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Lower Bounds of Blow Up Time for a Class of Slow Reaction Diffusion Equations with Inner Absorption Terms

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Abstract: In this paper, a class of slow reaction-diffusion equations with nonlocal source and inner absorption terms are studied. By using the technique of improved differential inequality, the lower bounds of blow up time for the system under either homogeneous Dirichlet or nonhomogeneous Neumann boundary conditions are obtained.

Key words: slow reaction diffusion equations; inner absorption terms; lower bounds of blow up time

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0 Introduction

In this paper, we consider the lower bound of blow up time for a class of slow reaction diffusion equations with nonlocal source and inner absorption terms

$$u_t = u^{h_1}(\Delta u^{m_1} + \int_\Omega v^{s_1}\mathrm{d}x - k_1 u^{r_1}),\ \ (x,t)\in\Omega\times(0,t^*) \quad (1)$$

$$v_t = v^{h_2}(\Delta v^{m_2} + \int_\Omega u^{s_2}\mathrm{d}x - k_2 v^{r_2}),\ \ (x,t)\in\Omega\times(0,t^*) \quad (2)$$

$$u(x,0)=u_0(x),\ \ v(x,0)=v_0(x),\ \ x\in\Omega \quad (3)$$

Under either homogeneous Dirichlet boundary condition

$$u(x,t)=v(x,t)=0,\ \ (x,t)\in\partial\Omega\times(0,t^*) \quad (4)$$

or nonhomogeneous Neumann boundary condition

$$\frac{\partial u^{m_1}}{\partial \upsilon}=u^{1-h_1},\ \frac{\partial v^{m_2}}{\partial \upsilon}=v^{1-h_2},\ \ (x,t)\in\partial\Omega\times(0,t^*) \quad (5)$$

where Δ is a Laplace operator, $\Omega\subset\mathbb{R}^3$ is a bounded region with smooth boundary $\partial\Omega$, $0<h_i<1$, $m_i>1$, r_i, s_i, $h_i+s_i>1$, $h_i+m_i>1, i=1,2$, υ is the unit external normal vector in the external normal direction of $\partial\Omega$.

Equations (1)-(5) can be used to describe slow diffusion phenomena in physics and chemistry, such as combustion of two mixtures in heat conduction process, reaction changes of two reactants in chemical reaction, and so on. u, v represent temperature of two media and concentration of reactants respectively etc.

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There are many research achievements on the lower bound of blow up time of the following single reaction-diffusion equation with absorption term,

$$u_t = \Delta u^m + a(x)u^p \int_\Omega u^q \mathrm{d}x - ku^s \tag{6}$$

When $m=1, a(x)=1, p=0$, Song[1] studied the lower bound of blow up time for solution of equation (6) with homogeneous Dirichlet and homogeneous Neumann boundary conditions; Liu[2] studied the lower bound of blow up time for the solution of equation (6) with nonlinear boundary conditions; Tang *et al*[3] has extended the results in equation (6) to higher dimensional cases, see Refs. [4-7] for other relevant achievements.

Relatively speaking, there are few studies on the lower bound of blow up time for the solution of cross coupled reaction diffusion equations with nonlocal sources and absorption terms. Ouyang *et al*[8] studied global existence and blow up of solutions for the following generalized nonlocal porous media equations with time dependent coefficients and absorption terms

$$u_t = \Delta u^a + k_1(t)h_1(u,v) - f_1(u)$$
$$v_t = \Delta v^b + k_2(t)h_2(u,v) - f_2(v)$$

Ouyang *et al*[9] studied the global existence of the solution and the lower bound of blow up time of the solution for the following equations

$$u_t = \Delta u + k_1(t)u^p \int_\Omega v^q \mathrm{d}x - u^\alpha$$
$$v_t = \Delta v + k_2(t)v^r \int_\Omega u^s \mathrm{d}x - v^\beta$$

when blow up occurs. Other similar equations or equation set are shown in Refs. [10-16].

Inspired by Refs. [8-16], this paper studies the lower bound of blow up time for solutions of slow reaction-diffusion equations (1)-(5) with different boundary conditions when $m>1, n>1$, and generalizes the existing results. The results can better describe the reaction diffusion problems with the cross influence of two variables in physical, chemical and hydrodynamic problems.

1 Some Important Inequalities

This part introduces some important inequalities used in this paper.

