Wuhan University
Journal of Natural Sciences

2023, Vol.28 No.5, 392-398

Article ID 1007-1202(2023)05-0392-07

DOI https://doi.org/10.1051/wujns/2023285392

Topological Uniform Descent and Judgement of A-Weyl's Theorem

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Abstract: In this paper, a-Browder's theorem and a-Weyl's theorem for bounded linear operators are studied by means of the property of the topological uniform descent. The sufficient and necessary conditions for a bounded linear operator defined on a Hilbert space holding a-Browder's theorem and a-Weyl's theorem are established. As a consequence of the main result, the new judgements of a-Browder's theorem and a-Weyl's theorem for operator function are discussed.

Key words: a-Browder's theorem; a-Weyl's theorem; topological uniform descent

CLC number: O177

0 Introduction

The research of Weyl type theorem is an important subject in spectral theory. In 1909, Weyl discovered Weyl's theorem when he studied the spectrum of the self-adjoint operator[1]. Then Harte and Lee defined Browder's theorem[2]. Rakočević gave two other variations of Weyl's theorem: a-Weyl's theorem and a-Browder's theorem[3,4]. These generalizations are called Weyl type theorems by scholars. The study on Weyl type theorems can well reflect the structural characteristics of spectrums. Hence the research of Weyl type theorem has attracted much attention and got many good results in resent years[5-7]. In this paper, we mainly study a-Browder's theorem and a-Weyl's theorem for bounded linear operators and operator functions by means of the property of the topological uniform descent.

In this paper, H denotes a complex separable infinite dimensional Hilbert space. Let $B(H)$ be the algebra of all bounded linear operators on H. For an operator $T\in B(H)$ we shall denote by $n(T)$ the dimension of the kernel $N(T)$, and by $d(T)$ the codimension of the range $R(T)$. We call $T\in B(H)$ is an upper semi-Fredholm operator if $R(T)$ is closed and $n(T)<\infty$. We say that T is a lower semi-Fredholm operator when $d(T)<\infty$. An operator $T\in B(H)$ is said to be Fredholm if $R(T)$ is closed and both $n(T)$ and $d(T)$ are finite. If $T\in B(H)$ is an upper (or a lower) semi-Fredholm operator, the index of T, ind(T), is defined to be $\mathrm{ind}(T)=n(T)-d(T)$. The ascent of T, asc(T), is the least non-negative integer n such that $N(T^n)=N(T^{n+1})$ and the descent, des(T), is the least non-negative integer n such that $R(T^n)=R(T^{n+1})$. The operator T is Weyl if it is Fredholm of index zero, and T is said to be Browder if it is Fredholm "of finite ascent and descent". We

Received date: 2023-03-26

Foundation item: Supported by the 2021 General Special Scientific Research Project of Education Department of Shaanxi Provincial Government (21JK0637), Science and Technology Planning Project of Weinan Science and Technology Bureau (2022ZDYFJH-11), and 2021 Talent Project of Weinan Normal University (2021RC16)

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call T a Drazin invertible operator if $\mathrm{asc}(T)=\mathrm{des}(T)<\infty$. Let $\sigma(T)$ be the spectrum of T. The approximate point spectrum of T is denoted by $\sigma_a(T)$. The Weyl spectrum $\sigma_w(T)$, the upper semi-Fredholm spectrum $\sigma_{SF_+}(T)$, the Browder spectrum $\sigma_b(T)$, the Drazin spectrum $\sigma_D(T)$, are defined by $\sigma_w(T)=\{\lambda\in\mathbb{C}:T-\lambda I$ is not Weyl$\}$($\mathbb{C}$ denotes the set of complex numbers), $\sigma_{SF_+}(T)=\{\lambda\in\mathbb{C}:T-\lambda I$ is not upper semi-Fredholm$\}$, $\sigma_b(T)=\{\lambda\in\mathbb{C}:T-\lambda I$ is not Browder$\}$, $\sigma_D(T)=\{\lambda\in\mathbb{C}:T-\lambda I$ is not Drazin invertible$\}$. Let $\rho(T)=\mathbb{C}\backslash\sigma(T)$, $\rho_a(T)=\mathbb{C}\backslash\sigma_a(T)$, $\rho_w(T)=\mathbb{C}\backslash\sigma_w(T)$, $\rho_{SF_+}(T)=\mathbb{C}\backslash\sigma_{SF_+}(T)$, $\rho_b(T)=\mathbb{C}\backslash\sigma_b(T)$, $\rho_D(T)=\mathbb{C}\backslash\sigma_D(T)$. For a set $E\subseteq\mathbb{C}$, we write iso$E$ (acc$E$) for the set of isolated (accumulation) points of E, and we denote int$E$ (∂E) for interior (boundary) points set of E. If $\lambda_0=\sigma(T)\cap\rho_D(T)$, then λ_0 is a pole of T. $T\in B(H)$ is called an a-isoloid operator if $\mathrm{iso}\sigma_a(T)\subseteq\sigma_p(T)$, where $\sigma_p(T)=\{\lambda\in\mathbb{C}:n(T-\lambda I)>0\}$.

$T\in B(H)$ satisfies a-Browder's theorem if

$$\sigma_{ab}(T)=\sigma_{ea}(T)$$

where $\sigma_{ab}(T)=\{\lambda\in\mathbb{C}:T-\lambda I$ is not upper semi-Fredholm or $\mathrm{asc}(T-\lambda I)=\infty\}$ and $\sigma_{ea}(T)=\{\lambda\in\mathbb{C}:T-\lambda I$ is not upper semi-Fredholm or $\mathrm{ind}(T-\lambda I)>0\}$. Let $\rho_{ea}(T)=\mathbb{C}\backslash\sigma_{ea}(T)$ and $\rho_{ab}(T)=\mathbb{C}\backslash\sigma_{ab}(T)$. The a-Weyl's theorem holds for T if and only if

$$\sigma_a(T)\backslash\sigma_{ea}(T)=\pi_{00}^a(T)$$

where we write $\pi_{00}^a(T)=\{\lambda\in\mathrm{iso}\sigma_a(T):0<n(T-\lambda I)<\infty\}$. It can be shown that a-Weyl's theorem $\Rightarrow$ a-Browder's theorem, but the converse is not true. Let $T\in B(\ell^2)$ be defined by $T(x_1,x_2,x_3,\cdots)=(0,0,\frac{x_2}{2},\frac{x_3}{3},\cdots)$. Then $\sigma(T)=\sigma_a(T)=\sigma_{ea}(T)=\sigma_{ab}(T)=\{0\}$ and $\pi_{00}^a(T)=\{0\}$. So T satisfies a-Browder's theorem, but a-Weyl's theorem does not hold for T.

If $T\in B(H)$ satisfies $N(T)\subseteq\bigcap_{n=1}^{\infty}R(T^n)$, then T is called a Sapher operator[8,9]. The Sapher spectrum is $\sigma_s(T)=\{\lambda\in\mathbb{C}:T-\lambda I$ is not Sapher operator$\}$. Goldberg defined $\sigma_c(T)=\{\lambda\in\mathbb{C}:R(T-\lambda I)$ is not closed$\}$[10]. T is called a Kato operator if $R(T)$ is closed and $N(T)\subseteq\bigcap_{n=1}^{\infty}R(T^n)$. Therefore, the Kato spectrum is $\sigma_k(T)=\sigma_c(T)\cup\sigma_s(T)$.

