A Note of the Interpolating Sequence in $Q_p \cap H^\infty$

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Abstract: In this paper, $\{z_i\}_{i=1}^\infty$ acts as an interpolating sequence for $Q_p \cap H^\infty$. An analytic function $f$ is constructed, and $f(z_i)=\sum_{j=1}^n \lambda_j f_j(z_i)=\lambda_n, n=1, 2, \cdots$ for any $\lambda_n \in l^p$, where $f$ and $f_j$ belong to $Q_p \cap H^\infty$. As a result, the study achieves a comparable outcome for $F(p,p-2,s) \cap H^\infty$.

Key words: $Q_p$ space; $H^\infty$ space; $F(p,p-2,s) \cap H^\infty$; interpolating sequence

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0 Introduction

In this paper, a unit disc $\{z: |z| < 1\}$ is denoted by $D$. The Möbius transformation of $D$ is defined by:

$$\phi_a(z) = \frac{a-z}{1-a \overline{z}}, a \in D.$$ 

Note that

$$\beta(z,w) = \frac{1}{2} \log \frac{1+\rho(z,w)}{1-\rho(z,w)}$$

is the hyperbolic metric for any $z,w \in D$, where $\rho(z,w) = |\phi_a(w)|$ is the pseudohyperbolic distance for any $z,w \in D$. For $0 < p < \infty$, an analytic function $f$ belongs to the space $Q_p$ if

$$\|f\|_{Q_p} = \sup_{z \in D} |f'(z)|^p (1-|\phi_a(z)|^2) dA(z) < \infty$$

(1)

where $dA(z)$ is an area measure on $D$ normalized, which makes $\int_D dA(z) = 1$.

Equipped with the norm $|f(0)| + \|f\|_{Q_p}$, the space $Q_p$ is Banach. Generally, $Q_p$ is the Bloch space if $p > 1$. If $p=1$, $Q_1$ coincides with BMOA, analytic function of bounded mean oscillation. $Q_\infty$ is just the Dirichlet space. Regarding the theory of $Q_p$ spaces, readers may refer to Refs. [1-4].

An analytic function $f$ belongs to the Bloch space, denoted by $B$, if

$$\|f\|_B = \sup_{z \in B} (1-|z|^2) |f'(z)| < \infty.$$ 

The space $H^\infty$ consists of all bounded analytic functions $f$ on $D$ with:

$$\sup_{z \in B} |f(z)| < \infty.$$ 

For a subarc $I \subset \partial D$, $\theta$ acts as the midpoint of $I$ and denotes the Carleson box:

$$S(I) = \left\{z \in D: 1 - |I| < |z| < 1, |\theta - \arg z| < \frac{|I|}{2} \right\}.$$ 

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for $|I| \leq 1$ and $S(I) = D$ for $|I| > 1$. A positive measure $\mu$ is a $p$-Carleson measure if

$$\|\mu\|_p = \sup_{I \subset D} \frac{\mu(S(I))}{|I|^p} < \infty,$$

where $|I|$ denotes the arc length of $I$. An analytic function $f \in Q_p$ if and only if the positive measure $|f'(z)|^p (1 - |z|^2) \, dA(z)$ is a $p$-Carleson measure.

The sequence space SCM$_p$ consists of all complex numbers $\{\lambda_n\}$ so that $\sum_{n=1}^\infty (1 - |z_i|^2)^p |\lambda_n| \delta_{\lambda_n}$ is a $p$-Carleson measure, where $\delta_{\lambda_n}$ denotes the unit point-mass measure at $z = D$ and $\{z_n\}_{n=1}^\infty \subset D$.

A sequence $\{z_n\}_{n=1}^\infty$ is called an interpolating sequence for $Q_p \cap H^\infty$ if, for each bounded sequence $\{\lambda_n\}_{n=1}^\infty$ of complex values, there exists an $f \in Q_p \cap H^\infty$ such that $f(z_n) = \lambda_n$ for all $n$.

A sequence $\{z_n\}_{n=1}^\infty$ in $D$ is separated if $\inf_{n \neq m} \rho(z_n, z_m) > 0$.

