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Uniform Asymptotics for Finite-Time Ruin Probabilities of Risk Models with Non-Stationary Arrivals and Strongly Subexponential Claim Sizes

□ **XU Chenghao[1], WANG Kaiyong[1†], PENG Jiangyan[2]**

1. School of Mathematical Sciences, Suzhou University of Science and Technology, Suzhou 215009, Jiangsu, China;
2. School of Mathematical Sciences, University of Electronic Science and Technology of China, Chengdu 611731, Sichuan, China

Abstract: This paper considers the one- and two-dimensional risk models with a non-stationary claim-number process. Under the assumption that the claim-number process satisfies the large deviations principle, the uniform asymptotics for the finite-time ruin probability of a one-dimensional risk model are obtained for the strongly subexponential claim sizes. Further, as an application of the result of one-dimensional risk model, we derive the uniform asymptotics for a kind of finite-time ruin probability in a two dimensional risk model sharing a common claim-number process which satisfies the large deviations principle.

Key words: one-dimensional risk model; two-dimensional risk model; large deviations principle; finite-time ruin probability; heavy-tailed distributions

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0 Introduction

In this section we consider a one-dimensional risk model, in which the surplus at time $t\geqslant 0$ is described as

$$U(t)=x+ct-\sum_{i=1}^{N(t)}X_i \tag{1}$$

where $x\geqslant 0$ is the initial surplus, $c>0$ is the constant premium rate and the claim size $\{X_i, i\geqslant 1\}$ are independent, identically distributed (i. i. d.) and nonnegative random variables with common distribution F and finite mean. $\{\tau_i, i\geqslant 1\}$ are the claim-arrival times, which constitute the claim-number process

$$N(t)=\sup\{i\geqslant 0:\tau_i\leqslant t\}, t\geqslant 0$$

with a finite mean function $\lambda(t)=E(N(t))$, $t\geqslant 0$, where $\sup\varnothing=0$ and $\tau_0=0$ by convention. The nonnegative random variables $\{\theta_i=\tau_i-\tau_{i-1}, i\geqslant 1\}$ are the claim inter-arrival times, which are independent of $\{X_i, i\geqslant 1\}$. For the risk model (1), the finite-time ruin probability up to

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Biography: XU Chenghao, male, Master candidate, research direction: actuarial science. E-mail: 534759246@qq.com

† Corresponding author. E-mail: beewky@vip.163.com

time $t \geqslant 0$ is defined as

$$\psi(x,t) = P\left(\inf_{0\leqslant s\leqslant t} U(s) < 0 | U(0) = x\right) \qquad (2)$$

The risk model (1) has been widely studied and results under various conditions are presented. For the uniform asymptotics of the finite-time ruin probability $\psi(x,t)$ as $x \to \infty$, when $\{\theta_i, i \geqslant 1\}$ are i.i.d., Tang [1] investigated the case that the claim sizes have consistently-varying-tailed distributions and obtained the asymptotics of $\psi(x,t)$ holds uniformly for $t \in \Lambda = \{t \geqslant 0: \lambda(t) > 0\}$. In the case where the distributions of the claim sizes are from a subclass of subexponential distribution class, Leipus and Šiaulys [2] presented the asymptotics of $\psi(x,t)$ holds uniformly for $t \in [f(x), \gamma x]$, where $f(x)$ is an infinitely increasing function and $\gamma > 0$ is a constant. Leipus and Šiaulys [3] and Kočetova *et al* [4] considered the claim sizes have strong subexponential distributions and showed the asymptotics of $\psi(x,t)$ holds uniformly for $t \in [f(x), \infty)$. Yang *et al* [5] and Wang *et al* [6] improved the above results by considering the dependent $\{\theta_i, i \geqslant 1\}$. Chen *et al* [7] established a two-dimensional risk model for (1) and obtained some corresponding results for i.i.d. $\{\theta_i, i \geqslant 1\}$. Chen *et al* [8] extended the results of Chen *et al* [7] by considering the dependent $\{\theta_i, i \geqslant 1\}$.

In the above literatures, they mainly considered the claim inter-arrival times $\{\theta_i, i \geqslant 1\}$ are i.i.d or have some dependence structures. Few articles have studied the claim-number process is non-stationary. In fact, a non-stationary claim-number process may be more practical. Stabile and Torrisi [9] derived the infinite and finite time ruin probabilities for the risk model with a non-stationary Hawkes process and light-tailed claim sizes. Recently, Refs.[10,11] considered the claim-number processes may not be stationary and ergodic and satisfy the large deviations principle (LDP for short). A family of probability measures $\{\mu_t\}_{t\in(0,\infty)}$ on a Hausdorff topological space $(M, \mathcal{F}_M)$ satisfies the LDP with rate function I: $M \to [0,\infty)$, if I is a lower semi-continuous function and the following inequalities hold for every Borel set B:

$$-\inf_{x\in B^o} I(x) \leqslant \liminf_{t\to\infty} \frac{1}{t}\log \mu_t(B)$$

$$\leqslant \limsup_{t\to\infty} \frac{1}{t}\log \mu_t(B) \leqslant -\inf_{x\in \bar{B}} I(x),$$

where B^o and $\overline{B}$ denote the interior and closure of B, respectively, see, e. g., Dembo *et al* [12] and Bordenave *et al* [13].

