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Optimal Asymmetric Quantum Codes from the Euclidean Sums of Linear Codes

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Abstract: In this paper, we first give the definition of the Euclidean sums of linear codes, and prove that the Euclidean sums of linear codes are Euclidean dual-containing. Then we construct two new classes of optimal asymmetric quantum error-correcting codes based on Euclidean sums of the Reed-Solomon codes, and two new classes of optimal asymmetric quantum error-correcting codes based on Euclidean sums of linear codes generated by Vandermonde matrices over finite fields. Moreover, these optimal asymmetric quantum error-correcting codes constructed in this paper are different from the ones in the literature.

Key words: Euclidean sums of linear codes; optimal asymmetric quantum errorcorrecting codes; vandermonde matrices; Reed-Solomon codes

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0 Introduction

Quantum error-correcting codes play an important role in quantum computing and quantum communication. Shor[1] and Steane[2] first investigated quantum error-correcting codes. Calderbank *et al*[3] established the connections between quantum error-correcting codes and classical codes. The establishment showed that quantum error-correcting codes can be constructed from classical linear codes with dual containing properties.

Asymmetric quantum error-correcting (AQEC) codes are quantum codes defined over quantum channels where qudit-flip errors and phase-shift errors may have different probabilities. In many quantum mechanical systems, the probabilities of occurrence of qudit-flip and phase-shift errors are quite different[4]. Wang *et al*[5] studied the characterization and constructions of AQEC codes. La Guardia[6, 7] utilized classical Bose-Chaudhuri-Hocquenghem (BCH) codes to construct new classes of AQEC codes. Later, several classes of optimal AQEC codes have been constructed[8-15]. Chen *et al*[8] studied optimal AQEC codes by using negacyclic codes. In Ref. [11], Chen *et al* constructed some classes of optimal AQEC codes from constacyclic codes. Wang *et al*[13] also constructed six classes of new optimal AQEC codes from dual-containing constacyclic codes over finite fields by using the Cascading Style Sheets (CSS) construction. Recently, Xu *et al*[14] obtained two new classes

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of optimal asymmetric quantum codes from constacyclic codes. One of them has length $n=\frac{q^2+1}{5}$ and $d_z>q+1$, where q is an odd prime power with the form $10m+3$ or $10m+7$ ($m\geqslant 0$ is integer). In Ref. [10], some classes of optimal AQEC codes was constructed by utilizing constacyclic codes with length $n=\frac{q^2+1}{10h}$, where q is an odd prime power with the form $q=10hm+t$ or $q=10hm+10h-t$, where m is a positive integer, and both h and t are odd with $10h=t^2+1$ and $t\geqslant 3$.

In the above work, researchers constructed AQEC codes by using constacylic codes, negacylic codes, and generalized Reed-Solomon codes. In this paper, we construct four classes of optimal AQEC codes by using the Euclidean sums of the Reed-Solomon codes and linear codes generated by Vandermonde matrices as follows:

1) Let $n=q-1$, $1\leqslant\delta<q-1$, and $2\delta-1\leqslant k<\left\lceil\frac{n+2\delta-1}{2}\right\rceil$. Then there exists a class of optimal AQEC codes Q with parameters $\left[\left[n, 2\delta-1, k-2\delta+\frac{2}{n}-k+1\right]\right]_q$.

2) Let $n\mid q-1, n\geqslant 2k-1$ and $k\geqslant 1$. Then there exists a class of optimal AQEC codes Q with parameters $\left[\left[n, 1, n-k+\frac{1}{k}\right]\right]_q$.

3) Let $n\mid q-1, n\geqslant 2k-1$ and $k\geqslant 1$. Then there exists a class of optimal AQEC codes Q with parameters $\left[\left[n, n-2k+1, k+\frac{1}{k}\right]\right]_q$.