Usually $\{z_n\}_{n=1}^\infty$ is an interpolating sequence for $H^\infty$ if and only if $\{z_n\}_{n=1}^\infty$ in $D$ is separated and

$$\sum_{n} (1 - |z_n|^2)^p \delta_{\lambda_n}$$

is a Carleson measure for $H^\infty$. See Ref. [5] for interpolating sequence in $H^\infty$. Readers can refer to Refs. [6,7] about the Hardy and Bergman space theory. See Ref. [8] for more interpolating sequences. Sundberg solved the interpolating question for BMOA in Ref. [9].

A necessary and sufficient condition is obtained for the interpolating sequence in the Bloch space [10]. Pascuas characterized the interpolating sequence in the Bloch space by the $p$-Carleson measure, as details in Ref. [11]. Ref. [12] gave a characterization of the interpolating sequence for $Q_p \cap H^\infty$ and $H^\infty \cap Q_{p,\infty}$. The main result is listed as follows:

**Theorem 1**[12] Let $p \in (0, 1)$. A sequence $\{z_n\}_{n=1}^\infty$ of points in the unit disc is an interpolating sequence for $Q_p \cap H^\infty$ if and only if $\{z_n\}_{n=1}^\infty$ in $D$ is separated and

$$\sum_{n} (1 - |z_n|^2)^p \delta_{\lambda_n}(z)$$

is a $p$-Carleson measure.

In the following analysis, $f \prec g$ (for two functions $f$ and $g$) if there is a constant $C$ such that $f \leq C g$. $f \approx g$ (that is, $f$ is comparable with $g$) whenever $g - f \prec g$.

## 1 Interpolating Sequence in $Q_p \cap H^\infty$

If the sequence $\{z_j\}$ is an interpolating sequence for the Bloch space, then

$$\beta(z_n, z_m) \approx C(1 - |z_n|^2)$$

(2)

The aforementioned condition represents the separation criterion. It is subsequently reformulated into the following condition

$$1 - \rho(z_n, z_m) \approx C(1 - |z_n|^2)$$

(3)

where $\lambda$ is the constant from the separation condition (2), coinciding with the one used in Ref. [13]. Readers can refer to Refs. [8] and [13] about the separation condition (3). The condition (2) holds, then $\{z_j\}$ is separated.

To a point $z$, a region $V_z$ is defined by

$$V_z = \{w : w \in D, |w - z'| \leq (1 - |z|^2)^\frac{\beta}{p} \}$$

(4)

where $z'$ denotes the radial projection $\frac{z}{|z|}$ of $z$ and $0 < \beta < 1$. If the two regions $V_{z_j}$, $V_{z_m}$ intersect and $|z_j| > |z_m|$, then $(1 - |z_j|)^\eta(1 - |z_m|)^\eta$ can be obtained and $z_m$ is outside of $V_{z_j}$. The constants $\beta(0 < \beta < 1)$ and $\eta(\eta > 1)$ are chosen so that $1 < \eta < \frac{2\beta - 1}{1 - \lambda}$ and $\eta\beta > 1$. Those implications of the separation condition can be found in Ref. [13].

This paper chooses a constant $\rho$ so that $1 > \rho > \beta$. The constant $\rho$ will be defined to determine the support of the function $g$ in the following Lemma 1. The paper constructs a function living essentially in a region $V_{z_n}$ with reasonable estimates of how it behaves for all points.

**Lemma 1** Let $s > -1$ and $0 < p < 2$. For any given point $b \in D$, there exists a function $g_b$ so that:

$$f_s(w) = \int_{D} \frac{g_s(u)(1 - |u|)^s}{(1 - u \cdot w)^{1-s}} \, dA(u)$$

satisfies $f_s(b) = 1$, and for points in $V_{z_n}$ the value is estimated by:

$$f_s(w) = c(y(w)) + C(1 - |b|)^{-\gamma}.$$  

Here $\gamma = \gamma(w)$ is defined by $|w - b| = (1 - |b|)^{1-\gamma}$.