This section still considers the claim-number process $\{N(t), t \geqslant 0\}$ satisfying the LDP and investigates the uniform asymptotics of the finite-time ruin probability $\psi(x,t)$ for the risk model (1). Section 1 presents the main results after introducing necessary preliminaries and the proofs of the main results are given. Section 2 studies a two-dimensional risk model and investigates a kind of finite-time ruin probability by using the results of Section 1.

1 Preliminaries and Main Results

Hereafter, all limit relationships hold as $x \to \infty$ unless stated otherwise. For two positive functions $a(x)$ and $b(x)$, we write $a(x) \lesssim b(x)$, if $\limsup a(x)/b(x) \leqslant 1$; write $a(x) \gtrsim b(x)$, if $\liminf a(x)/b(x) \geqslant 1$ and write $a(x) \sim b(x)$, if $\lim a(x)/b(x) = 1$. For two positive functions $a(x,t)$ and $b(x,t)$, we say that $a(x,t) \lesssim b(x,t)$ holds uniformly for $t \in \Delta \neq \varnothing$, If

$$\limsup_{x\to\infty} \sup_{t\in\Delta} \frac{a(x,t)}{b(x,t)} \leqslant 1;$$

say that $a(x,t) \gtrsim b(x,t)$ holds uniformly for $t \in \Delta \neq \varnothing$, if

$$\liminf_{x\to\infty} \inf_{t\in\Delta} \frac{a(x,t)}{b(x,t)} \geqslant 1;$$

and say that $a(x,t) \sim b(x,t)$ holds uniformly for $t \in \Delta \neq \varnothing$, if $a(x,t) \gtrsim b(x,t)$ and $a(x,t) \lesssim b(x,t)$ hold uniformly for $t \in \Delta \neq \varnothing$. $\mathbf{1}_A$ is the indicator function of a set A.

In this paper, we will consider the claim sizes have heavy-tailed distributions. Some subclasses of heavy-tailed distribution class will be given. Say that a distribution V on $(-\infty,\infty)$ is heavy-tailed if for any $\lambda > 0$,

$$\int_{-\infty}^{\infty} e^{\lambda t} V(dt) = \infty.$$

One of the important distribution classes of heavy-tailed distributions is the consistently-varying-tailed distribution class $\mathcal{C}$. By definition, a distribution V on $(-\infty,\infty)$ belongs to the class $\mathcal{C}$, denoted by $V \in \mathcal{C}$, if

$$\lim_{y\nearrow 1} \limsup_{x\to\infty} \frac{\overline{V}(xy)}{\overline{V}(x)} = 1,$$

or equivalently,

$$\lim_{y\searrow 1} \liminf_{x\to\infty} \frac{\overline{V}(xy)}{\overline{V}(x)} = 1.$$

A related distribution class is the dominated varying tailed distribution class $\mathcal{D}$. Say that a distribution V on $(-\infty,\infty)$ belongs to the class $\mathcal{D}$, denoted by $V \in \mathcal{D}$, if for any fixed $0 < y < 1$,

$$\limsup_{x\to\infty} \frac{\overline{V}(xy)}{\overline{V}(x)} < \infty.$$

A distribution V on $(-\infty,\infty)$ is said to be in the long-tailed distribution class $\mathcal{L}$, if for any fixed $y>0$,

$$\lim_{x\to\infty}\frac{\overline{V}(x+y)}{\overline{V}(x)}=1.$$

An important subclass of the class $\mathcal{L}$ is the subexponential distribution class $\mathcal{S}$. By definition, a distribution V on $[0,\infty)$ is said to be subexponential if

$$\lim_{x\to\infty}\frac{\overline{V*V}(x)}{\overline{V}(x)}=2,$$

where $V*V$ denotes the 2-fold convolution of V. In the case that a distribution V is on $(-\infty,\infty)$, we say that $V\in\mathcal{S}$ if the distribution $V(x)\mathbf{1}_{\{x\geqslant 0\}}$ belongs to the class $\mathcal{S}$. It is well-known that these distribution classes have the following inclusions

$$\mathcal{C}\subset\mathcal{L}\cap\mathcal{D}\subset\mathcal{S}\subset\mathcal{L},$$

see, e.g., Embrechts *et al*[14]. Korshunov[15] introduced another subclass of the subexponential distribution class, which is the strongly subexponential distribution class $\mathcal{S}_*$. Say that a distribution V on $(-\infty,\infty)$ belongs to the class $\mathcal{S}_*$, if $\int_0^\infty\overline{V}(y)\mathrm{d}y<\infty$ and the distribution V_u defined by

$$\overline{V_u}(x)=\begin{cases}\min\left\{1,\int_x^{x+u}\overline{V}(y)\mathrm{d}\,y\right\}, & x\geqslant 0,\\ 1, & x<0,\end{cases}$$

satisfies

$$\lim_{x\to\infty}\frac{\overline{V_u*V_u}(x)}{\overline{V_u}(x)}=2$$

uniformly for $u\in[1,\infty)$. Korshunov[15] pointed out that the Pareto distribution with parameter exceeding one, the lognormal distribution and the Weibull distribution with suitably chosen parameters belong to the class $\mathcal{S}_*$ and the class $\mathcal{S}_*$ almost coincides with the class of subexponential distributions with finite means. For the distributions with finite means the following relationships hold

$$\mathcal{L}\cap\mathcal{D}\subset\mathcal{S}_*\subset\mathcal{S},$$

see, e.g., Korshunov[15] and Kaas *et al*[16].

This paper mainly considers the claim-number process $\{N(t),t\geqslant 0\}$ satisfying the LDP. We first present the following assumption.