Lemma 1[17] (Young inequality) Let $1<p,q<\infty$, $\frac{1}{p}+\frac{1}{q}=1$, then $ab \leqslant \frac{a^p}{p}+\frac{b^q}{q}$, $(a,b>0)$.

Lemma 2[17] (Membrane inequality)

$$\lambda \int_\Omega w^2 \mathrm{d}x \leqslant \int_\Omega |\nabla w|^2 \mathrm{d}x$$

where λ is the first eigenvalue of $\Delta w = \lambda w = 0$, $w>0$, $(x\in\Omega)$, $w=0\ (x\in\partial\Omega)$.

Lemma 3[16] Let Ω be the bounded star region in $\mathbb{R}^N$, and $N\geqslant 2$, then

$$\int_{\partial\Omega} u^n \mathrm{d}s \leqslant \frac{N}{\rho_0}\int_\Omega u^n \mathrm{d}x + \frac{nd}{\rho_0}\int_\Omega u^{n-1}|\nabla u|\mathrm{d}x$$

where $\rho_0 = \min_{\partial\Omega}(x.n)>0, d=\max_{\partial\Omega}|x|$.

Lemma 4[17] (Special Young inequality) Let λ be an arbitrary constant, and $0<x<1$, then

$$a^x b^y = (\lambda a)^x \left(\frac{b^{\frac{y}{1-x}}}{\lambda^{\frac{x}{1-x}}}\right)^{1-x} \leqslant \lambda x a + (1-x)\lambda^{\frac{x}{x-1}} b^{\frac{y}{1-x}},\ (a,b>0).$$

2 Lower Bound of Blow up Time under Homogeneous Dirichlet Boundary Conditions

The lower bound of blow up time for solutions of equations under homogeneous Dirichlet boundary conditions is discussed below.

Theorem 1 Define an auxiliary functions

$$H(t) = \int_\Omega (u^l + v^l)\mathrm{d}x \tag{7}$$

for $l>\max\{1, s_i, h_i+m_i-1, 2h_i+3s_i-2, i=1,2\}$. If (u,v) is a non-negative classical solution of equations (1)-(4) and blow up occurs in the sense of measure $H(t)$ at time t^*, then the lower bound of t^* is

$$t^* \geqslant \int_{H(0)}^{\infty} \frac{\mathrm{d}\xi}{J_1\xi^{c_1} + (J_2+J_3)\xi}$$

where $H(0)=\int_\Omega ({u_0}^l + {v_0}^l)\mathrm{d}x$. The normal number J_1, J_2, J_3, c_1 are given in the following proof.

Proof

$$\begin{aligned}
H(t) &= l\int_\Omega u^{l-1}u_t \mathrm{d}x + l\int_\Omega v^{l-1}v_t\mathrm{d}x \\
&= l\int_\Omega u^{l+h_1-1}\Delta u^{m_1}\mathrm{d}x + l\int_\Omega u^{l+h_1-1}\int_\Omega v^{s_1}\mathrm{d}x\mathrm{d}x \\
&\quad -lk_1\int_\Omega u^{l+h_1+r_1-1}\mathrm{d}x + l\int_\Omega v^{l+h_2-1}\Delta v^{m_2}\mathrm{d}x \\
&\quad +l\int_\Omega v^{l+h_2-1}\int_\Omega u^{s_2}\mathrm{d}x\mathrm{d}x - lk_2\int_\Omega v^{l+h_2+r_2-1}\mathrm{d}x \\
&= -l\int_\Omega u^{l+h_1-1}\Delta u^{m_1}\mathrm{d}x + l\int_{\partial\Omega} u^{l+h_1-1}\frac{\Delta u^{m_1}}{\partial \upsilon}\mathrm{d}s \\
&\quad +l\int_\Omega u^{l+h_1-1}\int_\Omega v^{s_1}\mathrm{d}x\mathrm{d}x - lk_1\int_\Omega u^{l+h_1+r_1-1}\mathrm{d}x \\
&\quad -l\int_\Omega v^{l+h_2-1}\nabla v^{m_2}\mathrm{d}x + l\int_{\partial\Omega} v^{l+h_2-1}\frac{\Delta v^{m_2}}{\partial \upsilon}\mathrm{d}s \\
&\quad +l\int_\Omega v^{l+h_2-1}\int_\Omega u^{s_2}\mathrm{d}x\mathrm{d}x - lk_2\int_\Omega v^{l+h_2+r_2-1}\mathrm{d}x
\end{aligned}$$