$$c(y(w)) = \begin{cases} 0, & \gamma < \rho, \\ (1 - |b|)^{1-\gamma}, & \rho \leq \gamma \leq 1 \\ 1, & \gamma > 1. \end{cases}$$

For points outside of $V_{z_n}$, the bound of $f$ is obtained by:

$$|f_s(w)| \leq C(1 - |b|)^{\gamma}$$

(5)

Further,
\[
\int_{\partial} |g_s(u)|^y (1 - |u|)^y \, d\mathcal{A}(u) \leq C (1 - |b|)^y 
\] (6)

**Proof** In this proof, a technique that is borrowed from Ref. [8] has been adapted to enhance its efficiency. The relation defining \( g_s \) is as follows
\[
(1 - |\zeta|) y (1 - \bar{b} \bar{b}) = K \cdot \frac{1 - |b|}{|\zeta - b|^y},
\]
where \( \zeta \) lives in the annulus
\[
E_2 = \left\{ \zeta | (1 - |b|) \leq |\zeta - b| \leq (1 - |b|) \right\}
\]
and is further restricted to a cone with the vertex in \( b^* \) and a fixed small aperture. For all others \( \zeta, g_s \) is taken to be zero. There is the following equation:
\[
\int_{\partial} \frac{(1 - |\zeta|) y}{(1 - \bar{b} \bar{b})^{y-1}} g_s(\zeta) \, d\mathcal{A}(\zeta) = K (1 - |b|) \int_{(1 - |b|)}^{(1 - |\zeta|) y} \frac{1}{\bar{b}} \, d\zeta = K (1 - |b|)^{y-1}
\]
where \( K \) is chosen so that \( f_s(b) = 1 \). Observing the definition of \( g_n \) then we get:
\[
|g_n(\zeta)| \leq C (1 - |b|)^{y-1}.
\]

The function \( f_s(w) \) is to be estimated. Let us suppose that \( w \) is in \( V_\rho \). Let:
\[
E_1 = \left\{ \zeta | \zeta - b^* \leq (1 - |b|)^{1/2} \right\}
\]
and
\[
E_2 = \left\{ \zeta | \zeta - b^* > (1 - |b|)^{1/2} \right\}.
\]

The contributions on \( E_1 \) and \( E_2 \) are considered. For any \( \zeta \in E_1 \), there is \( 1 - \bar{w} \bar{b} \geq C (1 - |b|) \) by \( w - b^* = (1 - |b|)^{1/2} \). Then:
\[
\int_{\partial} \frac{(1 - |\zeta|) y}{(1 - \bar{w} \bar{b})^{y-1}} g_s(\zeta) \, d\mathcal{A}(\zeta) \leq C (1 - |b|)^{y-1} g_s(\zeta) \, d\mathcal{A}(\zeta) \leq C (1 - |b|)^{y-1}.
\]

For \( E_2 \), after a calculation,
\[
\int_{\partial} \frac{(1 - |\zeta|) y}{(1 - \bar{b} \bar{b})^{y-1}} g_s(\zeta) \, d\mathcal{A}(\zeta) = (1 - |b|)^{y-1} - (1 - |b|)^{y-1}. 
\]

Since
\[
(1 - \bar{w} \bar{b})^{1/2} \leq C |w - b| \cdot (1 - |\zeta|)^{1/2},
\]
and
\[
|b - w| \leq |b - b^*| + |w - b^*| \leq 2(1 - |b|)^{1/2},
\]
then
\[
\int_{\partial} \frac{(1 - |\zeta|) y}{(1 - \bar{w} \bar{b})^{y-1}} g_s(\zeta) \, d\mathcal{A}(\zeta) \leq C (1 - |b|)^{y-1} \int_{\partial} \frac{|g_s(\zeta)|}{(1 - \bar{b} \bar{b})^{y-1}} \, d\mathcal{A}(\zeta) \leq C (1 - |b|)^{y-1} + O (1 - |b|)^{1/3}.
\]

For a point \( w \) outside of \( V_\rho \), \( |1 - \bar{w} \bar{b}| \geq (1 - |b|)^{1/2} \) holds when \( \zeta \) belongs to the support of \( g_s \). Thus,
\[
|f_s(w)| \leq (1 - |b|)^{1/3}.
\]

\( p = (1 + s)(\rho - \beta) \) for sufficiently large \( s \) can be obtained so that the condition (5) holds. Finally, we can get (6) by the direct calculation.

