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Nonlinear A^*B+B^*A Type Derivations on *-Algebras

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Abstract: Let $\mathcal{A}$ be a unital *-algebra with the unit I and a nontrivial projection $P\in\mathcal{A}$. Suppose that $\mathcal{A}$ satisfies $X\mathcal{A}P=0$ implies $X=0$ and $X\mathcal{A}(I-P)=0$ implies $X=0$. In this paper, we prove that ϕ is a nonlinear A^*B+B^*A type derivation on $\mathcal{A}$ if and only if ϕ is an additive *-derivation. This is then applied to prime *-algebra, von Neumann algebras with no central summands of type I_1, factor von Neumann algebras and standard operator algebras.

Key words: A^*B+B^*A type derivations; *-derivation; von Neumann algebras

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0 Introduction

Let $\mathcal{A}$ be a *-algebra over the complex field $\mathbb{C}$. For $A,B\in\mathcal{A}$, we write $[A,B]_\diamond=A^*B-B^*A, A\bullet B=AB+BA^*$and $A\circ B=A^*B+B^*A$ for the bi-skew Lie product, *-Jordan product and A^*B+B^*A product, respectively. These products have recently attracted the attention of many authors (see Refs. [1-14]).

Recall that a map $\phi:\mathcal{A}\to\mathcal{A}$ is said to be an additive derivation if $\phi(A+B)=\phi(A)+\phi(B)$ and $\phi(AB)=\phi(A)B+A\phi(B)$ for $A,B\in\mathcal{A}$. Furthermore, ϕ is an additive *-derivation if it is an additive derivation and $\phi(A^*)=\phi(A)^*$ for $A\in\mathcal{A}$. A map $\phi:\mathcal{A}\to\mathcal{A}$(without the linearity assumption) is called a nonlinear A^*B+B^*A derivation if $\phi(A\circ B)=\phi(A)\circ B+A\circ\phi(B)$ for $A,B\in\mathcal{A}$, where $A\circ B=A^*B+B^*A$. Darvish *et al*[1] proved that every nonlinear A^*B+B^*A triple derivation on prime *-algebras is an additive *-derivation. A map $\phi:\mathcal{A}\to\mathcal{A}$ (without the linearity assumption) is called a nonlinear *-Jordan derivation if $\phi(A\bullet B)=\phi(A)\bullet B+A\bullet\phi(B)$ for $A,B\in\mathcal{A}$. The authors of Ref. [2] introduced the concept of *-Jordan-type derivation. Suppose that $n\geqslant 2$ is a fixed positive integer. Accordingly, a nonlinear *-Jordan-type derivation is a map ϕ: $\mathcal{A}\to\mathcal{A}$ satisfying the condition $\phi(A_1\bullet A_2\bullet\cdots\bullet A_n)=\sum_{k=1}^{n}A_1\bullet\cdots\bullet A_{k-1}\bullet\phi(A_k)\bullet A_{k+1}\bullet\cdots\bullet A_n$ for all $A_1,A_2,\cdots,A_n\in\mathcal{A}$, where $A_1\bullet A_2\bullet\cdots\bullet A_n=(\cdots((A_1\bullet A_2)\bullet A_3)\bullet\cdots\bullet A_n), A_i\bullet A_j=A_iA_j+A_jA_i^*$, $i,j\in\mathbb{N}$. Under some mild condition on a *-algebra $\mathcal{A}$, they showed that ϕ is a nonlinear *-Jordan-type derivation on $\mathcal{A}$ if and only if ϕ is an additive *-derivation.

Motivated by the above results, we introduce the A^*B+B^*A type derivations. Suppose that $n\geqslant 2$ is a fixed positive

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integer. A nonlinear A^*B+B^*A type derivation is a map $\phi:\mathcal{A}\to\mathcal{A}$ satisfying the condition $\phi(A_1\circ A_2\circ\cdots\circ A_n)=\sum_{k=1}^{n}A_1\circ\cdots\circ A_{k-1}\circ\phi(A_k)\circ A_{k+1}\circ\cdots\circ A_n$ for all $A_1,A_2,\cdots,A_n\in\mathcal{A}$, where $A_1\circ A_2\circ\cdots\circ A_n=(\cdots((A_1\circ A_2)\circ A_3)\circ\cdots\circ A_n), A_i\circ A_j=A_i^*A_j+A_j^*A_i,\ i,j\in\mathbb{N}$. In this paper, under some mild condition on a *-algebra $\mathcal{A}$, we prove that ϕ is a nonlinear A^*B+B^*A type derivation on $\mathcal{A}$ if and only if ϕ is an additive *-derivation.

