H. That is,

$$(u,v)=\sum_{j=1}^{3}\int_\Omega u_j(x)v_j(x)\mathrm{d}x,\quad \|u\|^2=(u,u),\ u,v\in H,$$

$((\cdot,\cdot))$ and $\|\cdot\|_V$ denote the inner product and norm in V, that is,

$$((u,v))=\sum_{i,j=1}^{3}\int_\Omega \frac{\partial u_j}{\partial x_i}\frac{\partial v_j}{\partial x_i}\mathrm{d}x,\ u,v\in V,$$

and

$$\|u\|_V^2=\sum_{i=1,j=1}^{3}\int_\Omega\left\|\partial_i u_j\right\|^2,\ u\in V.$$

We call $u\in L^\infty(0,T;H)\cap L^2(0,T;V)\cap L^4(0,T;\mathbf{L}^4(\Omega))$ a weak solution of problem (1) on $[0,T]$, if

$$\begin{cases}\dfrac{\mathrm{d}}{\mathrm{d}t}(u,v)+\gamma((u,v))+a(u,v)+(b|u|u,v)+(c|u|^2u,v)=(f,v),\forall v\in V,\forall t>0\\ u(0)=u_0\end{cases}\tag{2}$$

The weak form (2) is equivalent to the following functional equation:

$$\begin{cases}\dfrac{\mathrm{d}u}{\mathrm{d}t}+\gamma Au+au+B(u)=f,\ \forall t>0\\ u(0)=u_0\end{cases}\tag{3}$$

Here $Au=-\tilde{P}\Delta u$ is the Stokes operator defined as $<Au,v>=((u,v))$. $\tilde{P}$ is the orthogonal projection from $\mathbf{L}^2(\Omega)$ to H. $F(u)=b|u|u+c|u|^2u$, $B(u)=\tilde{P}F(u)$, $f\in H$.

Then we introduce some useful inequalities and lemmas.

Ladyzhenskaya's inequality[18]:

$$\|u\|_3\leqslant C\|u\|^{\frac{1}{2}}\|u\|_V^{\frac{1}{2}},\ \forall u\in V\tag{4}$$

$$\|u\|_4\leqslant C\|u\|^{\frac{1}{4}}\|u\|_V^{\frac{3}{4}},\ \forall u\in V\tag{5}$$

Sobolev's inequality[19]:

$$\|u\|_6\leqslant C\|u\|_V,\ \forall u\in V\tag{6}$$

Lemma 1 (Grownwall's inequality) If $y(t)$, $h(t)\in L^1_{\mathrm{loc}}([0,T];\mathbb{R})$, and $y(t)$ is an absolutely continuous function on $[0,T]$, and the following inequality holds:

$$y'(t)+ky(t)\leqslant h(t),$$

where $k\geqslant 0$, then

$$y'(t)\leqslant y(t_0)\mathrm{e}^{-k(t-t_0)}+\int_{t_0}^{t}\mathrm{e}^{-k(t-s)}h(s)\mathrm{d}s.$$

Especially if $h(t)=C$, then

$$y(t)\leqslant y(t_0)\mathrm{e}^{-k(t-t_0)}+Ck^{-1}.$$

Now we give the existence and uniqueness theorem of the strong solution of equation (1).

Theorem 1 Suppose $u_0\in V\bigcap\mathbf{L}^4(\Omega)$ and $f\in H$. Then there exists a strong solution of equation (1) satisfying

$$u\in L^\infty(0,T;V)\bigcap L^\infty(0,T;\mathbf{L}^4(\Omega))\bigcap L^2(0,T;\mathbf{H}^2(\Omega)),$$

$$\nabla u|u|\in L^2(0,T;H),\ u_t\in L^2(0,T;H).$$

Proof The proof of this theorem is similar to the proof of Theorem 3.2 in Ref.[15], so here we omit the proof process.

Then we give some uniform estimates of the solution below.

Proposition 1 Suppose $u_0\in V\cap\mathbf{L}^4(\Omega)$ and $f\in H$. Then there exists a time t_0, a positive constant ρ_1 such that

$$\|u(t)\|\leqslant\rho_1,\ \forall t>t_0.$$

Proposition 2 Suppose $u_0\in V\cap\mathbf{L}^4(\Omega)$ and $f\in H$. Then there exists a time t_1, a positive constant ρ_2 such that

$$\|\nabla u(t)\|^2+\|u\|_4^4\leqslant\rho_2,\ \forall t>t_1.$$

Proposition 3 Suppose $u_0\in V\cap\mathbf{L}^4(\Omega)$ and $f\in H$. Then there exists a time t_2, a positive constant ρ_3 such that

$$\|u_t(s)\|\leqslant\rho_3,\ \forall s>t_2.$$

The propositions 1, 2, and 3 are proved in Ref.[15], and it is easy to verify that they are still valid in the three-dimensional unbounded domain that satisfies the Poincaré's inequality.