Assumption A 1) $P(N(t)/t\in\cdot)$ satisfies the LDP with rate function $I(\cdot)$ such that $I(x)=0$ if and only if $x=z$, where z is a positive constant.

2) $I(\cdot)$ is increasing on $[z,\infty)$ and decreasing on $[0,z]$.

As noted in Remark 2.1 of Fu *et al*[10], the linear Hawkes process defined in Section 1 of Bordenave *et al*[13] satisfies Assumption A. One can see Lefevere *et al*[17], Macci *et al*[18] and Jiang *et al*[19] for some other counting processes satisfying the LDP.

The following is the main result of this section.

Theorem 1 Consider the risk model (1). Suppose that Assumption A holds. If $F\in\mathcal{S}_*$ and $v:=c/z-E\left(X_1\right)>0$, then

$$\psi(x,t)\sim\frac{1}{v}\int_x^{x+vzt}\overline{F}(y)\mathrm{d}y \tag{3}$$

holds uniformly for $t\in[f(x),\infty)$, where $f:[0,\infty)\to[0,\infty)$ is an infinitely increasing function.

Before giving the proof of Theorem 1, we first present a lemma, which follows from Lemmas 1 and 9 in Korshunov[15] (see also Lemma 2.2 in Leipus and Šiaulys[3]).

Lemma 1 Let $\{\xi_i,i\geqslant 1\}$ be i.i.d. random variables with common distribution V and finite mean $E\xi_1<0$.

1) If $V\in\mathcal{L}$, then for sufficiently large x,

$$P\left(\max_{1\leqslant k\leqslant n}\sum_{i=1}^k\xi_i>x\right)\geqslant\frac{1-\varepsilon_1(x)}{|E\xi_1|}\int_x^{x+n|E\xi_1|}\overline{V}(u)\mathrm{d}u$$

holds uniformly for integers $n\geqslant 1$;

2) If $V\in\mathcal{S}_*$, then for sufficiently large x,

$$P\left(\max_{1\leqslant k\leqslant n}\sum_{i=1}^k\xi_i>x\right)\leqslant\frac{1+\varepsilon_2(x)}{|E\xi_1|}\int_x^{x+n|E\xi_1|}\overline{V}(u)\mathrm{d}u$$

holds uniformly for integer $n\geqslant 1$, where $\varepsilon_1(x)$ and $\varepsilon_2(x)$ are some positive vanishing functions as $x\to\infty$.

In the following we prove Theorem 1.

Proof of Theorem 1 By Assumption A, for any fixed $w_1<z$ and $w_2>z$, there exist some constants $\delta_1>0$ and $\delta_2>0$ such that $I(w_1)-\delta_1>0, I(w_2)-\delta_2>0$ and for sufficiently large t,

$$P(N(t)/t\leqslant w_1)\leqslant \mathrm{e}^{-t(I(w_1)-\delta_1)} \tag{4}$$

and

$$P(N(t)/t\geqslant w_2)\leqslant \mathrm{e}^{-t(I(w_2)-\delta_2)} \tag{5}$$

where the facts $I(x)>0$ for $x\neq z$ and $I(\cdot)$ is decreasing on $[0,z]$ and increasing on $[z,\infty)$ have been used.

Note that for all $x\geqslant 0$ and $t\geqslant 0$,

$$\psi(x,t)=P\left(\sup_{1\leqslant k\leqslant N(t)}\left(\sum_{i=1}^k X_i-c\sum_{i=1}^k\theta_i\right)>x\right).$$

For any infinitely increasing function $f(x)$, we will prove

$$\psi(x,t)\gtrsim\frac{1}{v}\int_x^{x+vzt}\overline{F}(y)\mathrm{d}y \tag{6}$$

and

$$\psi(x,t)\lesssim\frac{1}{v}\int_x^{x+vzt}\overline{F}(y)\mathrm{d}y \tag{7}$$

hold uniformly for $t\in[f(x),\infty)$, respectively.

Firstly, we show the asymptotic upper bound (7).

For any $\varepsilon>0, x\geqslant 0$ and $t>0$, we have

$$\psi(x,t)=P\left(\sup_{1\leqslant k\leqslant N(t)}\left(\sum_{i=1}^{k}X_i-c\sum_{i=1}^{k}\theta_i\right)>x, N(t)\leqslant(1+\varepsilon)zt\right)$$
$$+P\left(\sup_{1\leqslant k\leqslant N(t)}\left(\sum_{i=1}^{k}X_i-c\sum_{i=1}^{k}\theta_i\right)>x, N(t)>(1+\varepsilon)zt\right)$$
$$=:\psi_1(x,t)+\psi_2(x,t).$$

For any $\delta\in(0,v/c)$, let

$$A=\sup_{1\leqslant k\leqslant(1+\varepsilon)zt}\sum_{i=1}^{k}\left(X_i-c\left(\frac{1}{z}-\delta\right)\right),$$

$$B=c\sup_{k\geqslant 1}\sum_{i=1}^{k}\left(\left(\frac{1}{z}-\delta\right)-\theta_i\right) \text{ and } B^+:=\max\{B,0\}.$$

It follows from the conditions of the risk model that A and B^+ are independent.