We mention that the optimal AQEC codes in the constructions of 1), 2) and 3) are new in the sense that their parameters are not covered by the codes available in the literature and many of the new codes have large minimum distance.

This paper is outlined as follows. In Section 1, we first recall some basic knowledge on linear codes and cyclic codes. Then we define the Euclidean sums of linear codes, and prove that the Euclidean sums of linear codes are Euclidean dual-containing. In Section 2, we briefly review some basic facts of AQEC codes. In Sections 3 and 4, we construct two new class of optimal AQEC codes by using Euclidean sums of Reed-Solomon codes, and two new classes of optimal AQEC codes by using the Euclidean sums of linear codes generated by Vandermonde matrices. Finally, a brief summary of this work is described in Section 5.

1 Preliminaries

In this section, we are going to give some basic concepts and results about linear codes that are needed in the rest of this paper. Throughout this paper, let F_q be the finite field with q elements, where q is a prime power. For a positive integer n, let F_q denote the vector space of all n-tuples over F_q. A linear $[n,k]_q$ code C over F_q is a k-dimensional subspace of F_q. The Hamming weight wt(c) of a codeword $c\in C$ is the number of nonzero components of c. The Hamming distance of two codewords $c_1, c_2\in C$ is $d(c_1,c_2)=\mathrm{wt}(c_2-c_1)$. The minimum Hamming distance $d(C)$ of C is the minimum Hamming distance between any two distinct codewords of C. An $[n,k,d]_q$ code is an $[n,k]_q$ code with the minimum Hamming distance d.

A linear code C with parameters $[n,k,d]_q$ over F_q is called a maximum distance separable (MDS) code if it satisfies $d=n-k+1$ (see Ref. [16]). For two vectors $\boldsymbol{a}=(a_1,a_2,\cdots,a_n)$ and $\boldsymbol{b}=(b_1,b_2,\cdots,b_n)$ in F_q, we define the Euclidean inner product $[\boldsymbol{a},\boldsymbol{b}]$ to be $[\boldsymbol{a},\boldsymbol{b}]=\sum_{i=1}^{n}a_ib_i$. For a linear $[n,k]_q$ code C over F_q, we define the Euclidean dual code as $C^{\perp}=\{b\in F_q \mid [\boldsymbol{a},\boldsymbol{b}]=0 \text{ for all } \boldsymbol{a}\in C\}$.

Definition 1 Let C_1 and C_2 be two linear codes of length n over F_q. Then $C_1+C_2=\{c_1+c_2 \mid c_1\in C_1, c_2\in C_2\}$ is called the sum of C_1 and C_2. The Euclidean sum of a linear code C over F_q is defined to be $\mathrm{Sum}(C)=C+C^{\perp}$.

Theorem 1 If C is a linear code over F_q, we have

1) $\mathrm{Sum}(C)^{\perp}=C\cap C^{\perp}$;

2) $\mathrm{Sum}(C)^{\perp}\subset C$, and $\mathrm{Sum}(C)\subset C^{\perp}$.

Proof 1) is a result from Ref. [17]. According to 1), 2) is obvious.

A linear code of length n over F_q is cyclic if the code invariant under the automorphism τ and $\tau(c_0,c_1,\cdots,c_{n-1})=(c_{n-1},c_0,c_1,\cdots,c_{n-2})$. Let i be an integer such that $0\leqslant i\leqslant n-1$, and let l be the smallest positive integer such that $iq^l\equiv i(\bmod\ n)$. Then $C_i=\{i, iq,\cdots,iq^{l-1}\}$ is the q-cyclotomic coset module n containing i. Since q is coprime with n, the irreducible factors of x^n-1 in $F_q[x]$ can be described by the q-cyclotomic cosets. Suppose that α is a primitive n-th root of unity over some extension field of F_q, and let $M_j(x)$ be the minimal polynoial of α^j with resect to F_q. Let $\{s_1,s_2,\cdots,s_t\}$ be a complete set of representatives of q-cyclotomic cosets. Then the polynomial x^n-1 factors uniquely into monic irreducible polynomial in $F_q[x]$ as $x^n - 1 = \prod_{j=1}^{t} M_{sj}(x)$ (see Ref. [18]).