Proposition 4 Suppose $u_0\in V\bigcap\mathbf{L}^4(\Omega)$ and $f\in H$. Then there exists a positive constant ρ_4 such that

$$\|\Delta u(t)\|\leqslant\rho_4,\ \forall t>t_2.$$

Proof Applying Minkowski's inequality, from (1) we have

$$\gamma\|\Delta u\|\leqslant\|u_t\|+a\|u\|+\|f\|+b\||u|u\|+c\||u|^2u\|\tag{7}$$

According to Ladyzhenskaya's inequality we have

$$b\||u|u\|=b\|u\|_4^2\leqslant C\|u\|^{\frac{1}{2}}\|u\|_V^{\frac{3}{2}},\ \forall u\in V\tag{8}$$

And by Sobolev's inequality, we get

$$c\||u|^2u\|=c\|u\|_6^3\leqslant C\|u\|_V^3,\ \forall u\in V\tag{9}$$

Substituting (8) and (9) into (7), we get

$$\gamma\|\Delta u\| \leqslant \|u_t\| + a\|u\| + \|f\| + C\|u\|^{\frac{1}{2}}\|u\|_V^{\frac{3}{2}} + C\|u\|_V^3.$$

So there must exist a positive constant ρ_4 such that $\|\Delta u(t)\| \leqslant \rho_4,\ \forall t > t_2$.

Proposition 5 When $t \geqslant 0$, the map $S(t)$: $V \to V$ is a Lipschitz continuous map on V.

Proof Assuming that u, v are two solutions of equation (1), with initial values u_0 and v_0, respectively. Let $w = u - v$, $w_0 = u_0 - v_0$, then we have

$$\frac{\mathrm{d}w}{\mathrm{d}t} + \gamma Aw + aw = -B(u) + B(v) \tag{10}$$

Taking the inner product of (10) with Aw in H, we get

$$\begin{aligned} \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|w\|_V^2 + \gamma\|Aw\|^2 + a\|w\|_V^2 &= -(F(u) - F(v), Aw) \\ &\leqslant \|F(u) - F(v)\|\|Aw\| \leqslant \frac{\gamma}{2}\|Aw\|^2 + \frac{1}{2\gamma}\|F(u) - F(v)\|^2 \end{aligned} \tag{11}$$

Since

$$\begin{aligned} \|F(u) - F(v)\|^2 &= \int_\Omega \left| b|u|u - b|v|v + c|u|^2u - c|v|^2v \right|^2 \mathrm{d}x \\ &\leqslant 2\left(\int_\Omega \left| c|u|^2u - c|v|^2v \right|^2 \mathrm{d}x + \int_\Omega \left| b|u|u - b|v|v \right|^2 \mathrm{d}x \right) \\ &\leqslant C\left(\int_\Omega [|u|^2|w| + \left||u|^2 - |v|^2\right||v|]^2 \mathrm{d}x + \int_\Omega [|u||w| + ||u| - |v|||v|]^2 \mathrm{d}x \right) \\ &\leqslant C\left(\int_\Omega |u|^4|w|^2 \mathrm{d}x + \int_\Omega (|u| + |v|)^2|v|^2|w|^2 \mathrm{d}x + \int_\Omega |u|^2|w|^2 \mathrm{d}x + \int_\Omega |v|^2|w|^2 \mathrm{d}x \right) \end{aligned} \tag{12}$$

and in the last step of the above inequality, we used the following simple inequality

$$\left|x^p - y^p\right| \leqslant Cp(|x|^{p-1} + |y|^{p-1})\left|x - y\right|,\ \forall x, y \geqslant 0$$

So we have

$$\begin{aligned} \|F(u) - F(v)\|^2 &\leqslant C\|u\|_6^4\|w\|_6^2 + C\||u| + |v|\|_6^2\|v\|_6^2\|w\|_6^2 + C\|u\|_4^2\|w\|_4^2 + C\|v\|_4^2\|w\|_4^2 \\ &\leqslant C\|u\|_6^4\|w\|_V^2 + C\||u| + |v|\|_6^2\|v\|_V^2\|w\|_V^2 + C\|u\|_V^2\|w\|_V^2 + C\|v\|_V^2\|w\|_V^2 \end{aligned} \tag{13}$$