We first estimate $\psi_1(x,t)$. Let $C:=E\left(X_1-c(1/z-\delta)\right)=c\delta-v<0$. Therefore, for all $x\geqslant 0, y\in(0,x/2]$ and $t>0$,

$$\psi_1(x,t)\leqslant P\left(\sup_{1\leqslant k\leqslant(1+\varepsilon)zt}\sum_{i=1}^{k}\left(X_i-c\left(\frac{1}{z}-\delta\right)\right)\right.$$
$$\left.+c\sup_{k\geqslant 1}\sum_{i=1}^{k}\left(\left(\frac{1}{z}-\delta\right)-\theta_i\right)>x\right)$$
$$\leqslant P\left(A+B^+>x\right)$$
$$\leqslant\int_0^{x-y}P(A>x-u)P\left(B^+\in \mathrm{d}u\right)+P\left(B^+>x-y\right)$$
$$=:\psi_{11}(x,t)+\psi_{12}(x,t).$$

Using the line of the proof of Proposition 2.1 of Leipus and Šiaulys[3], we know that for all $x\geqslant 0, y\in(0,x/2]$ and $t>0$,

$$\int_0^{x-y}P(A>x-u)P\left(B^+\in \mathrm{d}u\right)$$
$$\leqslant\frac{(1+\alpha(y))}{|C|}\int_0^{x-y}\left(\int_{x-u}^{x-u+vzt(1+\varepsilon)}\overline{F}(v)\mathrm{d}v\right)P\left(B^+\in\mathrm{d}u\right),$$

where $\alpha(\cdot)$ is a positive function satisfying $\lim_{y\to\infty}\alpha(y)=0$. Let J denote the integral of the right side in the above inequality and G_{B^+} be the distribution of the random variable B^+. By Fubini's theorem, for all $x\geqslant 0, y\in(0,x/2]$ and $t>0$,

$$J=\int_0^{x-y}\left(\int_x^{x+vzt(1+\varepsilon)}\overline{F}(w-u)\mathrm{d}w\right)G_{B^+}(\mathrm{d}u)\leqslant\int_x^{x+vzt(1+\varepsilon)}\overline{F*G_{B^+}}(w)\mathrm{d}w.$$

Let $w_2=(1/z-\delta)^{-1}$ in (5). Since $0<\delta<v/c<1/z$, it knows that $w_2>z$. By (5) there exists a constant $\delta_2>0$ such that for sufficiently large x,

$$P(B>x)\leqslant\sum_{k=1}^{\infty}P\left(\sum_{i=1}^{k}\theta_i<k\left(\frac{1}{z}-\delta\right)-\frac{x}{c}\right)$$
$$\leqslant\sum_{k\geqslant\frac{xz/c}{1-z\delta}}P\left(\sum_{i=1}^{k}\theta_i<k\left(\frac{1}{z}-\delta\right)\right)$$
$$\leqslant\sum_{k\geqslant\frac{xz/c}{1-z\delta}}P\left(N\left(k\left(\frac{1}{z}-\delta\right)\right)\geqslant k\right)$$
$$\leqslant\sum_{k\geqslant\frac{xz/c}{1-z\delta}}\exp\left(-k\left(\frac{1}{z}-\delta\right)\left(I\left(\left(\frac{1}{z}-\delta\right)^{-1}\right)-\delta_2\right)\right)$$
$$\leqslant\frac{\exp\left(-\frac{x}{c}\left(I\left(\left(\frac{1}{z}-\delta\right)^{-1}\right)-\delta_2\right)\right)}{1-\exp\left(-\left(\frac{1}{z}-\delta\right)\left(I\left(\left(\frac{1}{z}-\delta\right)^{-1}\right)-\delta_2\right)\right)}$$
$$=:d_1\exp(-d_2x) \tag{8}$$

where

$$d_1=\left(1-\exp\left(-\left(\frac{1}{z}-\delta\right)\left(I\left(\left(\frac{1}{z}-\delta\right)^{-1}\right)-\delta_2\right)\right)\right)^{-1}>0,$$

and

$$d_2=c^{-1}\left(I\left(\left(\frac{1}{z}-\delta\right)^{-1}\right)-\delta_2\right)>0.$$

Thus for $x>0$,

$$\overline{G}_{B^+}(x)=P\left(B^+>x\right)=P(B>x)\leqslant d_1\exp(-d_2x).$$

Since $F\in\mathcal{S}_*\subset\mathcal{S}$, it holds that $\overline{G}_{B^+}(x)=o\left(\overline{F}(x)\right)$. By Corollary 3.18 of Foss *et al*[20],

$$\overline{F*G_{B^+}}(x)\sim\overline{F}(x).$$

Consequently, there exists a positive function $\beta(x)\to 0$ such that for sufficiently large x,

$$J\leqslant(1+\beta(x))\int_x^{x+vzt(1+\varepsilon)}\overline{F}(u)\mathrm{d}u$$
$$=(1+\beta(x))\int_x^{x+vzt}\overline{F}(u)\mathrm{d}u\left(1+\frac{\int_{x+vzt}^{x+vzt(1+\varepsilon)}\overline{F}(u)\mathrm{d}u}{\int_x^{x+vzt}\overline{F}(u)\mathrm{d}u}\right)$$
$$\leqslant(1+\beta(x))(1+\varepsilon)\int_x^{x+vzt}\overline{F}(u)\mathrm{d}u \tag{9}$$

So, for all $t>0, y\in(0,x/2]$ and sufficiently large x,

$$\int_0^{x-y}P(A>x-u)P\left(B^+\in\mathrm{d}u\right)$$
$$\leqslant\frac{(1+\alpha(y))(1+\beta(x))(1+\varepsilon)}{|C|}\int_x^{x+vzt}\overline{F}(u)\mathrm{d}u \tag{10}$$