1 The Main Result and Its Proof

Theorem 1 Let $\mathcal{A}$ be a unital *-algebra with the unit I and a nontrivial projection $P\in\mathcal{A}$. Suppose that $\mathcal{A}$ satisfies (a) $X\mathcal{A}P=0$ implies $X=0$ and (b) $X\mathcal{A}(I-P)=0$ implies $X=0$.

If a map $\phi:\mathcal{A}\to\mathcal{A}$ satisfies $\phi(A_1\circ A_2\circ\cdots\circ A_n)=\sum_{k=1}^{n}A_1\circ\cdots\circ A_{k-1}\circ\phi(A_k)\circ A_{k+1}\circ\cdots\circ A_n$ for all $A_1,A_2\in\mathcal{A}$ and $A_3=A_4=\cdots=A_n=\frac{I}{2}$, then ϕ is an additive *-derivation.

Let $P_1=P$ and $P_2=I-P_1$. Let $\mathcal{A}_{ij}=P_i\mathcal{A}P_j, i,j=1,2$; then $\mathcal{A}=\sum_{i,j=1}^{2}\mathcal{A}_{ij}$. We can write every $A\in\mathcal{A}$ as $A=\sum_{i,j=1}^{2}A_{ij}$, where A_{ij} denotes an arbitrary element of $\mathcal{A}_{ij}$. Let $\mathcal{M}=\{A\in\mathcal{A}:A^*=A\}, \mathcal{M}_{12}=\{P_1MP_2+P_2MP_1:M\in\mathcal{M}\}, \mathcal{M}_{ii}=P_i\mathcal{M}P_i, i=1,2$. Then for all $M\in\mathcal{M}, M=M_{11}+M_{12}+M_{22}$, where $M_{12}\in\mathcal{M}_{12}, M_{ii}\in\mathcal{M}_{ii}, i=1,2$.

Proof The proof is completed by the following several claims.

Claim 1 $\phi(0)=0$.

$$\phi(0)=\phi\left(0\circ 0\circ\frac{I}{2}\circ\ldots\circ\frac{I}{2}\right)=0$$

Claim 2 $\phi\left(\frac{I}{2}\right)=0, \phi\left(\frac{\mathrm{i}}{2}I\right)=0, \phi\left(-\frac{\mathrm{i}}{2}I\right)=0.$

Using $\frac{I}{2}=\frac{I}{2}\circ\frac{I}{2}\circ\ldots\circ\frac{I}{2}$, we obtain

$$\begin{aligned}\phi\left(\frac{I}{2}\right)&=\phi\left(\frac{I}{2}\circ\frac{I}{2}\circ\cdots\circ\frac{I}{2}\right)=\phi\left(\frac{I}{2}\right)\circ\frac{I}{2}\circ\cdots\circ\frac{I}{2}+\frac{I}{2}\circ\phi\left(\frac{I}{2}\right)\circ\cdots\circ\frac{I}{2}+\cdots+\frac{I}{2}\circ\frac{I}{2}\circ\cdots\circ\phi\left(\frac{I}{2}\right)\\&=\frac{n}{2}\left(\phi\left(\frac{I}{2}\right)+\phi\left(\frac{I}{2}\right)^*\right)\end{aligned}$$

which implies $\phi\left(\frac{I}{2}\right)^*=\phi\left(\frac{I}{2}\right)$, and then $(n-1)\phi\left(\frac{I}{2}\right)=0$. Since $n\geqslant 2$, we obtain

$$\phi\left(\frac{I}{2}\right)=0 \tag{1}$$

From (1), we obtain

$$\begin{aligned}0=\phi\left(\frac{I}{2}\right)&=\phi\left(\frac{\mathrm{i}}{2}I\circ\frac{\mathrm{i}}{2}I\circ\frac{I}{2}\circ\cdots\circ\frac{I}{2}\right)=\phi\left(\frac{\mathrm{i}}{2}I\right)\circ\frac{\mathrm{i}}{2}I\circ\frac{I}{2}\circ\cdots\circ\frac{I}{2}+\frac{\mathrm{i}}{2}I\circ\phi\left(\frac{\mathrm{i}}{2}I\right)\circ\frac{I}{2}\circ\cdots\circ\frac{I}{2}\\&=\mathrm{i}\left(\phi\left(\frac{\mathrm{i}}{2}I\right)^*-\phi\left(\frac{\mathrm{i}}{2}I\right)\right)\end{aligned}$$

i.e.,

$$\phi\left(\frac{\mathrm{i}}{2}I\right)^*=\phi\left(\frac{\mathrm{i}}{2}I\right) \tag{2}$$