Let $T\in B(H)$, for each nonnegative integer n, T induces a linear transformation from the vector space $R(T^n)/R(T^{n+1})$ to $R(T^{n+1})/R(T^{n+2})$. We denote by $k_n(T)$ the dimension of the null space of the induced map and put $k(T)=\sum_{n=0}^{\infty}k_n(T)$. If there is a nonnegative integer d for which $k_n(T)=0$ for $n\geqslant d$ and $R(T^n)$ is closed in the operator range topology of $R(T^d)$ for $n\geqslant d$, then we say that T has topological uniform descent[11]. If T is upper semi-Fredholm, then T has topological uniform descent. Let $\rho_\tau(T)=\{\lambda\in\mathbb{C}:T-\lambda I$ has topological uniform descent$\}$, and $\sigma_\tau(T)=\mathbb{C}\backslash\rho_\tau(T)$. We will use the following property which is discovered by Grabiner (Ref.[11], Corollary 4.9):

Lemma 1 Let $T\in B(H)$, $\lambda\in\partial\sigma(T)$. If $T-\lambda I$ has topological uniform descent, then $\lambda\in\rho_D(T)$.

On the basis of analyzing distribution of various spectrums of bounded linear operators, the sufficient and necessary conditions holding a-Browder's theorem and a-Weyl's theorem are established by means of the property of the topological uniform descent. In addition, the new judgements of a-Browder's theorem and a-Weyl's theorem for operator function are discussed.

1 Judgement of A-Browder's Theorem and A-Weyl's Theorem for Bounded Linear Operator

First, we describe a-Browder's theorem by the relation between topological uniform descent and $\sigma_b(T)$.

Theorem 1 $T\in B(H)$ satisfies a-Browder's theorem if and only if $\sigma_b(T)=\sigma_\tau(T)\cup\mathrm{int}\sigma_{ea}(T)\cup\mathrm{acc}\{\lambda\in\rho_{ab}(T):n(T-\lambda I)\neq d(T-\lambda I)\}\cup\{\lambda\in\mathbb{C}:n(T-\lambda I)=\infty\}$.

Proof "$\Rightarrow$". Suppose

$$\lambda_0\notin\sigma_\tau(T)\cup\mathrm{int}\sigma_{ea}(T)\cup\mathrm{acc}\{\lambda\in\rho_{ab}(T):n(T-\lambda I)\neq d(T-\lambda I)\}\cup\{\lambda\in\mathbb{C}:n(T-\lambda I)=\infty\}.$$

Then there exists a deleted neighborhood $B^\circ(\lambda_0;\varepsilon)$ centered on λ_0 such that for any $\mu\in B^\circ(\lambda_0;\varepsilon)$, $\mu\notin\mathrm{acc}\{\lambda\in\rho_{ab}(T):n(T-\lambda I)\neq d(T-\lambda I)\}$. Moreover, for any deleted neighborhood $B^\circ(\lambda_0)$, there exists $\mu_0\in B^\circ(\lambda_0)$ such that $\mu_0\in\rho_{ea}(T)$. Let $B^\circ(\lambda_0)\subseteq B^\circ(\lambda_0;\varepsilon)$, then we will get that $T-\mu_0 I$ is Browder operator since T satisfies a-Browder's theorem and $\lambda_0\notin\mathrm{acc}\{\lambda\in\rho_{ab}(T):n(T-\lambda I)\neq d(T-\lambda I)\}$. It follows that $\lambda_0\in\rho(T)\cup\partial\sigma(T)$. Since $\lambda_0\in\rho_\tau(T)$, $n(T-\lambda_0 I)<\infty$, we know that

$\lambda_0 \notin \sigma_b(T)$ according to Lemma 1.

"$\Leftarrow$". It's clear that

$$\rho_{ea}(T)\cap[\sigma_\tau(T)\cup \mathrm{int}\sigma_{ea}(T)\cup\{\lambda\in\mathbb{C}:n(T-\lambda I)=\infty\}]=\varnothing$$

Suppose $\lambda_0\in\rho_{ea}(T)\cap\mathrm{acc}\{\lambda\in\rho_{ab}(T):n(T-\lambda I)\neq d(T-\lambda I)\}$. According to perturbation theorem of semi-Fredholm operator, there exists $\varepsilon>0$ such that $\mu\in\rho_a(T)$ if $0<|\mu-\lambda_0|<\varepsilon$. Then $\lambda_0\in\mathrm{iso}\sigma_a(T)\cap\rho_a(T)$. It follows that $\lambda_0\in\rho_{ab}(T)$. If $\lambda_0\in\rho_{ea}(T)$ and $\lambda_0\notin\mathrm{acc}\{\lambda\in\rho_{ab}(T):n(T-\lambda I)\neq d(T-\lambda I)\}$, then $T-\lambda_0 I$ is Browder operator. Therefore, a-Browder's theorem holds for T.

Remark 1 (i) In Theorem 1, suppose $T\in B(H)$ satisfies a-Browder's theorem, then each part of the decomposition of $\sigma_b(T)$ cannot be deleted.

(a) Let $T\in B(\ell^2)$ be defined by $T(x_1,x_2,x_3,\cdots)=(0,x_1,\frac{x_2}{2},\frac{x_3}{3},\cdots)$. Then $\sigma_{ea}(T)=\sigma_{ab}(T)=\sigma_b(T)=\{0\}$, T satisfies a-Browder's theorem.

But $\mathrm{int}\sigma_{ea}(T)\cup\mathrm{acc}\{\lambda\in\rho_{ab}(T):n(T-\lambda I)\neq d(T-\lambda I)\}\cup\{\lambda\in\mathbb{C}:n(T-\lambda I)=\infty\}=\varnothing$. Thus $\sigma_\tau(T)$ cannot be deleted.

(b) Let $T\in B(\ell^2)$ be defined by $T(x_1,x_2,x_3,\cdots)=(x_2,x_3,x_4,\cdots)$. We can get that $\sigma_{ea}(T)=\sigma_{ab}(T)=\sigma_b(T)=\{\lambda\in\mathbb{C}:|\lambda|\leqslant 1\}$, a-Browder's theorem holds for T. However, $\sigma_b(T)\neq\sigma_\tau(T)\cup\mathrm{acc}\{\lambda\in\rho_{ab}(T):n(T-\lambda I)\neq d(T-\lambda I)\}\cup\{\lambda\in\mathbb{C}:n(T-\lambda I)=\infty\}$, which means $\mathrm{int}\sigma_{ea}(T)$ cannot be deleted.

(c) Let $T\in B(\ell^2)$ be defined by $T(x_1,x_2,x_3,\cdots)=(0,x_1,x_2,\cdots)$. Then $\sigma_{ea}(T)=\sigma_{ab}(T)=\{\lambda\in\mathbb{C}:|\lambda|=1\}$. T satisfies a-Browder's theorem. But $\sigma_b(T)=\{\lambda\in\mathbb{C}:|\lambda|\leqslant 1\}\neq\sigma_\tau(T)\cup\mathrm{int}\,\sigma_{ea}(T)\cup\{\lambda\in\mathbb{C}:n(T-\lambda I)=\infty\}$. Therefore $\mathrm{acc}\{\lambda\in\rho_{ab}(T):n(T-\lambda I)\neq d(T-\lambda I)\}$ cannot be deleted.

(d) Let $T\in B(\ell^2)$ be defined by $T(x_1,x_2,x_3,\cdots)=(0,x_2,x_3,\cdots)$. We have $\sigma_{ea}(T)=\sigma_{ab}(T)=\sigma_b(T)=\{1\}$, which implies a-Browder's theorem holds for T. Since $\sigma_\tau(T)\cup\mathrm{int}\,\sigma_{ea}(T)\cup\mathrm{acc}\{\lambda\in\rho_{ab}(T):n(T-\lambda I)\neq d(T-\lambda I)\}=\varnothing$, $\{\lambda\in\mathbb{C}:n(T-\lambda I)=\infty\}$ cannot be deleted.