Substituting (13) into (11), we get

$$\frac{\mathrm{d}}{\mathrm{d}t}\|w\|_V^2 + \gamma\|Aw\|^2 + 2a\|w\|_V^2 \leqslant C\left(\|u\|_6^4 + (\|u\|_6^2 + \|v\|_6^2)\|v\|_V^2 + \|u\|_V^2 + \|v\|_V^2\right)\|w\|_V^2 \tag{14}$$

Applying Gronwall's inequality to (14), we have

$$\|w(t)\|_V^2 \leqslant \|w_0\|_V^2 \exp\left\{ C\int_0^t [\|u\|_6^4 + (\|u\|_6^2 + \|v\|_6^2)\|v\|_V^2 + \|u\|_V^2 + \|v\|_V^2] \mathrm{d}s \right\} \tag{15}$$

According to Sobolev's inequality: $\|u\|_6 \leqslant C\|u\|_V$, $\|v\|_6 \leqslant C\|v\|_V$, and because of $u, v \in L^\infty(0, T; V)$, it yields

$$\int_0^t [\|u\|_6^4 + (\|u\|_6^2 + \|v\|_6^2)\|v\|_V^2 + \|u\|_V^2 + \|v\|_V^2] \mathrm{d}s < +\infty.$$

The proof is completed.

2 Uniform Estimate on the Tail of the Solution

In this section, we will employ the technique of uniform estimate on the tail of solution to establish the $(H_0^1(\Omega))^3$-asymptotic compactness of the Brinkman-Forchheimer equation in a three-dimensional unbounded domain satisfying Poincaré's inequality.

Given $k > 0$, we denote by Ω_k the set $\Omega_k = \{x \in \Omega, |x| \leqslant k\}$ and Ω/Ω_k the complement of Ω_k. For our purpose, we choose a cut-off function θ with two order continuous derivative such that $0 \leqslant \theta(s) \leqslant 1$ and

$$\begin{cases} \theta(s) = 0, & \text{if } |s| < 1 \\ \theta(s) = 1, & \text{if } |s| > 2 \end{cases} \tag{16}$$

We have the following lemma.

Lemma 2 Suppose $f\in H$, and $u_0\in B$, which is a bounded set in $V\cap \mathbf{L}^4(\Omega)$. Then for every $\varepsilon\geqslant 0$, there exist $T'(\varepsilon)>0$ and $k'(\varepsilon)>0$ such that $\int_{\Omega/\Omega_k}|\nabla u(t)|^2\mathrm{d}x\leqslant\varepsilon,\ \forall t\geqslant T',\ k\geqslant k'$, where $T'(\varepsilon)$ and $k'(\varepsilon)$ depend on ε.

Proof We will use the tail estimation method, which has been used by Wang in Ref.[17] to establish the existence of global attractor for reaction-diffusion equation in unbounded domain. The method has also been used in Ref.[20] to discuss the existence of global attractor for Newton Boussinesq equation in two-dimensional channel.

Multiplying the first equation of (1) by $-\theta^2(\frac{|x|^2}{k^2})\Delta u$, we have

$$(u_t,-\theta^2(\frac{|x|^2}{k^2})\Delta u)+(-\gamma\Delta u,-\theta^2(\frac{|x|^2}{k^2})\Delta u)+(au,-\theta^2(\frac{|x|^2}{k^2})\Delta u)+(b|u|u,-\theta^2(\frac{|x|^2}{k^2})\Delta u)$$
$$+(c|u|^2u,-\theta^2(\frac{|x|^2}{k^2})\Delta u)+(\nabla p,-\theta^2(\frac{|x|^2}{k^2})\Delta u)=(f,-\theta^2(\frac{|x|^2}{k^2})\Delta u) \quad (17)$$

For the first term on the left-hand side of (17), applying Green's formula, we have

$$\begin{aligned}(u_t,-\theta^2(\frac{|x|^2}{k^2})\Delta u)&=\int_\Omega-\theta^2(\frac{|x|^2}{k^2})\Delta u\cdot\frac{\partial u}{\partial t}\mathrm{d}x=\sum_{i=1}^3\int_\Omega\frac{\partial u}{\partial x_i}\frac{\partial(\theta^2(\frac{|x^2|}{k^2})u_t)}{\partial x_i}\mathrm{d}x\\&=\sum_{i=1}^3\int_\Omega\frac{\partial u}{\partial x_i}[4\theta(\frac{|x|^2}{k^2})\theta'(\frac{|x|^2}{k^2})\frac{x_i}{k^2}u_t+\theta^2(\frac{|x|^2}{k^2})\frac{\partial^2u}{\partial t\partial x_i}]\mathrm{d}x\\&=\frac12\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega\theta^2(\frac{|x|^2}{k^2})|\nabla u|^2\mathrm{d}x+4\sum_{i=1}^3\int_\Omega\theta(\frac{|x|^2}{k^2})\theta'(\frac{|x|^2}{k^2})\frac{\partial u}{\partial x_i}u_t\frac{x_i}{k^2}\mathrm{d}x\end{aligned} \quad (18)$$