$$
\begin{aligned}
&=-m_1 l(l+h_1-1)\int_\Omega u^{l+h_1+m_1-3}|\nabla u|^2\mathrm{d}x+l\int_{\partial\Omega}u^{l+h_1-1}\frac{\Delta u^{m_1}}{\partial \upsilon}\mathrm{d}s\\
&\quad+l\int_\Omega u^{l+h_1-1}\int_\Omega v^{s_1}\mathrm{d}x\mathrm{d}x-lk_1\int_\Omega u^{l+h_1+r_1-1}\mathrm{d}x\\
&\quad-m_2 l(l+h_2-1)\int_\Omega v^{l+h_2+m_2-3}|\nabla v|^2\mathrm{d}x+l\int_{\partial\Omega}v^{l+h_2-1}\frac{\Delta v^{m_2}}{\partial \upsilon}\mathrm{d}s\\
&\quad+l\int_\Omega v^{l+h_2-1}\int_\Omega u^{s_2}\mathrm{d}x\mathrm{d}x-lk_2\int_\Omega v^{l+h_2+r_2-1}\mathrm{d}x\\
&=-\frac{4m_1 l(l+h_1-1)}{(k+h_1+m_1-1)^2}\left|\nabla u^{\frac{l+h_1+m_1-1}{2}}\right|^2\mathrm{d}x+l\int_{\partial\Omega}u^{l+h_1-1}\frac{\Delta u^{m_1}}{\partial \upsilon}\mathrm{d}s\\
&\quad+l\int_\Omega u^{l+h_1-1}\int_\Omega v^{s_1}\mathrm{d}x\mathrm{d}x-lk_1\int_\Omega u^{l+h_1+r_1-1}\mathrm{d}x\\
&=-\frac{4m_2 l(l+h_2-1)}{(l+h_2+m_2-1)^2}\left|\nabla v^{\frac{l+h_2+m_2-1}{2}}\right|^2\mathrm{d}x+l\int_{\partial\Omega}v^{l+h_2-1}\frac{\Delta v^{m_2}}{\partial \upsilon}\mathrm{d}s\\
&\quad+l\int_\Omega v^{l+h_2-1}\int_\Omega u^{s_2}\mathrm{d}x\mathrm{d}x-lk_2\int_\Omega v^{l+h_2+r_2-1}\mathrm{d}x
\end{aligned}\tag{8}
$$

When the equations (1) - (4) take homogeneous Dirichlet boundary conditions, equation (8) becomes

$$
H_1'(t)=\alpha_1\int_\Omega\left|\nabla u^a\right|^2\mathrm{d}x+l\int_\Omega u^{l+h_1-1}\mathrm{d}x\int_\Omega v^{s_1}\mathrm{d}x-lk_1\int_\Omega u^{l+h_1+r_1-1}\mathrm{d}x \tag{9}
$$

$$
H_2'(t)=\beta_1\int_\Omega\left|\nabla v^b\right|^2\mathrm{d}x+l\int_\Omega v^{l+h_2-1}\mathrm{d}x\int_\Omega u^{s_2}\mathrm{d}x-lk_2\int_\Omega v^{l+h_2+r_2-1}\mathrm{d}x \tag{10}
$$

here

$$
\alpha_1=-\frac{4m_1 l(l+h_1-1)}{(l+h_1+m_1-1)^2},\beta_1=-\frac{4m_2 l(l+h_2-1)}{(l+h_2+m_2-1)^2},
$$
$$
a=\frac{l+h_1+m_1-1}{2},b=\frac{l+h_2+m_2-1}{2}.
$$

First, Hölder's inequality is used to estimate the second term of $H_1'(t)$ in equation (9), and we obtain

$$
\int_\Omega u^{l+h_1-1}\mathrm{d}x\leqslant\left(\int_\Omega (u^{l+h_1-1})^{\frac{l}{l-s_1}}\mathrm{d}x\right)^{\frac{l-s_1}{l}}|\Omega|^{\frac{s_1}{l}},
$$
$$
\int_\Omega v^{s_1}\mathrm{d}x\leqslant\left(\int_\Omega (v^{s_1})^{\frac{l}{s_1}}\mathrm{d}x\right)^{\frac{s_1}{l}}|\Omega|^{\frac{l-s_1}{l}},
$$
$$
\int_\Omega u^{l+h_1-1}\mathrm{d}x\int_\Omega v^{s_1}\mathrm{d}x\leqslant\left(\int_\Omega (u^{l+h_1-1})^{\frac{l}{l-s_1}}\mathrm{d}x\right)^{\frac{l-s_1}{l}}\left(\int_\Omega (v^{s_1})^{\frac{l}{s_1}}\mathrm{d}x\right)^{\frac{s_1}{l}}|\Omega|.
$$