(ii) Since $\sigma(T)=\sigma_b(T)\cup\sigma_0(T)$, T satisfies a-Browder's theorem if and only if $\sigma(T)=\sigma_\tau(T)\cup\mathrm{int}\sigma_{ea}(T)\cup\mathrm{acc}\{\lambda\in\rho_{ab}(T):n(T-\lambda I)\neq d(T-\lambda I)\}\cup\{\lambda\in\mathbb{C}:n(T-\lambda I)=\infty\}\cup\sigma_0(T)$.

(iii) By Theorem 1, if $\sigma_b(T)=\sigma_\tau(T)$, then T satisfies a-Browder's theorem. But the converse is not true. Let $T\in B(\ell^2)$ be defined by $T(x_1,x_2,x_3,\cdots)=(0,x_1,x_2,\cdots)$. We can get that a-Browder's theorem holds for T, but $\sigma_\tau(T)=\{\lambda\in\mathbb{C}:|\lambda|=1\}\neq\sigma_b(T)$.

(iv) $\sigma_b(T)=\sigma_\tau(T)\Leftrightarrow T$ satisfies a-Browder's theorem and $\rho_\tau(T)\subseteq\rho_w(T)\cup\{\lambda\in\mathrm{iso}\sigma_w(T):n(T-\lambda I)<\infty\}$.

In fact, $\sigma_b(T)=\sigma_\tau(T)$ yields $\rho_\tau(T)=\rho_b(T)\subseteq\rho_w(T)\cup\{\lambda\in\mathrm{iso}\sigma_w(T):n(T-\lambda I)<\infty\}$.

For the converse, since $[\rho_w(T)\cup\{\lambda\in\mathrm{iso}\sigma_w(T):n(T-\lambda I)<\infty\}]\cap[\mathrm{int}\,\sigma_{ea}(T)\cup\mathrm{acc}\{\lambda\in\rho_{ab}(T):n(T-\lambda I)\neq d(T-\lambda I)\}\cup\{\lambda\in\mathbb{C}:n(T-\lambda I)=\infty\}]=\varnothing$, by Theorem 1 we get $\rho_\tau(T)\subseteq\rho_b(T)$.

(v) $\sigma_D(T)=\sigma_\tau(T)\Leftrightarrow T$ satisfies a-Browder's theorem and $\rho_\tau(T)=\rho_w(T)\cup E$, where E is denumerable.

"$\Rightarrow$". It is clear that $\sigma_b(T)=\sigma_D(T)\cup\{\lambda\in\mathbb{C}:n(T-\lambda I)=\infty\}$. Then we can get that T satisfies a-Browder's theorem by Theorem 1. Since $\rho_w(T)\subseteq\rho_\tau(T)=\rho_D(T)\subseteq\rho_w(T)\cup E$, we have that $\rho_\tau(T)=\rho_w(T)\cup E$. Since $E\subseteq\mathrm{iso}\sigma(T)$, it follows that E is denumerable.

"$\Leftarrow$". Suppose that $\lambda_0\in\rho_\tau(T)$. If $\lambda_0\in\rho_w(T)$, then $\lambda_0\notin\sigma_D(T)$ by a-Browder's theorem holds for T. If $\lambda_0\in E\cap\sigma_w(T)$, then there exists a neighborhood $B(\lambda_0;\varepsilon)$ centered on λ_0 such that $B(\lambda_0;\varepsilon)\subseteq\rho_\tau(T)=\rho_w(T)\cup E$. Since E is denumerable, for any $B(\lambda_0;\delta)\subseteq B(\lambda_0;\varepsilon)$, there exists $\mu_0\in B(\lambda_0;\delta)$ such that $T-\mu_0 I$ is Weyl operator. It follows that $\lambda_0\in\partial\sigma(T)$ since T satisfies a-Browder's theorem. We can also get $\lambda_0\notin\sigma_D(T)$.

By Theorem 1, the following results can be obtained.

Corollary 1 Let $T\in B(H)$. The following statements are equivalent:

(1) T satisfies a-Browder's theorem;

(2) $\sigma_b(T)=\sigma_\tau(T)\cup\mathrm{acc}\sigma_{ea}(T)\cup\mathrm{acc}\{\lambda\in\rho_{ab}(T):n(T-\lambda I)\neq d(T-\lambda I)\}\cup\{\lambda\in\mathbb{C}:n(T-\lambda I)=\infty\}$;

(3) $\sigma_b(T)=\sigma_\tau(T)\cup\mathrm{acc}\sigma_{ea}(T)\cup\mathrm{acc}\sigma_k(T)\cup\mathrm{acc}[\rho_a(T)\cap\sigma(T)]\cup\{\lambda\in\mathbb{C}:n(T-\lambda I)=\infty\}$;

(4) $\sigma_b(T)=\sigma_\tau(T)\cup\mathrm{int}\sigma_{ea}(T)\cup\mathrm{acc}\sigma_k(T)\cup\mathrm{acc}[\rho_a(T)\cap\sigma(T)]\cup\{\lambda\in\mathbb{C}:n(T-\lambda I)=\infty\}$;

(5) $\sigma_b(T)=\sigma_\tau(T)\cup\mathrm{acc}\sigma_{ea}(T)\cup\mathrm{int}\sigma_k(T)\cup\mathrm{acc}[\rho_a(T)\cap\sigma(T)]\cup\{\lambda\in\mathbb{C}:n(T-\lambda I)=\infty\}$.

Proof (1)$\Rightarrow$(2). Using Theorem 1, we have $\sigma_b(T)=\sigma_\tau(T)\cup\mathrm{int}\sigma_{ea}(T)\cup\mathrm{acc}\{\lambda\in\rho_{ab}(T):n(T-\lambda I\neq d(T-\lambda I)\}\cup\{\lambda\in$

$\mathbb{C}:n(T-\lambda I)=\infty\}$when T satisfies a-Browder's theorem. Since $\mathrm{int}\sigma_{ea}(T)\subseteq\mathrm{acc}\sigma_{ea}(T)$, it follows that (2) holds.

(2)⇒(3). Let $\lambda_0\in\{\lambda\in\rho_{ab}(T):n(T-\lambda I)\neq d(T-\lambda I)\}$ and $T-\lambda_0 I$ is Kato operator. Then $\lambda_0\in\rho_a(T)\cup\sigma(T)$ (Ref. [12], Lemma 3.4). Therefore $\{\lambda\in\rho_{ab}(T):n(T-\lambda I)\neq d(T-\lambda I)\}\subseteq\sigma_k(T)\cup[\rho_a(T)\cap\sigma(T)]$. By (2) we know that (3) holds.

(3)⇒(4). Suppose $\lambda_0\notin\sigma_\tau(T)\cup\mathrm{int}\sigma_{ea}(T)\cup\mathrm{acc}\sigma_k(T)\cup\mathrm{acc}[\rho_a(T)\cap\sigma(T)]\cup\{\lambda\in\mathbb{C}:n(T-\lambda I)=\infty\}$. Then there exists a deleted neighborhood $B^\circ(\lambda_0;\varepsilon)$ centered on λ_0 such that for any $\mu\in B^\circ(\lambda_0;\varepsilon)$, $\mu\notin\rho_a(T)\cap\sigma(T)$. Since $\lambda_0\notin\mathrm{int}\sigma_{ea}(T)$, for any $B^\circ(\lambda_0;\delta)$, there exists $\mu_0\in B^\circ(\lambda_0;\delta)$ such that $\mu_0\in\rho_{ea}(T)$. Let $\delta<\varepsilon$, it follows that

$$\mu_0\notin\sigma_\tau(T)\cup\mathrm{acc}\sigma_{ea}(T)\cup\mathrm{acc}\sigma_k(T)\cup\mathrm{acc}[\rho_a(T)\cap\sigma(T)]\cup\{\lambda\in\mathbb{C}:n(T-\lambda I)=\infty\}.$$

Thus we can get $T-\mu_0 I$ is Browder operator by (3). From the proof of Theorem 1, we have $\lambda_0\notin\sigma_b(T)$.