For the second term on the left-hand side of (17), we obtain

$$(-\gamma\Delta u,-\theta^2(\frac{|x|^2}{k^2})\Delta u)=\gamma\int_\Omega\theta^2(\frac{|x|^2}{k^2})|\Delta u|^2\mathrm{d}x \quad (19)$$

For the third term on the left-hand side of (17), we get

$$\begin{aligned}(au,-\theta^2(\frac{|x|^2}{k^2})\Delta u)&=-a\int_\Omega\Delta u\theta^2(\frac{|x|^2}{k^2})u\mathrm{d}x=a\sum_{i=1}^3\int_\Omega\frac{\partial u}{\partial x_i}[\frac{\partial u}{\partial x_i}\theta^2(\frac{|x|^2}{k^2})+2\theta(\frac{|x|^2}{k^2})\theta'(\frac{|x|^2}{k^2})\frac{2x_i}{k^2}u]\mathrm{d}x\\&=a\sum_{i=1}^3\int_\Omega\theta^2(\frac{|x|^2}{k^2})|\frac{\partial u}{\partial x_i}|^2\mathrm{d}x+4a\sum_{i=1}^3\int_\Omega\frac{\partial u}{\partial x_i}\theta(\frac{|x|^2}{k^2})\theta'(\frac{|x|^2}{k^2})\frac{x_i}{k^2}u\mathrm{d}x\\&=a\int_\Omega\theta^2(\frac{|x|^2}{k^2})|\nabla u|^2\mathrm{d}x+4a\sum_{i=1}^3\int_\Omega\frac{\partial u}{\partial x_i}\theta(\frac{|x|^2}{k^2})\theta'(\frac{|x|^2}{k^2})\frac{x_i}{k^2}u\mathrm{d}x\end{aligned} \quad (20)$$

Taking (17)-(20) into account, we get

$$\begin{aligned}&\frac12\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega\theta^2(\frac{|x|^2}{k^2})|\nabla u|^2\mathrm{d}x+a\int_\Omega\theta^2(\frac{|x|^2}{k^2})|\nabla u|^2\mathrm{d}x+\gamma\int_\Omega\theta^2(\frac{|x|^2}{k^2})|\Delta u|^2\mathrm{d}x\\&=-4\sum_{i=1}^3\int_\Omega\theta(\frac{|x|^2}{k^2})\theta'(\frac{|x|^2}{k^2})\frac{\partial u}{\partial x_i}u_t\frac{x_i}{k^2}\mathrm{d}x-4a\sum_{i=1}^3\int_\Omega\frac{\partial u}{\partial x_i}\theta(\frac{|x|^2}{k^2})\theta'(\frac{|x|^2}{k^2})\frac{x_i}{k^2}u\mathrm{d}x\\&\quad+b\int_\Omega\theta^2(\frac{|x|^2}{k^2})\Delta u|u|u\mathrm{d}x+c\int_\Omega\theta^2(\frac{|x|^2}{k^2})\Delta u|u|^2u\mathrm{d}x+\int_\Omega\theta^2(\frac{|x|^2}{k^2})\Delta u\nabla p\mathrm{d}x-\int_\Omega f\theta^2(\frac{|x|^2}{k^2})\Delta u\mathrm{d}x\end{aligned} \quad (21)$$

We now estimate the right-hand side of (21) term by term. Applying Young's inequality and Holder's inequality, we find

$$\begin{aligned}|\int_\Omega\theta(\frac{|x|^2}{k^2})\cdot\theta'(\frac{|x|^2}{k^2})\frac{\partial u}{\partial x_i}u_t\frac{x_i}{k^2}\mathrm{d}x|&=|\int_{\Omega(k\leqslant|x|\leqslant\sqrt2 k)}\theta(\frac{|x|^2}{k^2})\cdot\theta'(\frac{|x|^2}{k^2})\frac{\partial u}{\partial x_i}u_t\frac{x_i}{k^2}\mathrm{d}x|\\&\leqslant\frac Ck\int_{\Omega(k\leqslant|x|\leqslant\sqrt2 k)}|\frac{\partial u}{\partial x_i}||u_t|\mathrm{d}x\leqslant\frac Ck\|\nabla u\|\ \|u_t\|\end{aligned} \quad (22)$$