The defining set of the cyclic code $C = \langle f(x) \rangle$ is defined as $Z(C)=\{i \in Z_n | f(\alpha^i)=0\}$. Obviously, the defining set $Z(C)$ is a union of some q-cyclotomic cosets and $\dim(C)=n-|Z(C)|$. The following BCH bound for cyclic codes can be found in Refs. [19, 20].

Theorem 2 (The BCH bound for cyclic codes) Suppose that $\gcd(q,n)=1$. If the defining set of a cyclic code C of length n over F_q contains a subset $\{i | i=h, h+1, \cdots, h+\delta-1\}$, then the minimum distance of C is at least δ.

2 Some Basic Facts of AQEC Codes

In this section, we first introduce the definition of asymmetric quantum codes which can be found in Ref. [4]. Then we give the well-known CSS construction and Singleton bound for AQEC codes. More details about AQEC codes theory, please refer to Refs. [5-9, 13-15, 21].

Let V_n be the Hilbert space $V_n = C^{q^n} = C^q \otimes \cdots \otimes C^q$. Let $|\boldsymbol{x}\rangle$ be the vectors of an orthonormal basis of C^{q^n}, where the labels x are elements of F_q. Then V_n has the following orthonormal basis

$$\{|c\rangle = |c_1 c_2 \cdots c_n\rangle = |c_1\rangle \otimes |c_2 \otimes \cdots \otimes |c_n\rangle : c = (c_1, c_2, \cdots c_n) \in F_q^n\}$$

For $\boldsymbol{a}, \boldsymbol{b} \in F_q$, the unitary linear operators $X(\boldsymbol{a})$ and $Z(\boldsymbol{b})$ in C^q are defined by $X(\boldsymbol{a})|\boldsymbol{x}\rangle = |\boldsymbol{x}+\boldsymbol{a}\rangle$ and $Z(\boldsymbol{b})|\boldsymbol{x}\rangle = w^{\mathrm{tr}(\boldsymbol{bx})}|\boldsymbol{x}\rangle$, respectively, where $w = \exp(\frac{2\pi i}{p})$ is a primitive p-th root of unity and tr is the trace map from F_q to F_p.

Let $\boldsymbol{a}=(a_1, \cdots, a_n) \in F_q^n$, we write $X(\boldsymbol{a}) = X(a_1) \otimes \cdots \otimes X(a_n)$ and $Z(a)=Z(a_1) \otimes \cdots \otimes Z(a_n)$ for the tensor products of n error operators. The set $E_n = \{X(\boldsymbol{a})Z(\boldsymbol{b}) : \boldsymbol{a}, \boldsymbol{b} \in F_q^n\}$ is an error basis on the complex vector space C^{q^n} and we set $G_n = \{w^c X(\boldsymbol{a})Z(\boldsymbol{b}) : \boldsymbol{a}, \boldsymbol{b} \in F_q^n, c \in F_p\}$ is the error group associated with E_n.

For a quantum error $\alpha = w^c X(\boldsymbol{a})Z(\boldsymbol{b}) \in G_n$, the quantum weight $w_Q(\alpha)$, the X-weight $w_X(\alpha)$ and the Z-weight $w_Z(\alpha)$ of α are defined as:

$$w_Q(\alpha) = |\{i : 1 \leqslant i \leqslant n, (a_i, b_i) \neq (0,0)\}|,$$

$$w_X(\alpha) = |\{i : 1 \leqslant i \leqslant n, a_i \neq 0\}|,$$

$$w_Z(\alpha) = |\{i : 1 \leqslant i \leqslant n, b_i \neq 0\}|$$

Definition 2 An AQEC code Q of length n, denoted by $[[n, k, d_z/d_x]]_q$, is a q^k -dimensional subspace of the Hilbert space V_n and can control all qubit-flip errors up to $\left\lfloor \frac{d_x - 1}{2} \right\rfloor$ and all phase-flip errors up to $\left\lfloor \frac{d_z - 1}{2} \right\rfloor$. The code Q also detects $d_x - 1$ qubit-flip errors as well as detects $d_z - 1$ phase-shift errors.