An operator \( T_s \) is defined as follows:
\[
T_s(g)(z) = \int_{\partial} \frac{(1 - |w|) y}{(1 - \bar{w} \bar{b})^{y-1}} \, d\mathcal{A}(w)
\]
(8)
where \( g \) is a measurable function on \( D \). The following Lemma 2 is Lemma 3.1.2 in Ref. [2].

**Lemma 2** Let \( p > 0 \). If \( |g(z)|^y (1 - |z|)^{y-p} \, d\mathcal{A}(z) \) is a \( p \)-Carleson measure on \( D \), then
\[
\left| T_s(g)(z) \right|^y (1 - |z|)^{y-p} \, d\mathcal{A}(z)
\]
is also a \( p \)-Carleson measure on \( D \).

A finite number of points can be added to an interpolating sequence, rendering it interpolating. This fact is employed in Ref. [13] to derive the implications of the separation condition.

**Lemma 3** Let \( \{z_n\}_{n=1}^\infty \) be a sequence in \( D \) so that
\[
\sum_n |1 - |z_n||^y < \infty.
\]
When (3) holds and \( \{\lambda_n\} \in l^r \), it can be bounded by \( \{a_n\}_{n=1}^\infty \) and \( \delta \in (0, 1) \) so that \( f(z) = \sum a_n f_n \) approximates it in the sense
\[
\| f(z_n) - \lambda_n \|_p \leq \delta \| \lambda_n \|_p.
\]

The coefficients \( a_n \) as well as \( \| f \|_p \) are bounded by \( C \cdot \| \lambda_n \|_p \). Corresponding to \( z_n \), \( f_n \) is the function in Lemma 1.

**Proof** Without loss of generality, it is supposed \( \| \lambda_n \|_p = 1 \). The points are ordered in sequence by their distance to the boundary. To a point \( z_m \), an increasing chain of regions \( V_{z_m} \subset V_{z_{m+1}} \subset \cdots \subset V_{z_k} \) is chosen at each step, selecting the smallest region strictly containing all the previous ones. \( \beta_j = c_{y_j} \) is defined in Lemma 1, where \( y_j \) is given by
Then $z_1$ is determined by the equation:

$$a_i = \lambda_i - \sum a_i \beta_i$$

Note that

$$\sum a_i \beta_i = \beta_i \lambda_i + (1 - \beta_i) \sum a_i \beta_i.$$ 

Since $a_i$ is defined in the same way as $a_j$, $|a_i| \leq 2$. The asserted properties of $f$ is checked. A point $z$ is fixed, and a notation is kept as in the construction above. Here:

$$f(z_i) - \lambda_i = \sum a_i (f_i(z_i) - \beta_i) + \sum \beta_i f_i(z_i) \quad (9)$$

The first term of the right of (9) should be considered first. Since

$$|z_i - z_{i+1}| = |z_{i+1} - z_i| \leq 1$$

then

$$|f_i(z_i) - \beta_i| \leq C \left(1 - |z_i|\right)^{1-p}.$$

By repeatedly applying the separation condition, there comes:

$$\left(1 - |z_i|\right) \leq \left(1 - |z_n|\right)^{1-p}.$$ 

Then

$$\sum_{i=1}^{n} a_i (f_i(z_i) - \beta_i) \leq \sum_{i=1}^{n} a_i \left(1 - |z_i|\right)^{1-p} \leq \sum_{i=1}^{n} \left(1 - |z_i|\right)^{(1-p)+1-n} \leq C \left(1 - |z_n|\right)^{1-p}.$$