$$\begin{aligned}|\int_\Omega\frac{\partial u}{\partial x_i}\theta(\frac{|x|^2}{k^2})\theta'(\frac{|x|^2}{k^2})\frac{x_i}{k^2}u\mathrm{d}x|&=|\int_{\Omega(k\leqslant|x|\leqslant\sqrt2 k)}\frac{\partial u}{\partial x_i}\theta(\frac{|x|^2}{k^2})\theta'(\frac{|x|^2}{k^2})\frac{x_i}{k^2}u\mathrm{d}x|\\&\leqslant\frac Ck\int_{\Omega(k\leqslant|x|\leqslant\sqrt2 k)}|\frac{\partial u}{\partial x_i}||u|\mathrm{d}x\leqslant\frac Ck\|\nabla u\|\ \|u\|\end{aligned} \quad (23)$$

$$|b\int_\Omega \theta^2(\frac{|x|^2}{k^2})\Delta u|u|u\mathrm{d}x| \leqslant b(\int_\Omega \theta^2(\frac{|x|^2}{k^2})|\Delta u|^2\mathrm{d}x)^{\frac{1}{2}}(\int_\Omega \theta^2(\frac{|x|^2}{k^2})(|u|u)^2\mathrm{d}x)^{\frac{1}{2}}$$
$$\leqslant \frac{\gamma}{4}\int_\Omega \theta^2(\frac{|x|^2}{k^2})|\Delta u|^2\mathrm{d}x + C\int_\Omega \theta^2(\frac{|x|^2}{k^2})(|u|u)^2\mathrm{d}x.$$

Since $\int_\Omega \theta^2(\frac{|x|^2}{k^2})(|u|u)^2\mathrm{d}x \leqslant \int_{\Omega(|x|\geqslant k)}(|u|u)^2\,\mathrm{d}x$, so we have

$$|b\int_\Omega \theta^2(\frac{|x|^2}{k^2})\Delta u|u|u\mathrm{d}x| \leqslant \frac{\gamma}{4}\int_\Omega \theta^2(\frac{|x|^2}{k^2})|\Delta u|^2\mathrm{d}x + C\int_{\Omega(|x|\geqslant k)}(|u|u)^2\,\mathrm{d}x \tag{24}$$

Similar with (24), we have

$$|c\int_\Omega \theta^2(\frac{|x|^2}{k^2})\Delta u|u|^2u\mathrm{d}x| \leqslant \frac{\gamma}{4}\int_\Omega \theta^2(\frac{|x|^2}{k^2})|\Delta u|^2\mathrm{d}x + C\int_{\Omega(|x|\geqslant k)}(|u|^2u)^2\,\mathrm{d}x \tag{25}$$

For the sixth term and the fifth term on the right-hand side of (21) we have

$$|\int_\Omega f\theta^2(\frac{|x|^2}{k^2})\Delta u\mathrm{d}x| = |\int_{\Omega(|x|\geqslant k)} f\theta^2(\frac{|x|^2}{k^2})\Delta u\mathrm{d}x|$$
$$\leqslant (\int_{\Omega(|x|\geqslant k)} f^2\,\mathrm{d}x)^{\frac{1}{2}}(\int_{\Omega(|x|\geqslant k)}\theta^4(\frac{|x|^2}{k^2})|\Delta u|^2\mathrm{d}x)^{\frac{1}{2}} \leqslant C\int_{\Omega(|x|\geqslant k)}|f|^2\mathrm{d}x + \frac{\gamma}{4}\int_\Omega \theta^2(\frac{|x|^2}{k^2})|\Delta u|^2\mathrm{d}x \tag{26}$$

$$|\int_\Omega \theta^2(\frac{|x|^2}{k^2})\Delta u\nabla p\mathrm{d}x| \leqslant \frac{\gamma}{4}\int_\Omega \theta^2(\frac{|x|^2}{k^2})|\Delta u|^2\mathrm{d}x + C\int_\Omega \theta^2(\frac{|x|^2}{k^2})|\nabla p|^2\mathrm{d}x \leqslant \frac{\gamma}{4}\int_\Omega \theta^2(\frac{|x|^2}{k^2})|\Delta u|^2\mathrm{d}x + C\int_{\Omega(|x|\geqslant k)}|\nabla p|^2\mathrm{d}x$$