According to Lemma 1 and equation aboves, we get

$$
\int_\Omega u^{l+h_1-1}\mathrm{d}x\int_\Omega v^{s_1}\mathrm{d}x\leqslant\frac{|\Omega|(l-s_1)}{l}\int_\Omega u^{\frac{l(l+h_1-1)}{l-s_1}}\mathrm{d}x+\frac{|\Omega|s_1}{k}\int_\Omega v^l\mathrm{d}x \tag{11}
$$

Second, Hölder's inequality is used to estimate the first term on the right side of inequality (11), and it is obtained that

$$
\int_\Omega u^{\frac{l(l+h_1-1)}{l-s_1}}\mathrm{d}x=\int_\Omega u^a u^{\frac{l(l+h_1-1)}{l-s_1}-a}\mathrm{d}x\leqslant\left(\int_\Omega u^{4a}\mathrm{d}x\right)^{\frac{1}{4}}\left(\int_\Omega u^{\frac{2[\frac{2l(l+h_1-1)}{l-s_1}-(l+h_1+m_1-1)]}{3}}\mathrm{d}x\right)^{\frac{3}{4}} \tag{12}
$$

From the second term on the right side of inequality sign of (12) and Hölder's inequality, we can get

$$
\left\{\int_\Omega u^{\frac{2[\frac{2l(l+h_1-1)}{l-s_1}-(l+h_1+m_1-1)]}{3}}\mathrm{d}x\right\}^{\frac{3}{4}}\leqslant\left(\int_\Omega u^l\mathrm{d}x\right)^{\frac{1+\frac{2(h_1+s_1-1)}{l-s_1}-\frac{h_1+m_1-1}{l}}{2}}|\Omega|^{\frac{\frac{1}{2}-\frac{2(h_1+s_1-1)}{l-s_1}+\frac{h_1+m_1-1}{l}}{2}} \tag{13}
$$

From the first term on the right side of inequality sign of (12) and Hölder's inequality, we know

$$
\int_\Omega u^{4a}\mathrm{d}x=\int_\Omega u^a u^{3a}\mathrm{d}x\leqslant\left(\int_\Omega u^{2a}\mathrm{d}x\right)^{\frac{1}{2}}\left(\int_\Omega (u^a)^6\mathrm{d}x\right)^{\frac{1}{2}} \tag{14}
$$

Using the following Sobolev inequality[17]

$$
\left(\int_\Omega|\phi|^{\gamma_1}\mathrm{d}x\right)^{\frac{1}{\gamma_1}}\leqslant C\left(\int_\Omega|\nabla\phi|^{\gamma_2}\mathrm{d}x\right)^{\frac{1}{\gamma_2}}
$$

where $\gamma_1=6,\gamma_2=2,C=4^{\frac{1}{3}}3^{-\frac{1}{2}}\pi^{-\frac{2}{3}}$, and the second term of inequality (14) can be simplified to

$$
\left(\int_\Omega (u^a)^6\mathrm{d}x\right)^{\frac{1}{2}}\leqslant C^3\left(\int_\Omega|\nabla u^a|^2\mathrm{d}x\right)^{\frac{3}{2}} \tag{15}
$$

By synthesizing inequalities (14) and (15), we have

$$
\left(\int_\Omega u^{4a}\mathrm{d}x\right)^{\frac{1}{4}}\leqslant C^{\frac{3}{4}}\left(\int_\Omega u^{2a}\mathrm{d}x\right)^{\frac{1}{8}}\left(\int_\Omega|\nabla u^a|^2\mathrm{d}x\right)^{\frac{3}{8}} \tag{16}
$$

Based on lemma 2, inequality (16) becomes

$$
\left(\int_\Omega u^{4a}\mathrm{d}x\right)^{\frac{1}{4}}\leqslant C^{\frac{3}{4}}\lambda^{-\frac{1}{8}}\left(|\nabla u^a|^2\mathrm{d}x\right)^{\frac{1}{2}} \tag{17}
$$