From the classical linear codes, we can directly obtain a family of AQEC codes by using the called CSS given by the following theorem[4].

Theorem 3 (CSS Code Construction) Let C_1 and C_2 be two classical linear codes over F_q with parameters $[n, k_1, d_1]_q$ and $[n, k_2, d_2]_q$, respectively. If $C_1^{\perp} \subset C_2$, then there exists an AQEC code with parameters $[[n, k_1 + k_2 - n, d_z/d_x]]_q$, where $d_x = \mathrm{wt}(C_1 \backslash C_2^{\perp})$, $d_z = \mathrm{wt}(C_2 \backslash C_1^{\perp})$.

To see that an AQEC code Q is good in terms of its parameters, we give a bound for AQEC codes similar to the quantum Singleton bound[4].

Lemma 1 (Ref. [4], Lemma 3.3) Let Q be an AQEC code with parameters $[n, k, d_z/d_x]_q$. Then $d_x + d_z \leqslant n - k + 2$.

If an AQEC code with parameters $[n, k, d_z/d_x]_q$ attains the AQEC Singleton bound, i. e. $d_x + d_z = n - k + 2$, then it is called an optimal AQEC code.

3 New Optimal AQEC Codes from Reed-Solomon Codes

In this section, we give two classes of optimal AQEC codes from the Euclidean sums of Reed-Solomon codes.

We assume $\delta \geqslant 0$ and $1 \leqslant k \leqslant q-1$. A Reed-Solomon code (RS code) is a cyclic code of length $q-1$ generated by $f(x) = (x - \omega^{\delta})(x - \omega^{\delta+1}) \cdots (x - \omega^{\delta+n-k-1})$, denoted by $\mathrm{RS}(n, k, \delta)$, where ω is a primitive element of F_q[18].

Remark 1 It is easy to prove that $\mathrm{RS}(n,k,\delta)^{\perp} = \mathrm{RS}(n, n-k, n-\delta+1)$. Thus, $Z(\mathrm{RS}(n,k,\delta)^{\perp}) = \{n-\delta+1, n-\delta+2, \cdots, n-\delta+k\}$. By Ref. [16], Exercise 239, Chapter 8, we have the following lemma.

Lemma 2 Let C be cyclic code with defining set $Z(C)$. Then the defining set of $\mathrm{Sum}(C)$ is given by $Z(C) \cap Z(C^{\perp})$.

Theorem 4 If $n = q-1, \delta \geqslant 1$, and $2\delta - 1 \leqslant k <$

$\left\lceil \frac{n+2\delta-1}{2} \right\rceil$, then there exists an optimal AQEC code Q with parameters $\left[\left[n, 2\delta-1, k-2\delta+2/n-k+1\right]\right]_q$.

Proof Suppose that $C=\mathrm{RS}(n,k,\delta)$. Then we have $Z(C)=\{\delta, \delta+1, \cdots, n+\delta-k-1\}$, and C is an Maximum Distance Separable (MDS) code with parameter $\left[n,k,n-k+1\right]_q$. By Remark 1, we have $Z(C^{\perp})=\{n-\delta+1, n-\delta+2, \cdots, n-\delta+k\}$, and $C^{\perp}$ is an MDS code with parameter $\left[n,n-k,k+1\right]_q$.