And because

$$\int_{\Omega(|x|\geqslant k)}|\nabla p|^2\mathrm{d}x = \int_{\Omega(|x|\geqslant k)}\left|f - u_t + \gamma\Delta u - au - b|u|u - c|u|^2u\right|^2\mathrm{d}x$$
$$\leqslant C(\int_{\Omega(|x|\geqslant k)}|f|^2\mathrm{d}x + \int_{\Omega(|x|\geqslant k)}|u_t|^2\mathrm{d}x + \int_{\Omega(|x|\geqslant k)}|\Delta u|^2\mathrm{d}x$$
$$+\int_{\Omega(|x|\geqslant k)}|u|^2\mathrm{d}x + \int_{\Omega(|x|\geqslant k)}\left||u|u\right|^2\mathrm{d}x + \int_{\Omega(|x|\geqslant k)}\left||u|^2u\right|^2\mathrm{d}x)$$

So we have

$$|\int_\Omega \theta^2(\frac{|x|^2}{k^2})\Delta u\nabla p\mathrm{d}x| \leqslant \frac{\gamma}{4}\int_\Omega \theta^2(\frac{|x|^2}{k^2})|\Delta u|^2\mathrm{d}x + C(\int_{\Omega(|x|\geqslant k)}|f|^2\mathrm{d}x + \int_{\Omega(|x|\geqslant k)}|u_t|^2\mathrm{d}x$$
$$+\int_{\Omega(|x|\geqslant k)}|\Delta u|^2\mathrm{d}x + \int_{\Omega(|x|\geqslant k)}|u|^2\mathrm{d}x + \int_{\Omega(|x|\geqslant k)}\left||u|u\right|^2\mathrm{d}x + \int_{\Omega(|x|\geqslant k)}\left||u|^2u\right|^2\mathrm{d}x) \tag{27}$$

By (22)-(27) and (21) we get

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \theta^2(\frac{|x|^2}{k^2})|\nabla u|^2\mathrm{d}x + a\int_\Omega \theta^2(\frac{|x|^2}{k^2})|\nabla u|^2\mathrm{d}x$$
$$\leqslant \frac{C}{k}\|\nabla u\|\,\|u_t\| + \frac{C}{k}\|\nabla u\|\,\|u\| + C\int_{\Omega(|x|\geqslant k)}(|u|u)^2\,\mathrm{d}x + C\int_{\Omega(|x|\geqslant k)}(|u|^2u)^2\,\mathrm{d}x$$
$$+C\int_{\Omega(|x|\geqslant k)}|f|^2\mathrm{d}x + \int_{\Omega(|x|\geqslant k)}|\Delta u|^2\mathrm{d}x + \int_{\Omega(|x|\geqslant k)}|u|^2\mathrm{d}x + \int_{\Omega(|x|\geqslant k)}|u_t|^2\mathrm{d}x \tag{28}$$

Now, from Proposition 1, Proposition 2, Proposition 3 and Proposition 4, we have known that there exists some positive constant M, and time T_1, such that $\|u(t)\| \leqslant M$, $\|\nabla u(t)\| \leqslant M$, $\|u(t)\|_4 \leqslant M$, $\|u_t(t)\| \leqslant M$, $\|\Delta u(t)\| \leqslant M$, for all $t \geqslant T_1$. Hence, given $\varepsilon > 0$, there exists $k_1 = k_1(\varepsilon) > 0$ such that

$$\frac{C}{k}\|\nabla u\|\|u_t\| + \frac{C}{k}\|\nabla u\|\|u\| \leqslant \frac{\varepsilon}{5},\ \forall k \geqslant k_1(\varepsilon),\ t \geqslant T_1 \tag{29}$$

For the given ε, there also exists $k_2 = k_2(\varepsilon) > 0$, such that

$$C\int_{\Omega(|x|\geqslant k)}|\Delta u|^2\mathrm{d}x + C\int_{\Omega(|x|\geqslant k)}|u|^2\mathrm{d}x + C\int_{\Omega(|x|\geqslant k)}|u_t|^2\mathrm{d}x \leqslant \frac{\varepsilon}{5},\ \forall k \geqslant k_2(\varepsilon),\ t \geqslant T_1 \tag{30}$$

Applying Ladyzhenskaya's inequality, we have

$$\int_\Omega (|u|u)^2\,\mathrm{d}x = \|u\|_4^4 \leqslant C\|u\|\|\nabla u\|^3 \leqslant C,\ \forall t \geqslant T_1.$$

Therefore, for the given $\varepsilon>0$, there exists $k_3=k_3(\varepsilon)>0$, such that

$$C\int_{\Omega(|x|\geqslant k)}(|u|u)^2\,\mathrm{d}x\leqslant\frac{\varepsilon}{5},\ \forall k\geqslant k_3(\varepsilon),\ t\geqslant T_1 \tag{31}$$

Furthermore, according to Sobolev's inequality, we have

$$\int_\Omega(|u|^2u)^2\,\mathrm{d}x=\|u\|_6^6\leqslant C\|\nabla u\|^6\leqslant C,\ \forall t\geqslant T_1.$$