By (10), it holds for all $t>0, y\in(0,x/2]$ and sufficiently large x that

$$\psi_{11}(x,t)\leqslant\frac{(1+\alpha(y))(1+\beta(x))(1+\varepsilon)}{|C|}\int_x^{x+vzt}\overline{F}(u)\mathrm{d}u,$$

which shows that

$$\limsup_{\varepsilon\to 0}\limsup_{\delta\to 0}\limsup_{x\to\infty}\sup_{t\in[f(x),\infty)}\frac{\psi_{11}(x,t)}{\frac{1}{v}\int_x^{x+vzt}\overline{F}(u)\mathrm{d}u}\leqslant 1 \quad (11)$$

In the following, we deal with $\psi_{12}(x,t)$. Using (8), for sufficiently large x, we have

$$P\left(B^{+}>x-y\right)\leqslant\sum_{k\geqslant\frac{(x-y)z}{c(1-z\delta)}}\exp\left(-k\left(\frac{1}{z}-\delta\right)\left(I\left(\left(\frac{1}{z}-\delta\right)^{-1}\right)-\delta_2\right)\right)$$
$$\leqslant d_1\exp(-d_2(x-y)).$$

Since $t\geqslant f(x)$ and $f(x)\uparrow\infty$ as $x\to\infty$, for sufficiently large x, it holds that $t\geqslant 1/vz$. Therefore, by $F\in\mathcal{S}_*\subset\mathcal{S}\subset\mathcal{L}$, for all $t\geqslant f(x)$ and $y\in(0,x/2]$, it holds that

$$\frac{P\left(B^{+}>x-y\right)}{\int_x^{x+vzt}\overline{F}(u)\mathrm{d}u}\leqslant\frac{P\left(B^{+}>\frac{x}{2}\right)}{\int_x^{x+1}\overline{F}(u)\mathrm{d}u}\leqslant\frac{d_1\mathrm{e}^{-d_2x/2}}{\overline{F}(x+1)}\to 0 \quad (12)$$

Combining with (12) yields that

$$\limsup_{x\to\infty}\sup_{t\in[f(x),\infty)}\frac{\psi_{12}(x,t)}{\frac{1}{v}\int_x^{x+vzt}\overline{F}(y)\mathrm{d}y}=0 \quad (13)$$

By (11) and (13), we get

$$\limsup_{\varepsilon\to 0}\limsup_{x\to\infty}\sup_{t\in[f(x),\infty)}\frac{\psi_{1}(x,t)}{\frac{1}{v}\int_x^{x+vzt}\overline{F}(u)\mathrm{d}u}\leqslant 1 \quad (14)$$

Next, we will estimate the asymptotic upper bound of $\psi_2(x,t)$. We first deal with the process $\{N(t),t\geqslant 0\}$. By (5), for $\widetilde{w_2}=(1+\varepsilon)z>z$, there exists $\widetilde{\delta_2}>0$ such that $I((1+\varepsilon)z)-\widetilde{\delta_2}>0$ and for all $t\in\left[f(x),\infty\right),\gamma>0$ and sufficiently large x,

$$\sum_{n>(1+\varepsilon)zt}(1+\gamma)^n P(N(t)\geqslant n)\leqslant\sum_{n>(1+\varepsilon)zt}(1+\gamma)^n P\left(N\left(\frac{n}{(1+\varepsilon)z}\right)\geqslant n\right)$$
$$\leqslant\sum_{n>(1+\varepsilon)zt}(1+\gamma)^n\exp\left(-\frac{n}{(1+\varepsilon)z}\left(I\left((1+\varepsilon)z\right)-\widetilde{\delta_2}\right)\right) \quad (15)$$

It holds for all $x\geqslant 0$ and $t\geqslant 0$ that

$$\psi_2(x,t)\leqslant\sum_{n>(1+\varepsilon)zt}P\left(\sup_{1\leqslant k\leqslant n}\sum_{i=1}^{k}X_i>x,N(t)=n\right)$$
$$=\sum_{n>(1+\varepsilon)zt}\overline{F^{*n}}(x)P\left(N(t)=n\right) \quad (16)$$

Since $F\in\mathcal{S}_*\subset\mathcal{S}$, by Kesten's bound, for $0<\gamma<\exp\left(\frac{\left(I\left((1+\varepsilon)z\right)-\widetilde{\delta_2}\right)}{\left((1+\varepsilon)z\right)}\right)-1$, there exists $l=l(\gamma)>0$ such that for any $x\geqslant 0$ and $n\geqslant 1$,

$$\overline{F^{*n}}(x)\leqslant l(1+\gamma)^n\overline{F}(x) \quad (17)$$

Therefore, by $F\in\mathcal{S}_*\subset\mathcal{L}$ and (15)-(17),

$$\limsup_{x\to\infty}\sup_{t\in[f(x),\infty)}\frac{\psi_2(x,t)}{\int_x^{x+vzt}\overline{F}(u)\mathrm{d}u}\leqslant\limsup_{x\to\infty}\sup_{t\in[f(x),\infty)}\frac{\overline{F}(x)}{\int_x^{x+vzt}\overline{F}(u)\mathrm{d}u}$$
$$\cdot\limsup_{x\to\infty}\sup_{t\in[f(x),\infty)}l\sum_{n>(1+\varepsilon)zt}(1+\gamma)^n\exp\left(-\frac{n}{(1+\varepsilon)z}\left(I\left((1+\varepsilon)z\right)-\widetilde{\delta_2}\right)\right)$$
$$\leqslant\limsup_{x\to\infty}\frac{\overline{F}(x)}{\overline{F}(x+1)}$$
$$\cdot\limsup_{x\to\infty}\sup_{t\in[f(x),\infty)}l\sum_{n>(1+\varepsilon)zt}(1+\gamma)^n\exp\left(-\frac{n}{(1+\varepsilon)z}\left(I\left((1+\varepsilon)z\right)-\widetilde{\delta_2}\right)\right)$$
$$=0 \quad (18)$$

Thus, by (14) and (18),

$$\limsup_{x\to\infty}\sup_{t\in[f(x),\infty)}\frac{\psi(x,t)}{\frac{1}{v}\int_x^{x+vzt}\overline{F}(u)\mathrm{d}u}\leqslant 1.$$

This completes the proof of (7).