By $k \geqslant 2\delta-1$, we have $n+\delta-k-1 \leqslant n-\delta < n-\delta+1$. Then the first element in the defining set of $Z(C^{\perp})$ comes after the last element in $Z(C)$. Since $\delta \geqslant 1$, $k \geqslant 2\delta-1 \geqslant \delta$, we rewrite $Z(C^{\perp})$ as $Z(C^{\perp})=\{-\delta+1, -\delta+2, \cdots, -1, 0, 1, \cdots, k-\delta\}$. Then, by $2\delta-1 \leqslant k < \left\lceil \frac{n+2\delta-1}{2} \right\rceil$, we have $Z(C) \cap Z(C^{\perp})=\{\delta, \delta+1, \cdots, k-\delta\}$. According to Theorem 2, the code Sum(C) is an MDS code with parameters $\left[n, n-k+2\delta-1, k-2\delta+2\right]_q$. In addition, Sum($C^{\perp}$) is an MDS code with parameters $\left[n, k-2\delta+1, n-k+2\delta\right]_q$. Take $C_1=\mathrm{Sum}(C)$ and $C_2=\mathrm{RS}(n,k,\delta)$. Then we have $C_1^{\perp} \subset C_2$ by Theorem 1. Since $\delta \geqslant 1$, we have $k+1 > k-2\delta+2$ and $n-k+2\delta > n-k+1$. Thus $d_x = d(C_1 \backslash C_2^{\perp}) = k-2\delta+2$ and $d_z = d(C_2 \backslash C_1^{\perp}) = n-k+1$.

According to Theorem 3, there exists an AQEC code Q with parameters $[[n, 2\delta-1, k-2\delta+2/n-k+1]]_q$. Again by $d_x+d_z=n-2\delta+3=n-(2\delta-1)+2$, we know that the AQEC code Q with parameters $[[n, 2\delta-1, k-2\delta+2/n-k+1]]_q$ is optimal.

Remark 2 In Theorem 4, taking $q=9, \delta=1$, we obtain new optimal AQEC codes with parameters $[[8, 1, k/9-k]]_q$, where $1 \leqslant k \leqslant 4$.

4 Construction of AQEC Codes from Linear Codes Generated by Vandermonde Matrices

In this section, we construct two classes of optimal AQEC codes by using Vandermonde matrices over F_q.

A Vandermonde $n\times n$ matrix is a matrix of the form

$$\boldsymbol{V}_n = \begin{pmatrix} 1 & a_1 & a_1^2 & \cdots & a_1^{n-1} \\ 1 & a_2 & a_2^2 & \cdots & a_2^{n-1} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & a_n & a_n^2 & \cdots & a_n^{n-1} \end{pmatrix}$$

where $a_1, a_2, \cdots, a_n, a^n$ are elements of F_q.

Let $n \mid q-1$. A particularly nice Vandermonde matrix is when a_j is the different n-th root of unity, that is when $a_j=\alpha^j$ where $\alpha^n=1$ and $\alpha^i \neq 1$ for $1 \leqslant i < n$.

The Fourier $n\times n$ matrix, relative to α, is the $n\times n$ matrix $\boldsymbol{F}_n = \begin{pmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & \alpha & \alpha^2 & \cdots & \alpha^{n-1} \\ 1 & \alpha^2 & \alpha^{2(2)} & \cdots & \alpha^{2(n-1)} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & \alpha^{n-1} & \alpha^{(n-1)2} & \cdots & \alpha^{(n-1)(n-1)} \end{pmatrix}$.

The rows of a Fourier matrix $\boldsymbol{F}_n$ under consideration will be denoted by $\{\boldsymbol{g}_0, \boldsymbol{g}_1, \cdots, \boldsymbol{g}_{n-1}\}$. Thus $\boldsymbol{g}_j = (1, \alpha^j, \alpha^{j(2)}, \cdots, \alpha^{j(n-1)}\}$ for $j=0,1,\cdots,n-1$. It is easy to check that $\boldsymbol{g}_i\boldsymbol{g}_j^{\mathrm{T}}=0$ for $j \neq n-i$. We recall the following fact (see Ref. [22]).