(4)⇒(5). Let $\lambda_0\notin\sigma_\tau(T)\cup\mathrm{acc}\sigma_{ea}(T)\cup\mathrm{int}\sigma_k(T)\cup\mathrm{acc}[\rho_a(T)\cap\sigma(T)]\cup\{\lambda\in\mathbb{C}:n(T-\lambda I)=\infty\}$. Then there exists a deleted neighborhood $B^\circ(\lambda_0;\varepsilon)$ centered on λ_0 such that for any $\mu\in B^\circ(\lambda_0;\varepsilon)$, $\mu\notin\rho_a(T)\cap\sigma(T)$ and $\mu\in\rho_{ea}(T)$. Since $\lambda_0\notin\mathrm{int}\sigma_k(T)$, we know that for any $B^\circ(\lambda_0)\subseteq B^\circ(\lambda_0;\varepsilon)$, there exists

$$\mu_0\notin\sigma_\tau(T)\cup\mathrm{int}\sigma_{ea}(T)\cup\mathrm{acc}\sigma_k(T)\cup\mathrm{acc}[\rho_a(T)\cap\sigma(T)]\cup\{\lambda\in\mathbb{C}:n(T-\lambda I)=\infty\}.$$

It follows that $T-\mu_0 I$ is Browder operator. This implies that (5) holds.

(5)⇒(1). It is clear that

$$\rho_{ea}(T)\cap[\sigma_\tau(T)\cup\mathrm{acc}\sigma_{ea}(T)\cup\mathrm{int}\,\sigma_k(T)\cup\{\lambda\in\mathbb{C}:n(T-\lambda I)=\infty\}]=\varnothing.$$

If $\lambda_0\in\rho_{ea}(T)\cap\mathrm{acc}[\rho_a(T)\cap\sigma(T)]$, then $\lambda_0\in\mathrm{iso}\sigma_a(T)\cup\rho_a(T)$, therefore $\lambda_0\in\rho_{ab}(T)$. If $\lambda_0\in\rho_{ea}(T)$ and $\lambda_0\notin\mathrm{acc}\{\lambda\in\rho_{ab}(T):n(T-\lambda I)\neq d(T-\lambda I)\}$, then $T-\lambda_0 I$ is Browder operator. We can conclude that T satisfies a-Browder's theorem.

Corollary 2 Let $T\in B(H)$. The following statements are equivalent:

(1) T satisfies a-Browder's theorem;

(2) $\sigma_{ab}(T)=\sigma_\tau(T)\cup\mathrm{acc}\sigma_{ea}(T)\cup[\mathrm{acc}\{\lambda\in\mathbb{C}:n(T-\lambda I)<d(T-\lambda I)\}\cap\mathrm{acc}\sigma_k(T)]\cup\{\lambda\in\mathbb{C}:n(T-\lambda I)=\infty\}\cup[\mathrm{iso}\sigma_a(T)\cap\sigma_c(T)]$;

(3) $\sigma_b(T)=\sigma_\tau(T)\cup\mathrm{acc}\sigma_{ea}(T)\cup[\mathrm{acc}\{\lambda\in\mathbb{C}:n(T-\lambda I)<d(T-\lambda I)\}\cap\mathrm{acc}\sigma_k(T)]\cup\{\lambda\in\mathbb{C}:n(T-\lambda I)=\infty\}\cup[(\rho_a(T)\cup\mathrm{iso}\sigma_a(T))\cap\mathrm{acc}\sigma(T)]$.

Proof (1)⇒(2). We only need to prove

$$\sigma_{ab}(T)\subseteq\sigma_\tau(T)\cup\mathrm{acc}\sigma_{ea}(T)\cup[\mathrm{acc}\{\lambda\in\mathbb{C}:n(T-\lambda I)<d(T-\lambda I)\}\cap\mathrm{acc}\sigma_k(T)]\cup\{\lambda\in\mathbb{C}:n(T-\lambda I)=\infty\}\cup[\mathrm{iso}\sigma_a(T)\cap\sigma_c(T)].$$

Suppose $\lambda_0\notin\sigma_\tau(T)\cup\mathrm{acc}\sigma_{ea}(T)\cup[\mathrm{acc}\{\lambda\in\mathbb{C}:n(T-\lambda I)<d(T-\lambda I)\}\cap\mathrm{acc}\sigma_k(T)]\cup\{\lambda\in\mathbb{C}:n(T-\lambda I)=\infty\}\cup[\mathrm{iso}\sigma_a(T)\cap\sigma_c(T)]$. Then there exists a deleted neighborhood $B^\circ(\lambda_0;\varepsilon_1)$ centered on λ_0 such that for any $\mu\in B^\circ(\lambda_0;\varepsilon_1)$, $\mu\in\rho_{ea}(T)$. If $\lambda_0\notin\mathrm{acc}\{\lambda\in\mathbb{C}:n(T-\lambda I)<d(T-\lambda I)\}$, there exists $B^\circ(\lambda_0;\varepsilon_2)$ such that for any $\mu\in B^\circ(\lambda_0;\varepsilon_2)$, $n(T-\mu I)>d(T-\mu I)$. Let $\varepsilon=\min\{\varepsilon_1,\varepsilon_2\}$. It follows that for any $\mu\in B^\circ(\lambda_0;\varepsilon)$, $T-\mu I$ is Weyl operator. From the a-Browder's theorem holds for T and the proof of Theorem 1, we can get that $T-\lambda_0 I$ is Browder operator. If $\lambda_0\notin\mathrm{acc}\sigma_k(T)$, then there exists a deleted neighborhood $B^\circ(\lambda_0;\delta)$ centered on λ_0 such that for any $\mu\in B^\circ(\lambda_0;\delta)$, $\mu\in\rho_a(T)$ since T satisfies a-Browder's theorem. Therefore $\lambda_0\in\rho_a(T)\cup\mathrm{iso}\sigma_a(T)$. Moreover, by $n(T-\lambda_0 I)<0$ and $\lambda_0\in\rho_c(T)=\mathbb{C}\backslash\sigma_c(T)$, we know that $\lambda_0\notin\sigma_{ab}(T)$.

(2)⇒(3). It is clear that $\sigma_b(T)=\sigma_{ab}(T)\cup[\sigma_b(T)\cap\rho_{ab}(T)]=\sigma_\tau(T)\cup\mathrm{acc}\sigma_{ea}(T)\cup[\mathrm{acc}\{\lambda\in\mathbb{C}:n(T-\lambda I)<d(T-\lambda I)\cap\mathrm{acc}\sigma_k(T)]\cup\{\lambda\in\mathbb{C}:n(T-\lambda I)=\infty\}\cup[\mathrm{iso}\sigma_a(T)\cap\sigma_c(T)]\cup[\sigma_b(T)\cap\rho_{ab}(T)]$. Since $\mathrm{iso}\sigma_a(T)\cap\sigma_c(T)\cap\mathrm{iso}\sigma(T)=\mathrm{iso}\sigma(T)\cap\sigma_c(T)\subseteq[\mathrm{iso}\sigma(T)\cap\sigma_c(T)\cap\{\lambda\in\mathbb{C}:n(T-\lambda I)=\infty\}]\cup[\mathrm{iso}\sigma(T)\cap\sigma_c(T)\cap\{\lambda\in\mathbb{C}:n(T-\lambda I)<\infty\}]$, and $\mathrm{iso}\sigma(T)\cap\sigma_c(T)\cap\{\lambda\in\mathbb{C}:n(T-\lambda I)<\infty\}\subseteq\sigma_\tau(T)$, we can get that $\mathrm{iso}\sigma_a(T)\cap\sigma_c(T)\subseteq[\mathrm{iso}\sigma_a(T)\cap\mathrm{acc}\sigma(T)]\cup\{\lambda\in\mathbb{C}:n(T-\lambda I)=\infty\}\cup\sigma_\tau(T)$. Also, $\rho_{ab}(T)\cap\sigma_b(T)\subseteq[\rho_a(T)\cap\mathrm{acc}\sigma(T)]\cup[\mathrm{iso}\sigma_a(T)\cap\mathrm{acc}\sigma(T)]$. Then we have