Using (1) and (2), we have

$$0=\phi\left(\frac{\mathrm{i}}{2}I\circ\frac{I}{2}\circ\cdots\circ\frac{I}{2}\right)=\phi\left(\frac{\mathrm{i}}{2}I\right)\circ\frac{I}{2}\circ\cdots\circ\frac{I}{2}=\phi\left(\frac{\mathrm{i}}{2}I\right)$$

In the same manner, we obtain $\phi\left(-\frac{\mathrm{i}}{2}I\right)=0$.

Claim 3 For every $A\in\mathcal{A}$, we have $\phi(\mathrm{i}A)=\mathrm{i}\phi(A)$.

For all $A\in\mathcal{A}$, using $\mathrm{i}A\circ\frac{I}{2}\circ\frac{I}{2}\circ\cdots\circ\frac{I}{2}=A\circ-\frac{\mathrm{i}}{2}I\circ\frac{I}{2}\circ\cdots\circ\frac{I}{2}$ and Claim 2, we obtain

$$\frac{1}{2}(\phi(\mathrm{i}A)^*+\phi(\mathrm{i}A))=\phi(\mathrm{i}A)\circ\frac{I}{2}\circ\frac{I}{2}\circ\cdots\circ\frac{I}{2}=\phi\left(\mathrm{i}\mathrm{A}\circ\frac{I}{2}\circ\frac{I}{2}\circ\cdots\circ\frac{I}{2}\right)=\phi\left(A\circ-\frac{\mathrm{i}}{2}I\circ\frac{I}{2}\circ\cdots\circ\frac{I}{2}\right)$$
$$=\phi(A)\circ-\frac{\mathrm{i}}{2}I\circ\frac{I}{2}\circ\cdots\circ\frac{I}{2}=\frac{\mathrm{i}}{2}(\phi(A)-\phi(A)^*)$$

i.e.,

$$\phi(\mathrm{i}A)^*+\phi(\mathrm{i}A)=\mathrm{i}\phi(A)-\mathrm{i}\phi(A)^* \tag{3}$$

Using $\mathrm{i}A\circ\frac{\mathrm{i}}{2}\circ\frac{I}{2}\circ\cdots\circ\frac{I}{2}=A\circ\frac{I}{2}\circ\frac{I}{2}\circ\cdots\circ\frac{I}{2}$ and Claim 2, we have

$$\frac{\mathrm{i}}{2}(\phi(\mathrm{i}A)^*-\phi(\mathrm{i}A))=\phi(\mathrm{i}A)\circ\frac{\mathrm{i}}{2}\circ\frac{I}{2}\circ\cdots\circ\frac{I}{2}=\phi\left(\mathrm{i}A\circ\frac{\mathrm{i}}{2}\circ\frac{I}{2}\circ\cdots\circ\frac{I}{2}\right)=\phi\left(A\circ\frac{I}{2}\circ\frac{I}{2}\circ\cdots\circ\frac{I}{2}\right)$$
$$=\phi(A)\circ\frac{I}{2}\circ\frac{I}{2}\circ\cdots\circ\frac{I}{2}=\frac{1}{2}(\phi(A)+\phi(A)^*)$$

i.e.,

$$-\phi(\mathrm{i}A)^*+\phi(\mathrm{i}A)=\mathrm{i}\phi(A)+\mathrm{i}\phi(A)^* \tag{4}$$

From (3) and (4), we obtain $\phi(\mathrm{i}A)=\mathrm{i}\phi(A)$.

Claim 4 For every $A\in\mathcal{M}$, we have $\phi(A)^*=\phi(A)$.

For all $A\in\mathcal{M}$, we have

$$\phi(A)=\phi\left(A\circ\frac{I}{2}\circ\frac{I}{2}\circ\cdots\circ\frac{I}{2}\right)=\phi(A)\circ\frac{I}{2}\circ\frac{I}{2}\circ\cdots\circ\frac{I}{2}=\frac{1}{2}(\phi(A)+\phi(A)^*)$$

which indicates $\phi(A)^*=\phi(A)$.

Claim 5 For every $M_{11}\in\mathcal{M}_{11},M_{12}\in\mathcal{M}_{12},M_{22}\in\mathcal{M}_{22}$, we have (i) $\phi(M_{11}+M_{12})=\phi(M_{11})+\phi(M_{12})$; (ii) $\phi(M_{12}+M_{22})=\phi(M_{12})+\phi(M_{22})$.