Lemma 3 Let C be a code generated by taking k consecutive rows of a Fourier $n\times n$ matrix. Then C is an MDS code with parameters $\left[n,k,n-k+1\right]_q$.

Remark 3 Let C be the code with generator matrix $\boldsymbol{G} = \begin{pmatrix} \boldsymbol{g}_0 \\ \boldsymbol{g}_1 \\ \vdots \\ \boldsymbol{g}_{k-1} \end{pmatrix}$. Then C is an MDS code with parameters $\left[n,k,n-k+1\right]_q$ by Lemma 3, and $\boldsymbol{H} = \begin{pmatrix} \boldsymbol{g}_1 \\ \boldsymbol{g}_2 \\ \vdots \\ \boldsymbol{g}_{n-k} \end{pmatrix}$ is a check matrix for C.

Theorem 5 Let $n|q-1, n \geqslant 2k-1$ and $k \geqslant 1$. Then 1) there exists an optimal AQEC code Q with parameters $\left[\left[n, 1, n-k+1/k\right]\right]_q$; 2) there exists an optimal AQEC code Q with parameters $\left[\left[n, n-2k+1, k+1/k\right]\right]_q$.

Proof For $1 \leqslant k$, set

$$\boldsymbol{G}_c = \begin{pmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & \alpha & \alpha^2 & \cdots & \alpha^{n-1} \\ 1 & \alpha^2 & \alpha^{2(2)} & \cdots & \alpha^{2(n-1)} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & \alpha^{k-1} & \alpha^{(k-1)2} & \cdots & \alpha^{(k-1)(n-1)} \end{pmatrix} = \begin{pmatrix} \boldsymbol{g}_1 \\ \boldsymbol{g}_2 \\ \vdots \\ \boldsymbol{g}_{k-1} \end{pmatrix}$$

Then code C generated by the matrix $\boldsymbol{G}_c$ is an MDS code with parameters $[n,k,n-k+1]_q$ by Lemma 3.

According to Remark 3, the matrix

$$\boldsymbol{H}_{C^{\perp}} = \begin{pmatrix} \boldsymbol{g}_1 \\ \boldsymbol{g}_2 \\ \vdots \\ \boldsymbol{g}_{n-k} \end{pmatrix}$$

is a parity-check matrix for the code C.

By Theorem 1, we have $\mathrm{Sum}(C)^{\perp} = C \cap C^{\perp}$. Since $n \geqslant 2k-1$, i.e., $n-k \geqslant k-1$, we know that the matrices

$$\boldsymbol{G}_{\mathrm{Sum}(C)^{\perp}}=\begin{pmatrix}\boldsymbol{g}_1\\ \boldsymbol{g}_2\\ \vdots\\ \boldsymbol{g}_{k-1}\end{pmatrix} \text{ and } \boldsymbol{G}_{\mathrm{Sum}(C)}=\begin{pmatrix}\boldsymbol{g}_0\\ \boldsymbol{g}_1\\ \vdots\\ \boldsymbol{g}_{n-k}\end{pmatrix}$$

are generator matrices for codes $\mathrm{Sum}(C^{\perp})$ and $\mathrm{Sum}(C)$, respectively. Moreover, the codes $\mathrm{Sum}(C^{\perp})$ and $\mathrm{Sum}(C)$ are MDS codes with parameters $[n,k-1,n-k+2]_q$ and $[n,n-k+1,k]_q$ by Lemma 3.