$$\sigma_b(T)\subseteq\sigma_\tau(T)\cup\mathrm{acc}\sigma_{ea}(T)\cup[\mathrm{acc}\{\lambda\in\mathbb{C}:n(T-\lambda I)<d(T-\lambda I)\}\cap\mathrm{acc}\sigma_k(T)]\cup\{\lambda\in\mathbb{C}:n(T-\lambda I)=\infty\}\cup[(\rho_a(T)\cup\mathrm{iso}\sigma_a(T))\cap\mathrm{acc}\sigma(T)].$$

Hence (2)⇒(3) is true.

(3)⇒(1). We know that $\rho_{ea}(T)\cap[\sigma_\tau(T)\cup\mathrm{acc}\sigma_{ea}(T)\cup\{\lambda\in\mathbb{C}:n(T-\lambda I)=\infty\}]=\varnothing$, and $\rho_{ea}(T)\cap[\mathrm{acc}\{\lambda\in\mathbb{C}:n(T-\lambda I)<n(T-\lambda I)\}\cap\mathrm{acc}\sigma_k(T)]=\varnothing$. If $\lambda_0\in\rho_{ea}(T)\cap[(\rho_a(T)\cup\mathrm{iso}\sigma_a(T))\cap\mathrm{acc}\sigma(T)]$, then $\lambda_0\in\rho_{ab}(T)$. If $\lambda_0\in\rho_{ea}(T)$ and $\lambda_0\notin[\rho_a(T)\cup\mathrm{iso}\sigma_a(T)]\cap\mathrm{acc}\sigma(T)$, then $T-\lambda_0 I$ is Browder operator. Thus T satisfies a-Browder's theorem.

In the following, we will discuss the a-Weyl's theorem for T.

Theorem 2 Let $T\in B(H)$. The following statements are equivalent:

(1) T satisfies a-Weyl's theorem;

(2) $\sigma_{ab}(T)=[\sigma_\tau(T)\cap \mathrm{acc}\sigma_a(T)]\cup \mathrm{acc}\sigma_{ea}(T)\cup[\mathrm{acc}\{\lambda\in\mathbb{C}:n(T-\lambda I)<d(T-\lambda I)\}\cap \mathrm{acc}\sigma_k(T)]\cup\{\lambda\in\mathbb{C}:n(T-\lambda I)=\infty\}\cup\{\lambda\in\sigma_a(T):n(T-\lambda I)=0\}$;

(3) $\sigma_b(T)=[\sigma_\tau(T)\cap \mathrm{acc}\sigma_a(T)]\cup \mathrm{acc}\sigma_{ea}(T)\cup[\mathrm{acc}\{\lambda\in\mathbb{C}:n(T-\lambda I)<d(T-\lambda I)\}\cap \mathrm{acc}\sigma_k(T)]\cup\{\lambda\in\mathbb{C}:n(T-\lambda I)=\infty\}\cup[\rho_{ab}(T)\cap \mathrm{acc}\sigma(T)]\cup\{\lambda\in\sigma_a(T):n(T-\lambda I)=0\}$.

Proof (1)$\Rightarrow$(2). Since T satisfies a-Weyl's theorem, we know that $\pi_{00}^a(T)\cap\sigma_\tau(T)=\varnothing$, $\pi_{00}^a(T)\cap\sigma_c(T)=\varnothing$. From Corollary 2, we have $\sigma_{ab}(T)=\sigma_\tau(T)\cup \mathrm{acc}\sigma_{ea}(T)\cup[\mathrm{acc}\{\lambda\in\mathbb{C}:n(T-\lambda I)<d(T-\lambda I)\}\cap\mathrm{acc}\sigma_k(T)]\cup\{\lambda\in\mathbb{C}:n(T-\lambda I)=\infty\}\cup[\mathrm{iso}\sigma_a(T)\cap\sigma_c(T)]$. Moreover, $\sigma_\tau(T)=[\sigma_\tau(T)\cap\mathrm{acc}\sigma_a(T)]\cup[\sigma_\tau(T)\cap\mathrm{iso}\sigma_a(T)]$. Since $[\sigma_\tau(T)\cap\mathrm{iso}\sigma_a(T)]=[\sigma_\tau(T)\cap\mathrm{iso}\sigma_a(T)\cap\{\lambda\in\mathbb{C}:n(T-\lambda I)=\infty\}]\cup[\sigma_\tau(T)\cap\mathrm{iso}\sigma_a(T)\cap\{\lambda\in\mathbb{C}:n(T-\lambda I)=0\}]\cup[\sigma_\tau(T)\cap\mathrm{iso}\sigma_a(T)\cap\{\lambda\in\mathbb{C}:0<n(T-\lambda I)<\infty\}]\subseteq\{\lambda\in\mathbb{C}:n(T-\lambda I)=\infty\}\cup\{\lambda\in\sigma_a(T):n(T-\lambda I)=0\}\cup[\sigma_\tau(T)\cap\pi_{00}^a(T)]\subseteq\{\lambda\in\mathbb{C}:n(T-\lambda I)=\infty\}\cup\{\lambda\in\sigma_a(T):n(T-\lambda I)=0\}$. We get $\sigma_\tau(T)\subseteq[\sigma_\tau(T)\cap\mathrm{acc}\sigma_a(T)]\cup\{\lambda\in\mathbb{C}:n(T-\lambda I)=\infty\}\cup\{\lambda\in\sigma_a(T):n(T-\lambda I)=0\}$.

Also, $\mathrm{iso}\sigma_a(T)\cap\sigma_c(T)\subseteq[\mathrm{iso}\sigma_a(T)\cap\sigma_c(T)\cap\{\lambda\in\mathbb{C}:n(T-\lambda I)=\infty\}]\cup[\mathrm{iso}\sigma_a(T)\cap\sigma_c(T)\cap\{\lambda\in\mathbb{C}:n(T-\lambda I)=0\}]\cup[\mathrm{iso}\sigma_a(T)\cap\sigma_c(T)\cap\{\lambda\in\mathbb{C}:0<n(T-\lambda I)<\infty\}]\subseteq\{\lambda\in\mathbb{C}:n(T-\lambda I)=\infty\}\cup\{\lambda\in\sigma_a(T):n(T-\lambda I)=0\}\cup[\sigma_c(T)\cap\pi_{00}^a(T)]\subseteq\{\lambda\in\mathbb{C}:n(T-\lambda I)=\infty\}\cup\{\lambda\in\sigma_a(T):n(T-\lambda I)=0\}$. Hence $\sigma_{ab}(T)\subseteq[\sigma_\tau(T)\cap\mathrm{acc}\sigma_a(T)]\cup\mathrm{acc}\sigma_{ea}(T)\cup[\mathrm{acc}\{\lambda\in\mathbb{C}:n(T-\lambda I)<d(T-\lambda I)\}\cap\mathrm{acc}\sigma_k(T)]\cup\{\lambda\in\mathbb{C}:n(T-\lambda I)=\infty\}\cup\{\lambda\in\sigma_a(T):n(T-\lambda I)=0\}$. Then we know that (2) holds.