Setting $T=\phi(M_{11}+M_{12})-\phi(M_{11})-\phi(M_{12})$, let us prove that $T=0$. Based on Claim 4, we obtain $T^*=T$. Since $P_2\circ M_{11}\circ\frac{I}{2}\circ\cdots\circ\frac{I}{2}=0$, it follows from Claim 1 and Claim 2 that

$$\phi(P_2)\circ(M_{11}+M_{22})\circ\frac{I}{2}\circ\cdots\circ\frac{I}{2}+P_2\circ\phi(M_{11}+M_{22})\circ\frac{I}{2}\circ\cdots\circ\frac{I}{2}=\phi(P_2\circ(M_{11}+M_{22})\circ\frac{I}{2}\circ\cdots\circ\frac{I}{2})$$
$$=\phi(P_2\circ M_{11}\circ\frac{I}{2}\circ\cdots\circ\frac{I}{2})+\phi(P_2\circ M_{22}\circ\frac{I}{2}\circ\cdots\circ\frac{I}{2})$$
$$=\phi(P_2)\circ(M_{11}+M_{22})\circ\frac{I}{2}\circ\cdots\circ\frac{I}{2}+P_2\circ(\phi(M_{11})+\phi(M_{22}))\circ\frac{I}{2}\circ\cdots\circ\frac{I}{2}$$

i.e., $P_2\circ T\circ\frac{I}{2}\circ\cdots\circ\frac{I}{2}=0$. This together with $T^*=T$ shows that $P_1TP_2=P_2TP_1=P_2TP_2=0$. Using $(P_1-P_2)\circ M_{12}\circ\frac{I}{2}\circ\cdots\circ\frac{I}{2}=0$, Claim 1 and Claim 2, we obtain

$$\phi(P_1-P_2)\circ(M_{11}+M_{12})\circ\frac{I}{2}\circ\cdots\circ\frac{I}{2}+(P_1-P_2)\circ\phi(M_{11}+M_{12})\circ\frac{I}{2}\circ\cdots\circ\frac{I}{2}$$
$$=\phi((P_1-P_2)\circ(M_{11}+M_{12})\circ\frac{I}{2}\circ\cdots\circ\frac{I}{2})$$
$$=\phi((P_1-P_2)\circ M_{11}\circ\frac{I}{2}\circ\cdots\circ\frac{I}{2})+\phi((P_1-P_2)\circ M_{12}\circ\frac{I}{2}\circ\cdots\circ\frac{I}{2})$$
$$=\phi(P_1-P_2)\circ(M_{11}+M_{12})\circ\frac{I}{2}\circ\cdots\circ\frac{I}{2}+(P_1-P_2)\circ(\phi(M_{11})+\phi(M_{12}))\circ\frac{I}{2}\circ\cdots\circ\frac{I}{2}$$

i.e., $(P_1-P_2)\circ T\circ\frac{I}{2}\circ\cdots\circ\frac{I}{2}=0$. This together with $T^*=T$ shows that $P_1TP_1=0$. And then $T=0$.

In the second case, we can similarly prove that the conclusion is valid.

Claim 6 For every $M_{11}\in\mathcal{M}_{11},M_{12}\in\mathcal{M}_{12},M_{22}\in\mathcal{M}_{22}$, we have $\phi(M_{11}+M_{12}+M_{22})=\phi(M_{11})+\phi(M_{12})+\phi(M_{22})$.

Setting $T=\phi\left(M_{11}+M_{12}+M_{22}\right)-\phi\left(M_{11}\right)-\phi\left(M_{12}\right)-\phi\left(M_{22}\right)$, since $P_1\circ M_{22}\circ\frac{I}{2}\circ\cdots\circ\frac{I}{2}=0$, applying Claim 1, Claim 2 and Claim 5(i), we obtain