So for the given $\varepsilon>0$, there also exists $k_4=k_4(\varepsilon)>0$ such that

$$C\int_{\Omega(|x|\geqslant k)}(|u|^2u)^2\,\mathrm{d}x\leqslant\frac{\varepsilon}{5},\forall k\geqslant k_4(\varepsilon),\ t\geqslant T_1 \tag{32}$$

Since $f\in H$, so there exists $k_5=k_5(\varepsilon)>0$ such that

$$C\int_{\Omega(|x|\geqslant k)}|f|^2\mathrm{d}x\leqslant\frac{\varepsilon}{5},\ \forall k\geqslant k_5(\varepsilon) \tag{33}$$

Let $K=\max\{k_1,k_2,k_3,k_4,k_5\}$, by (28)-(33), $\forall k\geqslant K, t\geqslant T_1$, we have

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega\theta^2\left(\frac{|x|^2}{k^2}\right)|\nabla u|^2\mathrm{d}x+a\int_\Omega\theta^2\left(\frac{|x|^2}{k^2}\right)|\nabla u|^2\mathrm{d}x\leqslant\varepsilon \tag{34}$$

Applying Gronwall's inequality to (34), $\forall k\geqslant K$, we find

$$\begin{aligned}\int_\Omega\theta^2\left(\frac{|x|^2}{k^2}\right)|\nabla u|^2\mathrm{d}x&\leqslant \mathrm{e}^{-2a(t-T_1)}\int_\Omega\theta^2\left(\frac{|x|^2}{k^2}\right)|\nabla u(T_1)|^2\mathrm{d}x+\frac{1}{a}\varepsilon\\&\leqslant \mathrm{e}^{-2a(t-T_1)}\|\nabla u(T_1)\|^2+\frac{1}{a}\varepsilon\leqslant \mathrm{e}^{-2a(t-T_1)}M^2+\frac{1}{a}\varepsilon\leqslant\frac{2}{a}\varepsilon\end{aligned}$$

for all $t\geqslant T_2=T_1-\dfrac{1}{2a}\ln\dfrac{\varepsilon}{aM^2}$.

Therefore,

$$\int_{\Omega(|x|\geqslant\sqrt{2}k)}|\nabla u(t)|^2\mathrm{d}x\leqslant\int_\Omega\theta^2\left(\frac{|x|^2}{k^2}\right)|\nabla u(t)|^2\mathrm{d}x\leqslant\frac{2\varepsilon}{a},\forall t\geqslant T_2,k\geqslant K.$$

The proof is completed.

3 Existence of Global Attractor

In this section, we prove the existence of global attractor for problem (1) in V. To this end, we need to establish the asymptotic compactness of the solution operator which is stated as follows.

Lemma 3 Suppose $u_0\in V\cap\mathbf{L}^4(\Omega)$ and $f\in H$. Then the dynamical system $\{S(t)\}_{t\geqslant0}$ is asymptotically compact in V, i.e., if $t_n\to+\infty$ and $\{u_{0,n}\}_{n=1}^\infty$ is bounded in $V\cap\mathbf{L}^4(\Omega)$, then the sequence $\{S(t_n)u_{0,n}\}_{n=1}^\infty$ has a convergent subsequence in V.

Proof Since $\{u_{0,n}\}_{n=1}^\infty$ is bounded in $V\cap\mathbf{L}^4(\Omega)$, there exists a positive constant $R>0$ such that

$$\|u_{0,n}\|_V\leqslant R,\forall n\in\mathbb{Z}^+ \tag{35}$$

It is well known that A is an unbounded self-adjoint operator with domain:

$$D(A)=(H^2(\Omega))^3\cap(H_0^1(\Omega))^3.$$

And $\|Aw\|$ defines a norm in $D(A)$ which is equivalent to the norm in $(H^2(\Omega))^3$, in other words, there exists a constant $C>0$ depending only on Ω such that

$$\|w\|_{(H^2(\Omega))^3}\leqslant C\|Aw\|,\ \forall w\in D(A).$$

By Proposition 4, there is a positive constant M and a time T_3, such that for every $u_0\in V\cap\mathbf{L}^4(\Omega)$, the following holds

$$\|S(t)u_0\|_{(H^2(\Omega))^3}\leqslant C\|Au(t)\|\leqslant C\|\Delta u(t)\|\leqslant M,\forall t\geqslant T_3 \tag{36}$$