(2)$\Rightarrow$(3). Since $\sigma_b(T)=[\sigma_b(T)\cap\sigma_{ab}(T)]\cup[\sigma_b(T)\cap\rho_{ab}(T)]=\sigma_{ab}(T)\cup[\sigma_b(T)\cap\rho_{ab}(T)]$ and $\sigma_b(T)\cap\rho_{ab}(T)=\mathrm{acc}\sigma(T)\cap\rho_{ab}(T)$, it follows that $\sigma_b(T)=[\sigma_\tau(T)\cap\mathrm{acc}\sigma_a(T)]\cup\mathrm{acc}\sigma_{ea}(T)\cup[\mathrm{acc}\{\lambda\in\mathbb{C}:n(T-\lambda I)<d(T-\lambda I)\}\cap\mathrm{acc}\sigma_k(T)]\cup\{\lambda\in\mathbb{C}:n(T-\lambda I)=\infty\}\cup[\rho_{ab}(T)\cap\mathrm{acc}\sigma(T)]\cup\{\lambda\in\sigma_a(T):n(T-\lambda I)=0\}$.

(3)$\Rightarrow$(1). $[\sigma_a(T)\backslash\sigma_{ea}(T)]\cap[\sigma_\tau(T)\cap\mathrm{acc}\sigma_a(T)]=\varnothing$, $[\sigma_a(T)\backslash\sigma_{ea}(T)]\cap[\mathrm{acc}\sigma_{ea}(T)\cup\{\lambda\in\mathbb{C}:n(T-\lambda I)=\infty\}\cup\{\lambda\in\sigma_a(T):n(T-\lambda I)=0\}]=\varnothing$, $[\sigma_a(T)\backslash\sigma_{ea}(T)]\cap[\mathrm{acc}\{\lambda\in\mathbb{C}:n(T-\lambda I)<d(T-\lambda I)\cap\mathrm{acc}\sigma_k(T)]=\varnothing$. If $\lambda_0\in\sigma_a(T)\backslash\sigma_{ea}(T)$ and $\lambda_0\notin\rho_{ab}(T)\cap\mathrm{acc}\sigma(T)$, then $T-\lambda_0 I$ is Browder operator. Moreover, $[\sigma_a(T)\backslash\sigma_{ea}(T)]\cap[\rho_{ab}(T)\cap\mathrm{acc}\sigma(T)]\subseteq\pi_{00}^a(T)$. Therefore $\sigma_a(T)\backslash\sigma_{ea}(T)\subseteq\pi_{00}^a(T)$. Similarly, we can prove that $\pi_{00}^a(T)\subseteq\sigma_a(T)\backslash\sigma_{ea}(T)$. So the a-Weyl's theorem holds for T.

Remark 2 (i) By Theorem 2, we can get that if $\sigma_b(T)=\sigma_\tau(T)\cap\mathrm{acc}\sigma_a(T)$, then T satisfies a-Weyl's theorem. But the converse is not true. Let $T\in B(\ell^2)$ be defined by $T(x_1,x_2,x_3,\cdots)=(x_2,x_3,x_4,\cdots)$. Then T satisfies a-Weyl's theorem, but $\sigma_b(T)\neq\sigma_\tau(T)\cap\mathrm{acc}\sigma_a(T)$.

(ii) $\sigma_b(T)=\sigma_\tau(T)\cap\mathrm{acc}\sigma_a(T)\Leftrightarrow T$ satisfies a-Weyl's theorem, $\rho_\tau(T)\subseteq\pi_{00}^a(T)=\mathrm{iso}\sigma_a(T)$ and $\sigma_a(T)=\sigma(T)$.

In fact, suppose that $\sigma_b(T)=\sigma_\tau(T)\cap\mathrm{acc}\sigma_a(T)$. Then $\rho_a(T)\subseteq\rho_b(T)$, so $\sigma_a(T)=\sigma(T)$. Similarly, since $\mathrm{iso}\sigma_a(T)\subseteq\rho_b(T)$ and $\rho_\tau(T)\subseteq\rho_b(T)$, we know that $\rho_\tau(T)\subseteq\pi_{00}^a(T)=\mathrm{iso}\sigma_a(T)$.

For the converse, let $\lambda_0\notin\sigma_\tau(T)\cap\mathrm{acc}\sigma_a(T)$. If $\lambda_0\in\rho_a(T)$, then $T-\lambda_0 I$ is invertible. If $\lambda_0\in\mathrm{iso}\sigma_a(T)=\mathrm{iso}\sigma(T)$, we can get that $T-\lambda_0 I$ is Browder operator by $\pi_{00}^a(T)=\mathrm{iso}\sigma_a(T)$ and the a-Weyl's theorem holds for T. If $\lambda_0\in\rho_\tau(T)$, by $\rho_\tau(T)\subseteq\pi_{00}^a(T)=\pi_{00}(T)$ we can get $T-\lambda_0 I$ is Browder operator.

2 Judgement of A-Browder's Theorem and A-Weyl's Theorem for Operator Function

In the following, we will research the a-Browder's theorem and a-Weyl's theorem for operator function by means of the property of the topological uniform descent.

Theorem 3 Let $T\in B(H)$. Then for any polynomial p, $p(T)$ satisfies a-Browder's theorem if and only if

(1) T satisfies a-Browder's theorem;

(2) If $\rho_{\mathrm{SF}_+}^+(T)=\{\lambda\in\rho_{\mathrm{SF}_+}(T):\mathrm{ind}(T-\lambda I)>0\}\neq\varnothing$, then $\rho_\tau(T)\subseteq\mathrm{int}\,\sigma_{ea}(T)\cup\{\lambda\in\mathbb{C}:n(T-\lambda I)=\infty\}\cup\rho_b(T)$.

Proof Suppose $p(T)$ satisfies a-Browder's theorem for any polynomial p. We only need to prove (2) holds. We can assert that if $\rho_{\mathrm{SF}_+}^+(T)\neq\varnothing$, then for any $\lambda\in\rho_{\mathrm{SF}_+}(T)$, $\mathrm{ind}(T-\lambda I)\geqslant 0$. In fact, suppose there exists $\lambda_1\in\rho_{\mathrm{SF}_+}(T)$, $\mathrm{ind}(T-\lambda_1 I)=-m<0$ where m is finite or $m=+\infty$. Let $\lambda_2\in\rho_{\mathrm{SF}_+}^+(T)$, $\mathrm{ind}(T-\lambda_2 I)=n>0$. We can see that n is finite. If $m<+\infty$, let $p_0(T)=(T-\lambda_1 I)^n(T-\lambda_2 I)^m$. Moreover, let $p_0(T)=(T-\lambda_1 I)(T-\lambda_2 I)$ if $m=+\infty$. Then $0\in\rho_{ea}(p_0(T))=\rho_{ab}(p_0(T))$. It follows that $\mathrm{asc}(T-\lambda_2 I)<\infty$. It is in contradiction to the fact that $\mathrm{ind}(T-\lambda_2 I)>0$. Then we will prove that $\rho_\tau(T)\subseteq\mathrm{int}\,\sigma_{ea}(T)\cup\{\lambda\in\mathbb{C}:n(T-\lambda I)=\infty\}\cup\rho_b(T)$. If $\lambda_0\in\rho_\tau(T)$ and $\lambda_0\notin\mathrm{int}\,\sigma_{ea}(T)\cup\{\lambda\in\mathbb{C}:n(T-\lambda I)=\infty\}$, then for any deleted neighborhood $B^\circ(\lambda_0)$ centered on λ_0, there exists $\mu_0\in B^\circ(\lambda_0)$ such that $\mu_0\in\rho_{ea}(T)$. Since for any $\lambda\in\rho_{\mathrm{SF}_+}(T)$, $\mathrm{ind}(T-\lambda I)\geqslant 0$, we know that $T-\mu_0 I$ is Weyl operator. By T satisfing a-Browder's theorem, we can get $T-\mu_0 I$ is Browder operator. Thus $\lambda_0\in\partial\sigma(T)\cup\rho(T)$. From $\lambda_0\in\rho_\tau(T)$ and $n(T-\lambda_0 I)<\infty$, we conclude that $T-\lambda_0 I$ is Browder operator.