1) Take $C_1=\mathrm{Sum}(C)$ and $C_2=C$, we have $C_1^{\perp}=\mathrm{Sum}(C)^{\perp}\subset C=C_2$ by Theorem 1. Since $k+1>k$ and $n-k+2>n-k+1$, we have $d_x=\mathrm{wt}(C_1\backslash C_2^{\perp})=k, d_z=d(C_2\backslash C_1^{\perp})=n-k+1$. Thus, by Theorem 3, there exists an AQEC code Q with parameters $[[n,1,n-k+1/k]]_q$. Since $d_x+d_z=n-1+2$, the AQEC code Q with parameters $[[n,1,n-k+1/k]]_q$ is optimal.

2) Take $C_1=\mathrm{Sum}(C)$ and $C_2=C^{\perp}$, we have $C_2^{\perp}=\mathrm{Sum}(C)^{\perp}\subset C_2$ by Theorem 1. Since $n\geqslant 2k-1$, i.e., $n-k+1>k$ and $n-k+2>k+1$, we obtain $d_x=d(C_1\backslash C_2^{\perp})=k$ and $d_z=d(C_2\backslash C_1^{\perp})=k+1$. Thus, by Theorem 3, there exists an AQEC code Q with parameters $[[n,n-2k+1,\ k+1/k]]_q$. Since $d_x+d_z=2k+1=n-(n-2k+1)+2$, the AQEC code Q with parameters $[[n,n-2k+1,k+1/k]]_q$ is optimal.

5 Code Comparison and Conclusion

In this paper, by using Euclidean sums of linear codes, we have constructed four new classes of optimal AQEC codes, in which the lengths of two new classes of optimal AQEC codes are flexible. Moreover, we remark that the parameters of optimal AQEC codes listed below have not covered ones given in this paper.

1) $$\left[\left[\frac{q^2+1}{2},\frac{q^2+1}{2}-2(t+s),\ (2k+1)/(2t+1)\right]\right]_{q^2},$$

where q is an odd prime power, and $0\leqslant t\leqslant k\leqslant \frac{q-1}{2}$[8].

2) $$\left[\left[\frac{q^2+1}{5},\frac{q^2+1}{5}-2(t+s+2),\ 2s+\frac{3}{2}t+3\right]\right]_{q^2},$$

where q is an even prime power with $q=2^e$, e is an odd with $e\equiv 1 \bmod 4$, and $0\leqslant t\leqslant s\leqslant \frac{3q-16}{10}$[9].

3) $$\left[\left[\frac{q^2+1}{5},\frac{q^2+1}{5}-2(t+s+2), 2s+\frac{3}{2}t+3\right]\right]_{q^2},$$

where q is an even prime power with $q=2^e$, e is an odd with $e\equiv 3 \bmod 4$, and $0\leqslant t\leqslant s\leqslant \frac{3q-14}{10}$[9].

4) $$\left[\left[\frac{q^2+1}{10h},\frac{q^2+1}{10h}-2(\delta_1+\delta_2+2), 2\delta_1+\frac{3}{2}\delta_2+3\right]\right]_{q^2},$$

where q is an odd prime power of the form $10hm+t$, m is an odd, both h and t are odd with $10h=t^2+1$ and $t\geqslant 3$, both δ_1 and δ_2 are integers such that $0\leqslant\delta_1\leqslant\frac{q-10h-t}{20h}$ and $\frac{q-3}{2}\leqslant\delta_2\leqslant\frac{q-3}{2}+Q\delta_1$[10].

5) $$\left[\left[\frac{q^2+1}{10h},\frac{q^2+1}{10h}-2(\delta_1+\delta_2+2), 2\delta_1+\frac{3}{2}\delta_2+3\right]\right]_{q^2},$$

where q is an odd prime power of the form $10hm+t$, $m\geqslant 2$ is an even, both h and t are odd with $10h=t^2+1$ and $t\geqslant 3$, both δ_1 and δ_2 are integers such that $0\leqslant\delta_1\leqslant\frac{q-10h-t}{20h}$ and $\frac{q-3}{2}\leqslant\delta_2\leqslant\frac{q-3}{2}+Q\delta_1$[10].