Since $t_n\to+\infty$, there is $N_1>0$ such that $t_n\geqslant T_3$ for all $n>N_1$. Therefore we have, for $n>N_1$,

$$\|S(t_n)u_{0,n}\|_{(H^2(\Omega))^3}\leqslant M \tag{37}$$

By (37) we find that there is a $u\in(H^2(\Omega))^3$ such that, up to a subsequence,

$$S(t_n)u_{0,n} \rightharpoonup u \quad \text{in } (H_0^1(\Omega))^3 \text{ and } (H^2(\Omega))^3 \tag{38}$$

Given $\varepsilon > 0$, by Lemma 2, there are positive constants k' and T_4 such that for any $k \geqslant k'$ and $t \geqslant T_4$, $S(t)u_{0,n}$ with $\| u_{0,n} \|_V \leqslant R$ satisfies

$$\int_{\Omega/\Omega_k} |\nabla S(t)u_{0,n}|^2 \mathrm{d}x \leqslant \frac{\varepsilon}{5} \tag{39}$$

Let N_2 be large enough such that $t_n \geqslant T_4$ for all $n \geqslant N_2$. Then by (39) we obtain, for $n \geqslant N_2$,

$$\int_{\Omega/\Omega_k} |\nabla S(t_n)u_{0,n}|^2 \mathrm{d}x \leqslant \frac{\varepsilon}{5} \tag{40}$$

Notice that (37) implies that the sequence $\{S(t_n)u_{0,n}|_{\Omega_k}\}_{n=1}^{\infty}$ is bounded in $(H^2(\Omega_k))^3$ and hence precompact in $(H_0^1(\Omega_k))^3$. Therefore, there is $\tilde{u} \in (H_0^1(\Omega_k))^3$ such that, up to a subsequence,

$$S(t_n)u_{0,n} \to \tilde{u}, \quad \text{in } (H_0^1(\Omega_k))^3 \tag{41}$$

By (38) and (41), we find $\tilde{u} = u|_{\Omega_k}$, which means that for every $k \geqslant k'$,

$$S(t_n)u_{0,n}|_{\Omega_k} \to u|_{\Omega_k} \quad \text{in } (H_0^1(\Omega_k))^3 \tag{42}$$

In other words, for the given $\varepsilon > 0$, there is $N_3 > 0$ such that for all $k \geqslant k'$ and $n \geqslant N_3$,

$$\int_{\Omega_k} |\nabla(S(t_n)u_{0,n} - u)|^2 \mathrm{d}x \leqslant \frac{\varepsilon}{5} \tag{43}$$

Since $u \in (H_0^1(\Omega))^3$, there is $k'' > 0$ such that for all $k > k''$,

$$\int_{\Omega/\Omega_k} |\nabla u|^2 \mathrm{d}x \leqslant \frac{\varepsilon}{5} \tag{44}$$

Let $k_0 = \max\{k', k''\}$ and $N_0 = \max\{N_1, N_2, N_3\}$. Then for all $n \geqslant N_0$, we have

$$\begin{aligned} \int_{\Omega} |\nabla(S(t_n)u_{0,n} - u)|^2 \mathrm{d}x &= \int_{\Omega_{k_0}} |\nabla(S(t_n)u_{0,n} - u)|^2 \mathrm{d}x + \int_{\Omega/\Omega_{k_0}} |\nabla(S(t_n)u_{0,n} - u)|^2 \mathrm{d}x \\ &\leqslant \int_{\Omega_{k_0}} |\nabla(S(t_n)u_{0,n} - u)|^2 \mathrm{d}x + 2\int_{\Omega/\Omega_{k_0}} |\nabla S(t_n)u_{0,n}|^2 \mathrm{d}x + 2\int_{\Omega/\Omega_{k_0}} |\nabla u|^2 \mathrm{d}x \leqslant \varepsilon \end{aligned} \tag{45}$$

where the last inequality is obtained by (40), (43) and (44). Notice that (45) shows that

$$S(t_n)u_{0,n} \to u \quad \text{in } (H_0^1(\Omega))^3,$$

and hence $\{S(t)\}_{t \geqslant 0}$ is asymptotically compact in $(H_0^1(\Omega))^3$. The proof is completed.

Theorem 2 Suppose $u_0 \in V \cap \mathbf{L}^4(\Omega)$ and $f \in H$. Then the problem (1) has a global attractor A in V, which is compact, invariant and attracts every bounded set with respect to the norm of V.

Proof By Proposition 2, the dynamical system $\{S(t)\}_{t \geqslant 0}$ has a bounded absorbing set in V, and by Lemma 3, $\{S(t)\}_{t \geqslant 0}$ is asymptotically compact in $(H_0^1(\Omega))^3$. Then the existence of a global attractor follows immediately from the standard attractor theory (see Refs.[21]-[25]).

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