$$\begin{aligned}&\phi\left(P_1\right)\circ(M_{11}+M_{12}+M_{22})\circ\frac{I}{2}\circ\cdots\circ\frac{I}{2}+P_1\circ\phi(M_{11}+M_{12}+M_{22})\circ\frac{I}{2}\circ\cdots\circ\frac{I}{2}\\&=\phi(P_1\circ(M_{11}+M_{12}+M_{22})\circ\frac{I}{2}\circ\cdots\circ\frac{I}{2})\\&=\phi(P_1\circ(M_{11}+M_{12})\circ\frac{I}{2}\circ\cdots\circ\frac{I}{2})+\phi(P_1\circ M_{22}\circ\frac{I}{2}\circ\cdots\circ\frac{I}{2})\\&=\phi\left(P_1\right)\circ(M_{11}+M_{12}+M_{22})\circ\frac{I}{2}\circ\cdots\circ\frac{I}{2}+P_1\circ(\phi(M_{11})+\phi(M_{12})+\phi(M_{22}))\circ\frac{I}{2}\circ\cdots\circ\frac{I}{2}\end{aligned}$$

i.e., $P_1\circ T\circ\frac{I}{2}\circ\cdots\circ\frac{I}{2}=0$. This together with $T^*=T$ shows that $P_1TP_1=P_1TP_2=P_2TP_1=0$. In the same manner, by applying the above proof for P_2 instead of P_1 and Claim 5(ii) instead of Claim 5(i), we have $P_2TP_2=0$.

Claim 7 For every $M_{12},B_{12}\in\mathcal{M}_{12}$, we have $\phi\left(M_{12}+B_{12}\right)=\phi(M_{12})+\phi(B_{12})$.

Let $M_{12},B_{12}\in\mathcal{M}_{12}$, we obtain $M_{12}=U_{12}+U_{12}^*,B_{12}=V_{12}+V_{12}^*$, where $U_{12},V_{12}\in\mathcal{M}_{12}$. Since $M_{12}B_{12}^*+B_{12}M_{12}^*=U_{12}V_{12}^*+V_{12}U_{12}^*+U_{12}^*V_{12}+V_{12}^*U_{12}$, we set $U_{12}V_{12}^*+V_{12}U_{12}^*=M_{11}\in\mathcal{M}_{11}$, $U_{12}^*V_{12}+V_{12}^*U_{12}=M_{22}\in\mathcal{M}_{22}$, then

$$\begin{aligned}&(P_1+U_{12}+U_{12}^*)\circ(P_2+V_{12}+V_{12}^*)\circ\frac{I}{2}\circ\cdots\circ\frac{I}{2}\\&=(U_{12}+U_{12}^*)+(V_{12}+V_{12}^*)+(U_{12}V_{12}^*+V_{12}U_{12}^*+U_{12}^*V_{12}+V_{12}^*U_{12})\\&=M_{12}+B_{12}+M_{12}B_{12}^*+B_{12}M_{12}^*=M_{12}+B_{12}+M_{11}+M_{22}\end{aligned}$$

Using $U_{12}+U_{12}^*,V_{12}+V_{12}^*\in\mathcal{M}_{12}$ and Claim 6, we obtain

$$\begin{aligned}&\phi(M_{12}+B_{12})+\phi(M_{11})+\phi(M_{22})=\phi(M_{12}+B_{12}+M_{12}B_{12}^*+B_{12}M_{12}^*)\\&=\phi((P_1+U_{12}+U_{12}^*)\circ(P_2+V_{12}+V_{12}^*)\circ\frac{I}{2}\circ\cdots\circ\frac{I}{2})=(\phi(P_1)+\phi(U_{12}+U_{12}^*))\circ(P_2+V_{12}+V_{12}^*)\circ\frac{I}{2}\circ\cdots\circ\frac{I}{2}\\&+(P_1+U_{12}+U_{12}^*)\circ(\phi(P_2)+\phi(V_{12}+V_{12}^*))\circ\frac{I}{2}\circ\cdots\circ\frac{I}{2}+\ldots+(P_1+U_{12}+U_{12}^*)\circ(P_2+V_{12}+V_{12}^*)\circ\frac{I}{2}\circ\cdots\circ\phi(\frac{I}{2})\\&=\phi(P_1\circ P_2\circ\frac{I}{2}\circ\cdots\circ\frac{I}{2})+\phi(P_1\circ(V_{12}+V_{12}^*)\circ\frac{I}{2}\circ\cdots\circ\frac{I}{2})\\&+\phi((U_{12}+U_{12}^*)\circ P_2\circ\frac{I}{2}\circ\cdots\circ\frac{I}{2})+\phi((U_{12}+U_{12}^*)\circ(V_{12}+V_{12}^*)\circ\frac{I}{2}\circ\cdots\circ\frac{I}{2})\\&=\phi(M_{12})+\phi(B_{12})+\phi(M_{12}B_{12}^*+B_{12}M_{12}^*)=\phi(M_{12})+\phi(B_{12})+\phi(M_{11})+\phi(M_{22})\end{aligned}$$

i.e., $\phi\left(M_{12}+B_{12}\right)=\phi(M_{12})+\phi(B_{12})$.