Next we prove the asymptotic lower bound (6). For any $\varepsilon>0$, by (4), for $w_1=\frac{z}{1+z\varepsilon}<z$, there exists $\delta_1>0$ such that $I\left(\frac{z}{1+z\varepsilon}\right)-\delta_1>0$ and for sufficiently large M,

$$P\left(c\sup_{k\geqslant 1}\sum_{i=1}^{k}\left(\theta_i-(1/z+\varepsilon)\right)\geqslant M\right)$$
$$\leqslant\sum_{k=1}^{\infty}P\left(\sum_{i=1}^{k}\theta_i\geqslant\frac{M+kc(1/z+\varepsilon)}{c}\right)$$
$$\leqslant\sum_{k=1}^{\infty}P\left(N\left(\frac{M+kc(1/z+\varepsilon)}{c}\right)\leqslant k\right)$$
$$\leqslant\sum_{k=1}^{\infty}P\left(N\left(\frac{M+kc(1/z+\varepsilon)}{c}\right)\leqslant\frac{M+kc(1/z+\varepsilon)}{c}\cdot\frac{z}{1+z\varepsilon}\right)$$
$$\leqslant\sum_{k=1}^{\infty}\exp\left(-\frac{M+kc(1/z+\varepsilon)}{c}\left(I\left(z/(1+z\varepsilon)\right)-\delta_1\right)\right)$$
$$=\exp\left(-\frac{M}{c}\left(I\left(z/(1+z\varepsilon)\right)-\delta_1\right)\right)$$
$$\cdot\sum_{k=1}^{\infty}\exp\left(-k(1/z+\varepsilon)\left(I\left(z/(1+z\varepsilon)\right)-\delta_1\right)\right)$$
$$\to 0\quad\text{as}\quad M\to\infty \quad (19)$$

Thus, by (19), for any $0<\varepsilon<1$,

$$\liminf_{M\to\infty}P\left(c\inf_{k\geqslant 1}\sum_{i=1}^{k}\left(\frac{1}{z}+\varepsilon-\theta_i\right)>-M\right)$$
$$=\liminf_{M\to\infty}P\left(c\sup_{k\geqslant 1}\sum_{i=1}^{k}\left(\theta_i-\left(\frac{1}{z}+\varepsilon\right)\right)<M\right)=1 \quad (20)$$

For the above $\varepsilon>0$, let $\tilde{v}:=c(1/z+\varepsilon)-EX_1$, then $\tilde{v}>0$ and $\tilde{v}\to v$ as $\varepsilon\to 0$. Thus, for the above $\varepsilon>0$ and $M>0$, and for all $t\in[f(x),\infty)$, by Lemma 1,

$$\psi(x,t)$$
$$\geqslant P\left(\sup_{1\leqslant k\leqslant N(t)}\sum_{i=1}^{k}\left(X_i-c\left(\frac{1}{z}+\varepsilon\right)\right)+c\inf_{k\geqslant 1}\sum_{i=1}^{k}\left(\frac{1}{z}+\varepsilon-\theta_i\right)>x\right)$$

$$\geqslant P\left(\sup_{1\leqslant k\leqslant N(t)}\sum_{i=1}^{k}\left(X_i-c\left(\frac{1}{z}+\varepsilon\right)\right)>x+M,\right.$$

$$\left. c\inf_{k\geqslant 1}\sum_{i=1}^{k}\left(\frac{1}{z}+\varepsilon-\theta_i\right)>-M\right)$$

$$\geqslant \sum_{n\geqslant(1-\varepsilon)zt} P\left(\sup_{1\leqslant k\leqslant n}\sum_{i=1}^{k}\left(X_i-c\left(\frac{1}{z}+\varepsilon\right)\right)>x+M\right)$$

$$\times P\left(c\inf_{k\geqslant 1}\sum_{i=1}^{k}\left(\frac{1}{z}+\varepsilon-\theta_i\right)>-M, N(t)=n\right)$$

$$\gtrsim \sum_{n\geqslant(1-\varepsilon)zt}\frac{1}{\tilde{v}}\inf_{s>x}\frac{\overline{F}\left(s+M+c\left(\frac{1}{z}+1\right)\right)}{\overline{F}(s)}\int_x^{x+\tilde{v}(1-\varepsilon)zt}\overline{F}(y)\mathrm{d}y$$

$$\times P\left(c\inf_{k\geqslant 1}\sum_{i=1}^{k}\left(\frac{1}{z}+\varepsilon-\theta_i\right)>-M, N(t)=n\right)$$

$$\gtrsim (1-\varepsilon)P\left(c\inf_{k\geqslant 1}\sum_{i=1}^{k}\left(\frac{1}{z}+\varepsilon-\theta_i\right)>-M, N(t)\geqslant(1-\varepsilon)zt\right)$$

$$\times\frac{1}{\tilde{v}}\int_x^{x+vzt}\overline{F}(y)\mathrm{d}y \qquad (21)$$

where in the last step, we have used $F\in\mathcal{S}_*\subset\mathcal{L}$ and the inequality

$$\int_a^c g(x)\mathrm{d}x\leqslant\frac{c-a}{b-a}\int_a^b g(x)\mathrm{d}x,$$

where $a\leqslant b\leqslant c$ are some constants and $g(x)$ is a non-increasing function on $[a,c]$.