Now, let us look to the second term of the right of (9). If $z_i$ is not in $V_{s_i}$, by Lemma 1 the estimate is made as follows:

$$|f_i(z_i)| \leq C \left(1 - |z_i|\right)^{1-p}.$$

If $z_i$ is in $V_{s_i}$, $V_z$ is founded in the chain not contained in $V_z$ so that $|z_i| \geq |z_i|$. Since

$$\text{diam}(V_z) \leq 2 \left(1 - |z_i|\right)^{1-p}$$

then $|z_i - z_i| \geq C \left(1 - |z_i|\right)^{1-p}$. This gives the estimate

$$|f_i(z_i)| \leq C \left(1 - |z_i|\right)^{1-p}.$$

The above estimates show that

$$|f(z_i) - \lambda_i| \leq \sum_{i=1}^{n} a_i \|f_i(z_i) - \beta_i\| + \sum_{i=1}^{n} |\lambda_i| (1 - |z_i|)^{1-p} \leq \sum_{i=1}^{n} (1 - |z_i|)^{1-p} + \left(\sum_{i=1}^{n} (1 - |z_i|)^{1-p}\right).$$

Note: the last term is finite by the $p$-Carleson measure condition. The left expression can be made smaller than a $\delta < 1$ by removing finitely many points from the sequence.

Now, let us prove $f \in H^\infty$. If $z$ is not contained in $V_z$, then $|f(z)| \leq C$ is easy to get. If $z$ is included in some region $V_z$, it is assumed that $V_z$ is the smallest, and $|f(z)|$ is bounded by applying the estimates on $f_i(z_i)$.

This paper points out that the following result is essential Theorem 1. Here, a new method is used to construct the function sequence $f_i \in Q_i \cap H^\infty$ such that

$$f = \sum a_i f_i (Q_i \cap H^\infty).$$

**Theorem 2** A sequence $\{z_i\}_{i=1}^{\infty}$ is an interpolating sequence for $Q_i \cap H^\infty$ if and only if the condition (3) holds and $\sum (1 - |z_i|)^{1-p} \delta_i (z)$ is a $p$-Carleson measure. Furthermore, there is an analytic function $f = \sum a_i f_i \in Q_i \cap H^\infty$ such that $f(z_i) = \lambda_i$, $i = 1, 2, \ldots$, for any sequence $\{\lambda_i\} \in l^\infty$.

**Proof** Note the condition (3) holds. By repeatedly applying Lemma 3 we can find a function $f = \sum a_i f_i \in H^\infty$ with $f(z_i) = \lambda_i$ satisfying $|a_i| \leq C \|\lambda_i\|$. Since each $f_i$ comes from a $g_i$, as in Lemma 1, $f$ is given by $g = \sum a_i g_i$, and

$$f'(w) = T_g (w) = \int_{S(1)} g(u) dA(u).$$

As a consequence of the separation condition, the support of $g$ is disjoint. Also, if the support of $g_i$ intersects a Carleson box $S(I)$, then $z_i$ is contained in $S(2I)$. 

$$\sum a_i \delta_i (z) \leq C \left(1 - |z|\right)^{1-p} \delta_i (z).$$
A Note of the Interpolating Sequence in $Q_s \cap H^p$

Just for this condition, a relation can be obtained:

$$\int \left| \sum_{j \in I} a_j g_{z_j} \right|^p \left(1 - |z|^2 \right)^p d\mu(z) \leq C \sum_{n \in I} \left| \sum_{j \in I} a_j \right|^p \left(1 - |z_n|^2 \right)^p \leq C |I|^p.$$ 

Then $f = \sum \lambda_j e^{\frac{f_{z_j}}{p}}$ is an interpolating sequence for Bloch space, by Theorem 1.

Conversely, $\sum \left(1 - |z_n|^2 \right) \delta_{\lambda_n}(z)$ is a $p$-Carleson measure by Theorem 1. Since $\{\lambda_n\}$ is also an interpolating sequence for Bloch space, the condition (3) holds by (2).