Claim 8 For each $C_{ii},D_{ii}\in\mathcal{M}_{ii},i=1,2$, we have (i) $\phi\left(C_{11}+D_{11}\right)=\phi\left(C_{11}\right)+\phi\left(D_{11}\right)$; (ii) $\phi\left(C_{22}+D_{22}\right)=\phi\left(C_{22}\right)+\phi\left(D_{22}\right)$.

Setting $T=\phi\left(C_{11}+D_{11}\right)-\phi\left(C_{11}\right)-\phi\left(D_{11}\right)$, we obtain

$$\begin{aligned}&\phi\left(P_2\right)\circ(C_{11}+D_{11})\circ\frac{I}{2}\circ\cdots\circ\frac{I}{2}+P_2\circ\phi(C_{11}+D_{11})\circ\frac{I}{2}\circ\cdots\circ\frac{I}{2}\\&=\phi(P_2\circ(C_{11}+D_{11})\circ\frac{I}{2}\circ\cdots\circ\frac{I}{2})=\phi(P_2\circ C_{11}\circ\frac{I}{2}\circ\cdots\circ\frac{I}{2})+\phi(P_2\circ D_{11}\circ\frac{I}{2}\circ\cdots\circ\frac{I}{2})\\&=\phi\left(P_2\right)\circ(C_{11}+D_{11})\circ\frac{I}{2}\circ\cdots\circ\frac{I}{2}+P_2\circ(\phi(C_{11})+\phi(D_{11}))\circ\frac{I}{2}\circ\cdots\circ\frac{I}{2}\end{aligned}$$

i.e., $P_2\circ T\circ\frac{I}{2}\circ\cdots\circ\frac{I}{2}=0$. This together with the fact $T^*=T$ shows that $P_1TP_2=P_2TP_1=P_2TP_2=0$. For all $A_{12}\in\mathcal{A}_{12}$, take $M=A_{12}+A_{12}^*$. Then $M\circ C_{11}\circ\frac{I}{2}\circ\cdots\circ\frac{I}{2}\in\mathcal{M}_{12},M\circ D_{11}\circ\frac{I}{2}\circ\cdots\circ\frac{I}{2}\in\mathcal{M}_{12}$. We get from Claim 7 that

$$\begin{aligned}&\phi\left(M\right)\circ(C_{11}+D_{11})\circ\frac{I}{2}\circ\cdots\circ\frac{I}{2}+M\circ\phi(C_{11}+D_{11})\circ\frac{I}{2}\circ\cdots\circ\frac{I}{2}\\&=\phi(M\circ(C_{11}+D_{11})\circ\frac{I}{2}\circ\cdots\circ\frac{I}{2})=\phi(M\circ C_{11}\circ\frac{I}{2}\circ\cdots\circ\frac{I}{2})+\phi(M\circ D_{11}\circ\frac{I}{2}\circ\cdots\circ\frac{I}{2})\\&=\phi\left(M\right)\circ(C_{11}+D_{11})\circ\frac{I}{2}\circ\cdots\circ\frac{I}{2}+M\circ(\phi(C_{11})+\phi(D_{11}))\circ\frac{I}{2}\circ\cdots\circ\frac{I}{2}\end{aligned}$$

i.e., $M\circ T\circ \frac{I}{2}\circ\cdots\circ\frac{I}{2}=0$. Then $P_1TA_{12}+A_{12}^*TP_1=0$ for all $A_{12}\in\mathcal{A}_{12}$. Hence $P_1TP_1AP_2=0$ for all $A\in\mathcal{A}$. From (b), we obtain $P_1TP_1=0$ and then $T=0$.

In the second case, we can similarly prove that the conclusion is valid.

Claim 9 ϕ is additive on $\mathcal{M}$.

By Claims 6-8, ϕ is additive on $\mathcal{M}$.

Claim 10 ϕ is additive on $\mathcal{A}$ and $\phi(A^*)=\phi(A)^*$ for all $A\in\mathcal{A}$.