By (4), $\widetilde{w_1}=(1-\varepsilon)z<z$, there exists $\widetilde{\delta_1}>0$ such that $I\left((1-\varepsilon)z\right)-\tilde{\delta}_1>0$ and for sufficiently large t,

$$P\left(N(t)<(1-\varepsilon)zt\right)\leqslant\exp\left(-t\left(I((1-\varepsilon)z)-\widetilde{\delta_1}\right)\right)$$

$$\to 0 \quad \text{as} \quad t\to\infty \qquad (22)$$

Since

$$P\left(c\inf_{k\geqslant 1}\sum_{i=1}^{k}\left(\frac{1}{z}+\varepsilon-\theta_i\right)>-M, N(t)\geqslant(1-\varepsilon)zt\right)$$

$$\geqslant P\left(c\inf_{k\geqslant 1}\sum_{i=1}^{k}\left(\frac{1}{z}+\varepsilon-\theta_i\right)>-M\right)-P\left(N(t)<(1-\varepsilon)zt\right),$$

which combining with (20) and (22) yields that

$$\liminf_{M\to\infty}\liminf_{x\to\infty}\inf_{t\in[f(x),\infty)}P\left(c\inf_{k\geqslant 1}\sum_{i=1}^{k}\left(\frac{1}{z}+\varepsilon-\theta_i\right)\right.$$

$$\left.>-M, N(t)\geqslant(1-\varepsilon)zt\right)=1 \qquad (23)$$

By (21) and (23), letting $M\to\infty$ and $\varepsilon\to 0$, it holds that

$$\liminf_{x\to\infty}\inf_{t\in[f(x),\infty)}\frac{\psi(x,t)}{\frac{1}{v}\int_x^{x+vzt}\overline{F}(y)\mathrm{d}y}\geqslant 1.$$

This completes the proof of (6).

2 Two-Dimensional Risk Model

In this section, we will apply Theorem 1 to deal with a two-dimensional risk model and derive the asymptotics of the finite-time ruin probability of a two-dimensional risk model.

2.1 Risk Model

In recent years, more and more scholars begin to study different two-dimensional risk models. In this section, we consider the following two-dimensional risk model in which the surplus at time $t\geqslant 0$ is described as

$$\begin{pmatrix}U_1(t)\\U_2(t)\end{pmatrix}=\begin{pmatrix}x_1\\x_2\end{pmatrix}+\begin{pmatrix}c_1 t\\c_2 t\end{pmatrix}-\begin{pmatrix}\sum_{i=1}^{N(t)}X_{1i}\\\sum_{i=1}^{N(t)}X_{2i}\end{pmatrix} \qquad (24)$$

where $\vec{\boldsymbol{x}}=(x_1,x_2)^{\mathrm{T}}$ is the initial surplus vectors; $\vec{\boldsymbol{c}}=(c_1,c_2)^{\mathrm{T}}$ is the vector of constant premium rates; the claim size vectors $\left\{\left(X_{1i},X_{2i}\right),i\geqslant 1\right\}$ are i. i. d. copies of $\left(X_1,X_2\right)$ with nonnegative independent component and marginal distributions $F_i, i=1,2,$ respectively; $\left\{\tau_i, i\geqslant 1\right\}$ are the claim-arrival times, which constitute the claim-number process $\left\{N(t),t\geqslant 0\right\}$.

The claim inter-arrival times $\left\{\theta_i=\tau_i-\tau_{i-1}, i\geqslant 2, \theta_1=\tau_1\right\}$ are independent of $\{(X_{1i},X_{2i}),i\geqslant 1\}$. For the risk model (24), some kinds of finite-time ruin probabilities up to time $t\geqslant 0$ are defined as

$$\psi_{\max}(\vec{x},t)=P\left(t_{\max}\leqslant t|U_i(0)=x_i, i=1,2\right),$$

where $t_{\max}=\inf\{s\geqslant 0:\max\{U_1(s),U_2(s)\}<0\}$ and

$$\psi_{\mathrm{sum}}(\vec{x},t)=P\left(t_{\mathrm{sum}}\leqslant t|U_i(0)=x_i, i=1,2\right) \qquad (25)$$

where $t_{\mathrm{sum}}=\inf\{s\geqslant 0: U_1(s)+U_2(s)<0\}$.

In some earlier works on the asymptotics of finite-time ruin probabilities, an important assumption is that the two kinds of businesses share a common claim-number process and the inter-arrival times are independent or have some dependence structure, see, e.g., Li *et al*[21], Chen *et al*[7], Chen *et al*[8], Lu *et al*[22] and so on. Recently many researchers have paid more attention to some generalizations of risk model (24), such as a risk model with a constant force of interest or stochastic return, see, e.g., Konstantinides *et al*[23], Li *et al*[24], Li[25], Yang *et al*[26], Cheng and Yu[27], Cheng *et al*[28], Yang *et al*[29] and so on.