Combining inequalities (13) and (17), inequality (12) becomes

$$
\int_\Omega u^{\frac{l(l+h_1-1)}{l-s_1}}\mathrm{d}x\leqslant\alpha_2\left(\int_\Omega|\nabla u^a|^2\mathrm{d}x\right)^{\frac{1}{2}}\left(\int_\Omega u^l\mathrm{d}x\right)^{d_1} \tag{18}
$$

where

$$
\alpha_2=C^{\frac{3}{4}}\lambda^{-\frac{1}{8}}|\Omega|^{\frac{1}{2}[\frac{1}{2}-\frac{2(h_1+s_1-1)}{l-s_1}+\frac{h_1+m_1-1}{l}]},
$$
$$
d_1=\frac{1}{2}[\frac{1}{2}-\frac{2(h_1+s_1-1)}{l-s_1}-\frac{h_1+m_1-1}{l}].
$$

Secondly, for $h_1+\gamma_1>1$, the order-preserving property of integrals is quoted in the third term of $H_1'(t)$ in equation (9), we have

$$
-lk_1\int_\Omega u^{l+h_1+r_1-1}\mathrm{d}x\leqslant -lk_1\int_\Omega u^l\mathrm{d}x \tag{19}
$$

By synthesizing inequalities (11), (18), (19) with an undetermined positive weight factor φ_1, $H_1'(t)$ of inequality (9) becomes

$$H_1'(t)\leqslant \alpha_1\int_\Omega |\nabla u^a|^2\,\mathrm{d}x$$
$$+\alpha_2|\Omega|(l-s_1)\left(\varphi_1[\int_\Omega |\nabla u^a|^2\mathrm{d}x]^{\frac{1}{2}}\varphi_1^{-1}[(\int_\Omega u^l\mathrm{d}x)^{2d_1}]^{\frac{1}{2}}\right)$$
$$+s_1|\Omega|\int_\Omega v^l\mathrm{d}x-lk_1\int_\Omega u^l\mathrm{d}x \tag{20}$$

where φ_1 is given in the later proof, according to the arithmetic geometric mean inequality

$$a^q b^p<qa+pb\,(a,b,p,q\geqslant 0,p+q=1).$$

Inequality (20) becomes

$$H_1'(t)\leqslant\left(\alpha_1+\frac{\alpha_2|\Omega|(l-s_1)\varphi_1}{2}\right)\int_\Omega|\nabla u^a|^2\mathrm{d}x$$
$$+\frac{\alpha_2|\Omega|(l-s_1)\varphi_1^{-1}}{2}\left(\int_\Omega u^l\mathrm{d}x\right)^{2d_1}+s_1|\Omega|\int_\Omega v^l\mathrm{d}x-lk_1\int_\Omega u^l\mathrm{d}x \tag{21}$$

The same derivation method is used to estimate the $H_2'(t)$ term in inequality (10),

$$H_2'(t)\leqslant\left(\beta_1+\frac{\beta_2|\Omega|(l-s_2)\varphi_2}{2}\right)\int_\Omega|\nabla v^b|^2\mathrm{d}x$$
$$+\frac{\beta_2|\Omega|(l-s_2)\varphi_2^{-1}}{2}\left(\int_\Omega v^l\mathrm{d}x\right)^{2d_2}+s_2|\Omega|\int_\Omega u^l\mathrm{d}x-lk_2\int_\Omega v^l\mathrm{d}x \tag{22}$$

where

$$\beta_2=C^{\frac{3}{4}}\lambda^{-\frac{1}{8}}|\Omega|^{\frac{1}{2}[\frac{1}{2}-\frac{2(h_2+s_2-1)}{l-s_2}+\frac{h_2+m_2-1}{l}]},$$
$$d_2=\frac{1}{2}[\frac{2(h_2+s_2-1)}{l-s_2}-\frac{h_2+m_2-1}{l}].$$

φ_2 is given in the later proof, in order to deal with the gradient terms in (21) and (22), set

$$\varphi_1=-\frac{2\alpha_1}{\alpha_2|\Omega|(l-s_1)},\varphi_2=-\frac{2\beta_1}{\beta_2|\Omega|(l-s_2)}.$$