For all $H,K\in\mathcal{M}$, from Claim 2 we obtain

$$\phi(H)=\phi((H+\mathrm{i}K)\circ\frac{I}{2}\circ\cdots\circ\frac{I}{2})=\phi(H+\mathrm{i}K)\circ\frac{I}{2}\circ\cdots\circ\frac{I}{2}=\frac{1}{2}(\phi(H+\mathrm{i}K)+\phi(H+\mathrm{i}K)^*) \tag{5}$$

On the other hand, from Claim 2, we have

$$\phi(K)=\phi((H+\mathrm{i}K)\circ\frac{\mathrm{i}}{2}I\circ\frac{I}{2}\circ\cdots\circ\frac{I}{2})=\phi(H+\mathrm{i}K)\circ\frac{\mathrm{i}}{2}I\circ\frac{I}{2}\circ\cdots\circ\frac{I}{2}=-\frac{\mathrm{i}}{2}(\phi(H+\mathrm{i}K)-\phi(H+\mathrm{i}K)^*)$$

i.e.,

$$\mathrm{i}\phi(K)=\frac{1}{2}(\phi(H+\mathrm{i}K)-\phi(H+\mathrm{i}K)^*) \tag{6}$$

By adding (5) and (6), from Claim 3, we obtain

$$\phi(H+\mathrm{i}K)=\phi(H)+\mathrm{i}\phi(K) \tag{7}$$

For all $A\in\mathcal{A}$, we have $A=A_1+\mathrm{i}A_2$ with $A_1,A_2\in\mathcal{M}$. From (7), Claim 4 and Claim 9, we obtain

$$\phi(A)^*=\phi(A_1+\mathrm{i}A_2)^*=(\phi(A_1)+\mathrm{i}\phi(A_2))^*=\phi(A_1)-\mathrm{i}\phi(A_2)=\phi(A_1)+\mathrm{i}\phi(-A_2)=\phi(A_1-\mathrm{i}A_2)=\phi(A^*)$$

For all $A,B\in\mathcal{A}$, we have $A=A_1+\mathrm{i}A_2, B=B_1+\mathrm{i}B_2$ with $A_1,A_2,B_1,B_2\in\mathcal{M}$. From (7) and Claim 9, we obtain

$$\begin{aligned}\phi(A+B)&=\phi((A_1+B_1)+\mathrm{i}(A_2+B_2))=\phi(A_1+B_1)+\mathrm{i}\phi(A_2+B_2)\\&=\phi(A_1)+\phi(B_1)+\mathrm{i}(\phi(A_2)+\phi(B_2))=(\phi(A_1)+\mathrm{i}\phi(A_2))+(\phi(B_1)+\mathrm{i}\phi(B_2))=\phi(A)+\phi(B)\end{aligned}$$

Claim 11 ϕ is an additive *-derivation on $\mathcal{A}$.

For all H, $K\in\mathcal{M}$, from Claim 2 and Claim 4, we have

$$\begin{aligned}&\phi(HK+KH)=\phi(H\circ K\circ\frac{I}{2}\circ\cdots\circ\frac{I}{2})\\&=\phi(H)\circ K\circ\frac{I}{2}\circ\cdots\circ\frac{I}{2}+H\circ\phi(K)\circ\frac{I}{2}\circ\cdots\circ\frac{I}{2}=\phi(H)K+K\phi(H)+H\phi(K)+\phi(K)H\end{aligned}$$

On the other hand, from (7), Claim 2 and Claim 4, we obtain

$$\begin{aligned}&\mathrm{i}\phi(HK-KH)=\phi(\mathrm{i}(HK-KH))=\phi(H\circ\mathrm{i}K\circ\frac{I}{2}\circ\cdots\circ\frac{I}{2})\\&=\phi(H)\circ\mathrm{i}K\circ\frac{I}{2}\circ\cdots\circ\frac{I}{2}+H\circ\phi(\mathrm{i}K)\circ\frac{I}{2}\circ\cdots\circ\frac{I}{2}=\mathrm{i}(\phi(H)K-K\phi(H)+H\phi(K)-\phi(K)H)\end{aligned}$$

So we have $\phi(HK)=\phi(H)K+H\phi(K)$.