**Remark 1** The analytic function $f$ in Theorem 1 is not unique. It is assumed that $f = \lambda_n, n = 1, 2, \ldots$ for any sequence $\{\lambda_n\} \in I^*$, causing the following relation,

$$\sup_{a \in B} \left\{ f(z) \right\}^p \left(1 - |f(z)|^2 \right)^p \delta \left( z \right) < \infty.$$ 

Since the inner function

$$B(z) = \left| \frac{1 - z}{1 - z_n} \right| \delta_{\lambda_n}(z) \in Q_s$$ 

by Theorem 5.2.1 in Ref. [1], then $fB \in Q_s$ by Lemma 1 of Ref. [14]. $h(z) = f(z)(1 + B(z))$ is defined. Then $h(z) = \lambda_n, n = 1, 2, \ldots$, and $f \in Q_s \cap H^p$.

## Interpolating Sequence in $F(p, q, s) \cap H^p$

For $0 < p < \infty, 0 < s < \infty$, an analytic function $F$ belongs to the space $F(p, p - 2, s)$ if

$$\sup_{z \in D} \left| f(z) \right|^p \left(1 - |z|^2 \right)^{q - 2} \left(1 - |f(z)|^2 \right)^{q - 2} d\mu(z) < \infty \quad (10)$$

We know in this paper that the space $F(p, p - 2, s)$ is a subspace of $B$ by Corollary 2.8 of Ref. [15]. An analytic function $f \in F(p, p - 2, s)$ if and only if the positive measure $|g(z)|^p \left(1 - |z|^2 \right)^{q - 2} d\mu(z)$ is an $s$-Carleson measure. $F(p, p - 2, s)$ belongs to a general of function space $F(p, q, s)$. See Ref. [15].

A sequence $\{z_n\}$ in the unit disc is an interpolating sequence for $F(p, p - 2, s) \cap H^p$ if each bounded sequence $\{\lambda_n\}$ of complex values; there exists an $f \in F(p, p - 2, s) \cap H^p$ so that $f(z_n) = \lambda_n$ for all $n$. Yuan and Tong gave a characterization of the interpolating sequences for $F(p, p - 2, s) \cap H^p$ in Ref. [16].

**Lemma 4** Let $1 < p < \infty, 0 < s < 1$. Then $\{z_n\}_{n=1}^{\infty} \subset D$ is an interpolating sequence for $F(p, p - 2, s) \cap H^p$ if and only if $\{\lambda_n\}_{n=1}^{\infty}$ is a separated sequence and $\sum_{n=1}^{\infty} (1 - |z_n|^2) \delta_{\lambda_n}$ is an $s$-Carleson measure.

The paper obtains a result about the space $F(p, p - 2, s)$.

**Theorem 3** Let $1 < p < \infty, 0 < s < 1$. Let $\{z_n\}$ be a sequence on $D$. Then $\{z_n\}_{n=1}^{\infty}$ is an interpolating sequence for $F(p, p - 2, s) \cap H^p$ if and only if the condition (3) holds and $\sum_{n=1}^{\infty} (1 - |z_n|^2) \delta_{\lambda_n}$ is an $s$-Carleson measure. Furthermore, there is an analytic function $f = \sum \lambda_n e^{\frac{f_{z_n}}{p}}$ such that $f(z_n) = \lambda_n, n = 1, 2, \ldots$ for any sequence $\{\lambda_n\} \in I^*$.

The proof of Theorem 3 can be omitted, as it bears similarity to the proof of Theorem 2.

We know in this paper that $F(p, p - 2, s)$ contains only constant functions if $p + s \leq 1$. Qian and Ye gave the following result in Ref. [17].

**Lemma 5** Let $0 < s < 1$, max $\{s, 1 - s\} < p < 1$. Suppose $\{z_n\}_{n=1}^{\infty}$ is a sequence in $D$. Then $\{z_n\}_{n=1}^{\infty}$ is an interpolating sequence for $F(p, p - 2, s) \cap H^p$ if and only if $\{z_n\}_{n=1}^{\infty}$ is a separated sequence and $\sum_{n=1}^{\infty} (1 - |z_n|^2) \delta_{\lambda_n}$ is an $s$-Carleson measure.

We have a similar result for $F(p, p - 2, s) \cap H^p$, where $0 < s < 1$, max $\{s, 1 - s\} < p < 1$. We omitted the detail analysis.

## References