For the converse, if $\rho^+_{\mathrm{SF}_+}(T)=\varnothing$, then for any $\lambda\in\rho_{\mathrm{SF}_+}(T)$, $\mathrm{ind}(T-\lambda I)\leqslant 0$. If $\rho^+_{\mathrm{SF}_+}(T)\neq\varnothing$, we can get that $\rho^-_{\mathrm{SF}_+}(T)=\{\lambda\in\rho_{\mathrm{SF}_+}(T):\mathrm{ind}(T-\lambda I)<0\}=\varnothing$ since $\rho^-_{\mathrm{SF}_+}(T)\subseteq\rho_\tau(T)$ but $\rho^-_{\mathrm{SF}_+}(T)\cap[\mathrm{int}\,\sigma_{ea}(T)\cup\{\lambda\in\mathbb{C}:n(T-\lambda I)=\infty\}]=\varnothing$. Let $\mu_0\in\rho_{ea}(p(T))$ and $p(T)-\mu_0 I=a(T-\lambda_1 I)^{n_1}(T-\lambda_2 I)^{n_2}\cdots(T-\lambda_t I)^{n_t}$, where $\lambda_i\neq\lambda_j\,(i\neq j)$ and $\mu_0=p(\lambda_i)$, $1\leqslant i\leqslant t$. It follows that $T-\lambda_i I$ is upper semi-Fredholm operator and $\mathrm{ind}(T-\lambda_i I)\leqslant 0$ for all λ_i. From T satisfing a-Browder's theorem, we have that $\mathrm{asc}(T-\lambda_i I)<\infty(1\leqslant i\leqslant t)$. Hence $\mu_0\in\rho_{ab}(p(T))$, $p(T)$ satisfies a-Browder's theorem.

By Theorem 3 and the proof procedure, we can get the following results.

Corollary 3 Let $T\in B(H)$. Then for any polynomial p, $p(T)$ satisfies a-Browder's theorem if and only if

(1) T satisfies a-Browder's theorem;

(2) If $\rho^+_{\mathrm{SF}_+}(T)\neq\varnothing$, then $\rho_\tau(T)\subseteq\mathrm{acc}\sigma_{ea}(T)\cup\{\lambda\in\mathbb{C}:n(T-\lambda I)=\infty\}\cup\rho_b(T)$.

Remark 3 (i) Suppose $\rho_\tau(T)\subseteq\mathrm{int}\,\sigma_{ea}(T)\cup\{\lambda\in\mathbb{C}:n(T-\lambda I)=\infty\}\cup\rho_b(T)$, then T satisfies a-Browder's theorem and $\rho^-_{\mathrm{SF}_+}(T)=\varnothing$. This implies that for any polynomial p, $p(T)$ satisfies a-Browder's theorem. However, the converse is not true. Let $T(x_1,x_2,x_3,\cdots)=(0,x_1,x_2,\cdots)$. The a-Browder's theorem holds for $p(T)$, but $\rho_\tau(T)\not\subset\mathrm{int}\,\sigma_{ea}(T)\cup\{\lambda\in\mathbb{C}:n(T-\lambda I)=\infty\}\cup\rho_b(T)$.

$\rho_\tau(T)\subseteq\mathrm{int}\,\sigma_{ea}(T)\cup\{\lambda\in\mathbb{C}:n(T-\lambda I)=\infty\}\cup\rho_b(T)\Leftrightarrow$ for any polynomial p, $p(T)$ satisfies a-Browder's theorem, $\sigma_{ea}(T)=\sigma_w(T)$.

In fact, suppose $\rho_\tau(T)\subseteq\mathrm{int}\,\sigma_{ea}(T)\cup\{\lambda\in\mathbb{C}:n(T-\lambda I)=\infty\}\cup\rho_b(T)$. This implies that $\rho^-_{\mathrm{SF}_+}(T)=\varnothing$. So $\sigma_{ea}(T)=\sigma_w(T)$.

For the converse, let $\lambda_0\in\rho_\tau(T)$ and $\lambda_0\notin\mathrm{int}\,\sigma_{ea}(T)\cup\{\lambda\in\mathbb{C}:n(T-\lambda I)=\infty\}$. Then for any deleted neighborhood $B^\circ(\lambda_0)$ centered on λ_0, there exists $\mu_0\in B^\circ(\lambda_0)$ such that $\mu_0\in\rho_{ea}(T)=\rho_w(T)$. Thus $T-\mu_0 I$ is Browder operator. So $\lambda_0\in\partial\sigma(T)\cup\rho(T)$. By $\lambda_0\in\rho_\tau(T)$ and $n(T-\lambda_0 I)<\infty$, we know that $T-\lambda_0 I$ is Browder operator.

(ii) $\rho_\tau(T)\subseteq\mathrm{acc}\sigma_{ea}(T)\cup\{\lambda\in\mathbb{C}:n(T-\lambda I)=\infty\}\cup\rho_b(T)\Leftrightarrow$ for any polynomial p, $p(T)$ satisfies a-Browder's theorem, $\sigma_{ea}(T)=\sigma_w(T)$.

In the following, we will establish sufficient and necessary conditions for operator functions holding a-Weyl's theorem.

Theorem 4 Let $T\in B(H)$. Then T is a-isoloid and for any polynomial p, $p(T)$ satisfies a-Weyl's theorem if and only if:

(1) $\sigma_b(T)=[\sigma_\tau(T)\cap\mathrm{acc}\sigma_a(T)]\cup\mathrm{acc}\sigma_{ea}(T)\cup[\mathrm{acc}\{\lambda\in\mathbb{C}:n(T-\lambda I)<d(T-\lambda I)\}\cap\mathrm{acc}\sigma_k(T)]\cup\{\lambda\in\mathbb{C}:n(T-\lambda I)=\infty\}\cup[\rho_{ab}(T)\cap\mathrm{acc}\sigma(T)]$;

(2) If $\rho^+_{\mathrm{SF}_+}(T)\neq\varnothing$, then $\rho_\tau(T)\subseteq\mathrm{int}\,\sigma_{ea}(T)\cup\{\lambda\in\mathbb{C}:n(T-\lambda I)=\infty\}\cup\rho_b(T)$.