6) $$\left[\left[\frac{q^2+1}{10h},\frac{q^2+1}{10h}-2(\delta_1+\delta_2+2),\ 2\delta_1+\frac{3}{2}\delta_2+3\right]\right]_{q^2},$$

where q is an odd prime power of the form $10hm+10h-t$, m is an odd, both h and t are odd with $10h=t^2+1$ and $t\geqslant 3$, both δ_1 and δ_2 are integers such that $0\leqslant\delta_1\leqslant\frac{q-10h-t}{20h}$ and $\frac{q-3}{2}\leqslant\delta_2\leqslant\frac{q-3}{2}+Q\delta_1$[10].

7) $$\left[\left[\frac{q^2+1}{10h},\frac{q^2+1}{10h}-2(\delta_1+\delta_2+2),\ 2\delta_1+\frac{3}{2}\delta_2+3\right]\right]_{q^2},$$

where q is an odd prime power of the form $10hm+10h-t$, $m\geqslant 2$ is an even, both h and t are odd with $10h=t^2+1$ and $t\geqslant 3$, both δ_1 and δ_2 are integers such that $0\leqslant\delta_1\leqslant\frac{q-10h-t}{20h}$ and $\frac{q-3}{2}\leqslant\delta_2\leqslant\frac{q-3}{2}+Q\delta_1$[10].

8) $$\left[\left[\frac{q^2-1}{3},\frac{q^2-1}{3}-(\delta_1+\delta_2),\ (\delta_1+1/(\delta_2+1)\right]\right]_{q^2},$$

where q is an odd prime power with $3\mid(q+1)$, δ_1 and δ_2 are positive integers, and $1\leqslant\delta_2\leqslant\delta_1\leqslant\frac{2q-4}{3}$[11].

9) $$\left[\left[\frac{q^2-1}{5},\frac{q^2-1}{5}-(\delta_1+\delta_2),\ (\delta_1+1/(\delta_2+1)\right]\right]_{q^2},$$

where q is an odd prime power with $5\mid(q+1),\delta_1$ and δ_2 are positive integers, and $1\leqslant\delta_2\leqslant\delta_1\leqslant\frac{3q+3}{5}-2$[11].

10) $$\left[\left[\frac{q^2-1}{7},\frac{q^2-1}{7}-(\delta_1+\delta_2),\ (\delta_1+1/(\delta_2+1)\right]\right]_{q^2},$$

where q is an odd prime power with $7\mid(q+1)$, δ_1 and δ_2 are positive integers, and $1\leqslant\delta_2\leqslant\delta_1\leqslant\frac{4(q+1)}{7}-2$[11].

11) $[[n, j, d_z/d_x]]_q$, where $q > 3$ is a prime power, $n \leqslant q, k \leqslant n-2, j \neq n-k-1$ and $\{d_z, d_x\} = \{n-k-j+1, k+1\}$[12].

12) $\left[\left[\frac{q^2+1}{5}, \frac{q^2+1}{5} - 2(s+t+1), (2s+2/(2t+2)\right]\right]_{q^2}$, $q = 20m+3$ or $q = 20m+7$ with m a positive integer, $0 \leqslant t \leqslant s \leqslant \frac{q+1}{4}$ is even[13].

13) $\left[\left[\frac{q^2+1}{5}, \frac{q^2+1}{5} - 2(s+t+1), (2s+2/(2t+2)\right]\right]_{q^2}$, $q = 20m-3$ or $q = 20m-7$ with m a positive integer, $0 \leqslant t \leqslant s \leqslant \frac{q+1}{4}$ is even[13].

14) $\left[\left[\frac{q^2-1}{5}, \frac{q^2-1}{5} - k - t, (k+1/(t+1)\right]\right]_{q^2}$, where $q \geqslant 5$ is an odd prime power, and $0 \leqslant t \leqslant s \leqslant q-1$[15].

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