For all $A,B\in\mathcal{A}$, we have $A=A_1+\mathrm{i}A_2, B=B_1+\mathrm{i}B_2$, where $A_1,A_2,B_1,B_2\in\mathcal{M}$. From Claim 10 and (7), we obtain

$$\begin{aligned}\phi(AB)&=\phi(A_1B_1+\mathrm{i}A_1B_2+\mathrm{i}A_2B_1-A_2B_2)=\phi(A_1B_1)+\phi(\mathrm{i}A_1B_2)+\phi(\mathrm{i}A_2B_1)-\phi(A_2B_2)\\&=\phi(A_1)B_1+A_1\phi(B_1)+\mathrm{i}\phi(A_1)B_2+\mathrm{i}A_1\phi(B_2)+\mathrm{i}\phi(A_2)B_1+\mathrm{i}A_2\phi(B_1)-\phi(A_2)B_2-A_2\phi(B_2)\\&=\phi(A_1)B_1+A_1\phi(B_1)+\phi(A_1)\mathrm{i}B_2+A_1\phi(\mathrm{i}B_2)+\phi(\mathrm{i}A_2)B_1+\mathrm{i}A_2\phi(B_1)+\phi(\mathrm{i}A_2)\mathrm{i}B_2+\mathrm{i}A_2\phi(\mathrm{i}B_2)\\&=(\phi(A_1)+\phi(\mathrm{i}A_2))(B_1+\mathrm{i}B_2)+(A_1+\mathrm{i}A_2)(\phi(B_1)+\phi(\mathrm{i}B_2))=\phi(A)B+A\phi(B)\end{aligned}$$

From this and Claim 10, we have proved that ϕ is an additive *-derivation. This completes the proof.

2 Corollaries

Now we give some applications of Theorem 1 to operator algebras. We say that $\mathcal{A}$ is prime when for $A,B\in\mathcal{A}$, if $A\mathcal{A}B=\{0\}$, then $A=0$ or $B=0$. It is easy to show that prime *-algebras satisfy (a) and (b), and the following corollary is immediate.

Corollary 1 Let $\mathcal{A}$ be a prime *-algebra with unit I and a nontrivial projection. Then ϕ is a nonlinear A^*B+B^*A type derivation on $\mathcal{A}$ if and only if ϕ is an additive *-derivation.

Recall that a von Neumann algebra $\mathcal{A}$ is weakly closed, self-adjoint algebra of operators on a Hilbert space $\mathcal{H}$ containing the identity operator I. By Ref.[3], if a von Neumann algebra has no central summands of type I_1, then $\mathcal{A}$ satisfies (a) and (b). So the following corollary is obvious.

Corollary 2 Let $\mathcal{A}$ be a von Neumann algebra with no central summands of type I_1. Then ϕ is a nonlinear A^*B+B^*A type derivation on $\mathcal{A}$ if and only if ϕ is an additive *-derivation.

$\mathcal{A}$ is a factor von Neumann algebra means that its center only contains the scalar operators. Clearly, $\mathcal{A}$ is prime. So the following corollary is obvious from Corollary 1.

Corollary 3 Let $\mathcal{A}$ be a factor von Neumann algebra. Then ϕ is a nonlinear A^*B+B^*A type derivation on $\mathcal{A}$ if and only if ϕ is an additive *-derivation.

$\mathcal{B}(\mathcal{H})$ is the algebra of all bounded linear operators on a complex Hilbert space $\mathcal{H}$. $\mathcal{F}(\mathcal{H})\subseteq\mathcal{B}(\mathcal{H})$ is all bounded finite rank operators. $\mathcal{A}\subseteq\mathcal{B}(\mathcal{H})$ is said to a standard operator algebra if it contains $\mathcal{F}(\mathcal{H})$. When $\mathcal{A}$ is a standard operator algebra, a more concrete form is achieved.

Corollary 4 Let $\mathcal{H}$ be an infinite dimensional complex Hilbert space and $\mathcal{A}$ be a standard operator algebra on $\mathcal{H}$ containing the identity operator I. Assume that $\mathcal{A}$ is closed under the adjoint operation. Let $\phi:\mathcal{A}\to\mathcal{A}$ be a nonlinear A^*B+B^*A type derivation. Then there exists $T\in\mathcal{B}(\mathcal{H})$ satisfying $T+T^*=0$ such that $\phi(A)=AT-TA$ for all $A\in\mathcal{A}$.

Proof Since $\mathcal{A}$ is prime, from Corollary 1, we obtain ϕ is an additive *-derivation. According to the result of Ref.[4], ϕ is linear, and then it is inner. Thus there exists $S\in\mathcal{B}(\mathcal{H})$ such that $\phi(A)=AS-SA$. Hence $A^*S-SA^*=\phi(A^*)=\phi(A)^*=S^*A^*-A^*S^*$ for all $A\in\mathcal{A}$. This indicates that $S+S^*=\lambda I$ for certain $\lambda\in\mathbb{R}$. Take $T=S-\frac{1}{2}\lambda I$, then $T+T^*=0$ and $\phi(A)=AT-TA$ for all $A\in\mathcal{A}$.

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