Proof "$\Leftarrow$". First, we will prove that T is a-isoloid. If there exists $\lambda_0\in\mathrm{iso}\sigma_a(T)$ and $n(T-\lambda_0 I)=0$, then $\lambda_0\notin[\sigma_\tau(T)\cap\mathrm{acc}\sigma_a(T)]\cup\mathrm{acc}\sigma_{ea}(T)\cup[\mathrm{acc}\{\lambda\in\mathbb{C}:n(T-\lambda I)<d(T-\lambda I)\}\cap\mathrm{acc}\sigma_k(T)]\cup\{\lambda\in\mathbb{C}:n(T-\lambda I)=\infty\}$. If $\lambda_0\in\rho_{ab}(T)\cap\mathrm{acc}\sigma(T)$, then $\lambda_0\in\rho_a(T)$. It is a contradiction. If $\lambda_0\notin\rho_{ab}(T)\cap\mathrm{acc}\sigma(T)$, then $T-\lambda_0 I$ is Browder operator, so $\lambda_0\in\rho(T)$. It is a contradiction too. Thus T is a-isoloid. From (1) we know that T satisfies a-Browder's theorem. By Theorem 3, we have that for any polynomial p, $p(T)$ satisfies a-Browder's theorem. Suppose $\mu_0\in\pi^a_{00}(p(T))$, let $p(T)-\mu_0 I=a(T-\lambda_1 I)^{n_1}(T-\lambda_2 I)^{n_2}\cdots(T-\lambda_t I)^{n_t}$, where $\lambda_i\neq\lambda_j\,(i\neq j)$ and $\mu_0=p(\lambda_i)$, $1\leqslant i\leqslant t$. Because T is a-isoloid, we get that $\lambda_i\in\pi^a_{00}(T)$. From (1) and the proof of Theorem 3, we have $\pi^a_{00}(T)\subseteq\sigma_a(T)\backslash\sigma_{ea}(T)$. Then $\lambda_i\in\rho_{ab}(T)$, hence $\mu_0\in\rho_{ab}(p(T))$. Thus for any polynomial p, $p(T)$ satisfies a-Weyl's theorem.

"$\Rightarrow$". By Theorem 3 we know (2) holds. Suppose $\lambda_0\notin[\sigma_\tau(T)\cap\mathrm{acc}\sigma_a(T)]\cup\mathrm{acc}\sigma_{ea}(T)\cup[\mathrm{acc}\{\lambda\in\mathbb{C}:n(T-\lambda I)<d(T-\lambda I)\}\cap\mathrm{acc}\sigma_k(T)]\cup\{\lambda\in\mathbb{C}:n(T-\lambda I)=\infty\}\cup[\rho_{ab}(T)\cap\mathrm{acc}\sigma(T)]$. If $\lambda_0\notin\mathrm{acc}\sigma_a(T)$, then $\lambda_0\in\rho_{ab}(T)$. In fact, if $\lambda_0\in\mathrm{iso}\sigma_a(T)$, from T is a-isoloid we can get that $\lambda_0\in\pi^a_{00}(T)$. Hence $\lambda_0\in\rho_{ab}(T)$. Since $\lambda_0\notin\rho_{ab}(T)\cap\mathrm{acc}\sigma(T)$, it follows that $T-\lambda_0 I$ is Browder operator. If $\lambda_0\notin\sigma_\tau(T)$, from the proof of Theorem 2, we get (1) holds.

Remark 4 (i) By Theorem 2, we can get that if $\sigma_b(T)=\sigma_\tau(T)\cap\mathrm{acc}\sigma_a(T)$, then for any polynomial p, $p(T)$ satisfies a-Weyl's theorem. But from Remark 2(ii), we know that the converse is not true. However,

$\sigma_b(T)=\sigma_\tau(T)\cap\mathrm{acc}\sigma_a(T)\Leftrightarrow$ for any polynomial p, $p(T)$ satisfies a-Weyl's theorem, $\rho_\tau(T)\subseteq\pi^a_{00}(T)=\mathrm{iso}\sigma_a(T)$ and $\sigma_a(T)=\sigma(T)$.

(ii) Let $\pi_{0f}^{a}(T)=\{\lambda\in \text{iso}\sigma_a(T):n(T-\lambda I)<\infty\}$. If $\pi_{0f}^{a}(T)\subseteq\rho_\tau(T)\subseteq \text{int}\,\sigma_{ea}(T)\cup\{\lambda\in\mathbb{C}:n(T-\lambda I)=\infty\}\cup\rho_b(T)$, then for any polynomial p, $p(T)$ satisfies a-Weyl's theorem. In fact, if $\mu_0\in\pi_{00}^{a}(p(T))$, let $p(T)-\mu_0 I=a(T-\lambda_1 I)^{n_1}(T-\lambda_2 I)^{n_2}\cdots(T-\lambda_t I)^{n_t}$, where $\lambda_i\neq\lambda_j\,(i\neq j)$ and $\mu_0=p(\lambda_i)$, $1\leqslant i\leqslant t$. Then $\lambda_i\in\pi_{0f}^{a}(T)$. Since $\rho_\tau(T)\subseteq\text{int}\,\sigma_{ea}(T)\cup\{\lambda\in\mathbb{C}:n(T-\lambda I)=\infty\}\cup\rho_b(T)$, we have $\lambda_i\in\text{iso}\sigma(T)$. By $\lambda_i\in\rho_\tau(T)$ we know that $T-\lambda_i I$ is Browder operator. Thus $p(T)-\mu_0 I$ is Browder operator. By Remark 3, we can get that $p(T)$ satisfies a-Weyl's theorem for any polynomial p. But the converse is not true. Let $T(x_1,x_2,x_3,\cdots)=(0,x_1,x_2,\cdots)$. Then for any polynomial p, $p(T)$ satisfies a-Weyl's theorem. However, $\rho_\tau(T)\not\subset\text{int}\,\sigma_{ea}(T)\cup\{\lambda\in\mathbb{C}:n(T-\lambda I)=\infty\}\cup\rho_b(T)$.

$\pi_{0f}^{a}(T)\subseteq\rho_\tau(T)\subseteq\text{acc}\sigma_{ea}(T)\cup\{\lambda\in\mathbb{C}:n(T-\lambda I)=\infty\}\cup\rho_b(T)$ if and only if for any polynomial p, $p(T)$ satisfies a-Weyl's theorem, T is a-isoloid and $\sigma_{ea}(T)=\sigma_w(T)$.

In fact, if $\pi_{0f}^{a}(T)\subseteq\rho_\tau(T)\subseteq\text{acc}\sigma_{ea}(T)\cup\{\lambda\in\mathbb{C}:n(T-\lambda I)=\infty\}\cup\rho_b(T)$, we only need to prove T is a-isoloid. If $\lambda_0\in\text{iso}\sigma_a(T)$ and $n(T-\lambda_0 I)=0$. Then $\lambda_0\in\pi_{0f}^{a}(T)\subseteq\rho_\tau(T)\subseteq\text{acc}\sigma_{ea}(T)\cup\{\lambda\in\mathbb{C}:n(T-\lambda I)=\infty\}\cup\rho_b(T)$. It follows that $T-\lambda_0 I$ is Browder operator. So $T-\lambda_0 I$ is invertible. It is in contradiction to the fact that $\lambda_0\in\text{iso}\sigma_a(T)$.

For the converse, we only need to prove that $\pi_{0f}^{a}(T)\subseteq\rho_\tau(T)$. Since T is a-isoloid, this implies that $\pi_{0f}^{a}(T)=\pi_{00}^{a}(T)$. By a-Weyl's theorem holds for T, we get that $\pi_{0f}^{a}(T)\subseteq\rho_\tau(T)$.

Corollary 4 Let $T\in B(H)$. Then T is a-isoloid and for any polynomial p, $p(T)$ satisfies a-Weyl's theorem if and only if:

(1) $\sigma_b(T)=[\sigma_\tau(T)\cap\text{acc}\sigma_a(T)]\cup\text{acc}\sigma_{ea}(T)\cup[\text{int}\{\lambda\in\mathbb{C}:n(T-\lambda I)<d(T-\lambda I)\}\cap\text{acc}\sigma_k(T)]\cup\{\lambda\in\mathbb{C}:n(T-\lambda I)=\infty\}\cup[\rho_{ab}(T)\cap\text{acc}\sigma(T)]$;

(2) $\rho_{\text{SF}_+}^{+}(T)\neq\varnothing$, $\rho_\tau(T)\subseteq\text{acc}\sigma_{ea}(T)\cup\{\lambda\in\mathbb{C}:n(T-\lambda I)=\infty\}\cup\rho_b(T)$.

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