Finally, by synthesizing inequalities (21) and (22), we can compute

$$H'(t)\leqslant-\frac{[\alpha_2|\Omega|(l-s_1)]^2}{4a_1}\left(\int_\Omega u^l\mathrm{d}x\right)^{2d_1}$$
$$-\frac{[\beta_2|\Omega|(l-s_2)]^2}{4\beta_1}\left(\int_\Omega v^l\mathrm{d}x\right)^{2d_2}+|\Omega|\left(s_1\int_\Omega v^l\mathrm{d}x+s_2\int_\Omega u^l\mathrm{d}x\right)$$
$$-l\left(k_1\int_\Omega u^l\mathrm{d}x+k_2\int_\Omega v^l\mathrm{d}x\right) \tag{23}$$

take

$$J_1=-\frac{[\alpha_2|\Omega|(l-s_1)]^2}{4\alpha_1}-\frac{[\beta_2|\Omega|(l-s_2)]^2}{4\beta_1},$$

$J_2=|\Omega|(s_1+s_1), J_3=-l(k_1+k_2), c_1=\max\{2d_1,2d_2\}>1$, inequality (23) becomes

$$H'(t)\leqslant J_1H^{c_1}(t)+(J_2+J_3)H(t) \tag{24}$$

Integrating (24) from 0 to t^*, we obtain

$$t^*\geqslant\int_{H(0)}^{\infty}\frac{\mathrm{d}\xi}{J_1\xi^{c_1}+(J_2+J_3)\xi} \tag{25}$$

This completes the proof of the theorem.

3 Lower Bound of Blow up Time under Nonhomogeneous Neumann Boundary Conditions

Theorem 2 Define the same measure as (7) and the same condition as l. If (u,v) is a nonnegative classical solution to the equations (1)-(3) and (5), then the lower bound of t^* is

$$t^*\geqslant\int_{H(0)}^{\infty}\frac{\mathrm{d}\xi}{J_1\xi^{c_1}+J_5\xi^{c_2}+(J_2+J_3+J_4)\xi}$$

where $H(0)=\int_\Omega(u_0^l+v_0^l)\mathrm{d}x$, the normal number $J_1,J_2,J_3,J_4,J_5,c_1,c_2$ are given in the following proof.

Proof Lemma 3 is used to estimate two boundary terms in inequality (8), then

$$\int_{\partial\Omega}u^{l+h_1-1}\frac{\partial u^{m_1}}{\partial \upsilon}\mathrm{d}s=\int_{\partial\Omega}u^l\mathrm{d}s$$
$$\leqslant\frac{3}{\rho_0}\int_\Omega u^l\mathrm{d}x+\frac{ld}{\rho_0}\int_\Omega u^{l-1}|\nabla u|\mathrm{d}x \tag{26}$$

where $\rho_0=\min\limits_{\partial\Omega}(x\cdot n)>0, d=\max\limits_{\partial\Omega}|x|$.

From the second term on the right side of inequality (26) and Hölder's inequality and Lemma 4, we can know

$$\int_\Omega u^{l-1}|\nabla u|\mathrm{d}x\leqslant\left(\int_\Omega u^{l+h_1+m_1-3}|\nabla u|^2\mathrm{d}x\right)^{\frac{1}{2}}\left(\int_\Omega u^{l-h_1-m_1+1}\mathrm{d}x\right)^{\frac{1}{2}}$$
$$\leqslant\frac{\lambda_1}{2}\left(\int_\Omega u^{k+p_1+m_1-3}|\nabla u|^2\mathrm{d}x\right)+\frac{1}{2\lambda_1}\int_\Omega u^{k-p_1-m_1+1}\mathrm{d}x$$
$$=-\frac{2\lambda_1}{(l+h_1+m_1-3)^2}\int_\Omega|\nabla u^{\frac{l+h_1+m_1-1}{2}}|^2\mathrm{d}x$$
$$+\frac{1}{2\lambda_1}\int_\Omega u^{l-h_1-m_1+1}\mathrm{d}x \tag{27}$$

where λ_1 is an arbitrary constant.