Recently, Fu and Li[10] considered the risk model (24) sharing a common claim-number process satisfying the LDP (i.e. Assumption A). They obtained the uniform asymptotics of the finite-time ruin probability $\psi_{\max}(\vec{x}, t)$ for the claim sizes belonging to the class $\mathcal{C}$. In the following we still consider the risk model (24) with a claim-number process $\{N(t), t \geqslant 0\}$, which satisfies the LDP and investigate the uniform asymptotics of the finite-time ruin probability $\psi_{\text{sum}}(\vec{x}, t)$ for the strongly subexponential claim sizes by using Theorem 1.

For the risk model (24), we assume that $\{X_{1i}, i \geqslant 1\}$, $\{X_{2i}, i \geqslant 1\}$ and $\{N(t), t \geqslant 0\}$ are independent. The following is the main result of this section.

Theorem 2 Consider the two-dimensional risk model (24). Suppose that Assumption A holds. If

$$F_i \in \mathcal{S}_*, i=1,2, \overline{F}_1(x)=O\left(\overline{F}_2(x)\right)\left(\text{or } \ \overline{F}_2(x)=O\left(\overline{F}_1(x)\right)\right)$$

and

$$v := (c_1+c_2)/z - E\left(X_1+X_2\right) > 0$$

then

$$\psi_{\text{sum}}(\vec{x}, t) \sim \frac{1}{v}\int_{x_1+x_2}^{x_1+x_2+vzt}\left(\overline{F}_1(y)+\overline{F}_2(y)\right)\mathrm{d}y \tag{26}$$

holds uniformly for $t \in [f(x_1+x_2), \infty)$ as $x_1+x_2 \to \infty$, where $f: [0,\infty) \to [0,\infty)$ is an infinitely increasing function.

The proof of the main result will be given in the following subsection.

2.2 Proof of Theorem 2

The following lemma is crucial to prove Theorem 2.

Lemma 2 Let ξ and η be nonnegative random variables with distributions V and W, respectively. If $V, W \in \mathcal{S}_*$ and $\overline{V}(x)=O\left(\overline{W}(x)\right)$, then $V*W \in \mathcal{S}_*$ and

$$\overline{V*W}(x) \sim \overline{V}(x)+\overline{W}(x) \tag{27}$$

Proof Since $\mathcal{S}_* \subset \mathcal{S}$, by Corollary 3.16 of Foss *et al* [20] we know that (27) holds. Hence, by (27) for sufficiently large x,

$$\overline{(V*W)_u}(x) = \int_x^{x+u}\overline{V*W}(y)\mathrm{d}y \sim \int_x^{x+u}\overline{V}(y)\mathrm{d}y + \int_x^{x+u}\overline{W}(y)\mathrm{d}y$$
$$= \overline{V_u}(x)+\overline{W_u}(x) \tag{28}$$

holds uniformly for $u \in [1, \infty)$.

According to the definition of $\mathcal{S}_*$, we get that $V_u \in \mathcal{S} \subset \mathcal{L}$ and $W_u \in \mathcal{S} \subset \mathcal{L}$. So $\overline{V_u}+\overline{W_u}$ is long-tailed. It follows from $\overline{V}(x)=O\left(\overline{W}(x)\right)$ that $\overline{V_u}(x)=O\left(\overline{W_u}(x)\right)$. Again using Corollary 3.16 of Foss *et al* [20], by (28) we have $V_u * W_u \in \mathcal{S}$ and

$$\overline{V_u*W_u}(x) \sim \overline{V_u}(x)+\overline{W_u}(x) \sim \overline{(V*W)_u}(x).$$

Thus, $(V*W)_u \in \mathcal{S}$ by Corollary 3.13 of Foss *et al* [20], which means $V*W \in \mathcal{S}_*$.

Proof of Theorem 2 Note that

$$\psi_{\text{sum}}(\vec{x}, t) = P\left(\sup_{1\leqslant k\leqslant N(t)}\left(\sum_{i=1}^{k}(X_{1i}+X_{2i}) - (c_1+c_2)\sum_{i=1}^{k}\theta_i\right) > x_1+x_2\right)$$

Since $F_i \in \mathcal{S}_*, i=1,2$ and $\overline{F}_1(y)=O\left(\overline{F}_2(y)\right)$, by Lemma 2 we get $F_1*F_2 \in \mathcal{S}_*$ and

$$\overline{F_1*F_2}(x) \sim \overline{F}_1(x)+\overline{F}_2(x) \tag{29}$$

Thus, by Theorem 1 and (29) for sufficiently large x_1+x_2, it holds uniformly for $t \in [f(x_1+x_2), \infty)$ that

$$\psi_{\text{sum}}(\vec{x}, t) \sim \frac{1}{v}\int_{x_1+x_2}^{x_1+x_2+vzt}\overline{F_1*F_2}(y)\mathrm{d}y$$
$$\sim \frac{1}{v}\int_{x_1+x_2}^{x_1+x_2+vzt}\left(\overline{F}_1(y)+\overline{F}_2(y)\right)\mathrm{d}y.$$

This completes the proof of Theorem 2.

3 Conclusion

We consider the risk models with a non-stationary claim-number process and obtain the uniform asymptotics for the finite-time ruin probability of a one-dimensional risk model for the strongly subexponential claim sizes when the claim-number process satisfies the large deviations principle. Further, applying Theorem 1 the uniform asymptotics for a kind of finite-time ruin probability in a two-dimensional risk model have been presented.

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