The Hölder's inequality is used to estimate the second term on the right side of inequality (27), and we have

$$\int_\Omega u^{l-(h_1+m_1-1)}\mathrm{d}x\leqslant\left(\int_\Omega u^l\mathrm{d}x\right)^{\frac{l-(h_1+m_1-1)}{l}}|\Omega|^{\frac{h_1+m_1-1}{l}} \tag{28}$$

Substituting (27) and (28) into inequality (26), we get

$$\int_{\partial\Omega}u^{l+h_1-1}\frac{\partial u^m}{\partial\upsilon}\mathrm{d}s\leqslant\frac{3}{\rho_0}\int_\Omega u^l\mathrm{d}x+\alpha_3\int_\Omega|\nabla u^a|^2\mathrm{d}x+\alpha_4\left(\int_\Omega u^l\mathrm{d}x\right)^{d_3} \tag{29}$$

where

$$\alpha_3 = -\frac{2\lambda_1 ld}{\rho_0(l+h_1+m_1+1)^2},$$
$$\alpha_4 = \frac{ld}{2\lambda_1\rho_0}|\Omega|^{\frac{h_1+m_1-1}{k}},$$
$$d_3 = \frac{l-h_1-m_1+1}{l}.$$

Similarly, another boundary term in inequality (8) is estimated as follows

$$\int_{\partial\Omega} u^{l+h_2-1}\frac{\partial u^{m_2}}{\partial \upsilon}\mathrm{d}s \leqslant \frac{3}{\rho_0}\int_\Omega v^l \mathrm{d}x + \beta_3\int_\Omega |\nabla v^b|^2 \mathrm{d}x + b_4\left(\int_\Omega v^l \mathrm{d}x\right)^{d_4} \quad (30)$$

$$\beta_3 = -\frac{2\lambda_2 ld}{\rho_0(l+h_2+m_2+1)^2},$$
$$\beta_4 = \frac{ld}{2\lambda_2\rho_0}|\Omega|^{\frac{h_2+m_2-1}{l}},$$
$$d_4 = \frac{l-h_2-m_2+1}{l},$$

λ_2 is an arbitrary constant.

Substituting (21), (22), (29) and (30) into inequality (8), we compute

$$H'(t) \leqslant \left(\alpha_1 - \alpha_3 + \frac{\alpha_2|\Omega|(l-s_1)\varphi_3}{2}\right)\int_\Omega |\nabla u^a|^2 \mathrm{d}x + \left(\beta_1 - \beta_3 + \frac{\beta_2|\Omega|(l-s_2)\varphi_4}{2}\right)\int_\Omega |\nabla v^\beta|^2 \mathrm{d}x$$
$$- \frac{[\alpha_2|\Omega|(l-s_1)]^2}{4a_1}\left(\int_\Omega u^l \mathrm{d}x\right)^{2d_1} - \frac{[\beta_2|\Omega|(k-s_2)]^2}{4\beta_1}\left(\int_\Omega v^k \mathrm{d}x\right)^{2d_2}$$
$$+\alpha_4\left(\int_\Omega u^k \mathrm{d}x\right)^{d_3} + \beta_4\left(\int_\Omega v^k \mathrm{d}x\right)^{d_4}$$
$$+\left(\frac{3}{\rho_0} + |\Omega|s_2 - lk_1\right)\int_\Omega u^l\,\mathrm{d}x + \left(\frac{3}{\rho_0} + |\Omega|s_1 - lk_2\right)\int_\Omega v^l\,\mathrm{d}x \quad (31)$$

Let

$$\varphi_3 = -\frac{2(\alpha_1+\alpha_3)}{\alpha_2|\Omega|(l-s_1)}, \varphi_4 = -\frac{2(\beta_1+\beta_3)}{\beta_2|\Omega|(l-s_2)},$$
$$J_1 = -\frac{[\alpha_2|\Omega|(l-s_1)]^2}{4\alpha_1} - \frac{[\beta_2|\Omega|(l-s_2)]^2}{4\beta_1},$$
$$J_2 = |\Omega|(s_1+s_2), J_3 = -l(k_1+k_2), J_5 = \alpha_4+\beta_4, J_4 = \frac{3}{\rho_0},$$
$$c_1 = \max\{2d_1, 2d_2\} > 1, c_2 = \max\{d_3, d_4\} > 0,$$

inequality (31) becomes

$$H'(t) \leqslant J_1 H^{c_1}(t) + J_5 H^{c_2}(t) + (J_2+J_3+J_4)H(t) \quad (32)$$

Integrating (32) from 0 to t^*, we obtain

$$t^* \geqslant \int_{H(0)}^{\infty}\frac{\mathrm{d}\xi}{J_1\xi^{c_1} + J_5\xi^{c_2} + (J_2+J_3+J_4)\xi},$$

where $H(0) = \int_\Omega (u_0^L + v_0^L)\mathrm{d}x$.

This completes the proof of the theorem.

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