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Blow-up for a Porous Medium Equation with Local Linear Boundary Dissipation

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Abstract: This article investigates the blow-up behaviors for a porous medium equation with a superlinear source and local linear boundary dissipation. Making use of the concavity method, we establish sufficient conditions to guarantee the occurrence of the finite time blow-up phenomenon. Meanwhile, we show the existence of the finite time blow-up solutions for arbitrarily high initial energy. Finally, we derive the life span bounds (i.e., the lower and upper bounds of the blow-up time).

Key words: porous medium equation; blow-up behavior; life span bounds

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0 Introduction

In the present article, our attention is focused on the blow-up behavior of the following porous medium equation with superline source and local linear boundary dissipation

$$\begin{cases} u_t = \Delta u^m + |u|^{p-2}u, & x \in \Omega,\ t>0, \\ u(x,t)=0, & x \in \Gamma_0,\ t>0, \\ \dfrac{\partial u^m}{\partial \boldsymbol{n}} = -u_t, & x \in \Gamma_1,\ t>0, \\ u(x,0)=u_0(x), & x \in \Omega, \end{cases} \tag{1}$$

where $m>1$, $p>m+1$, Ω is a bounded open subset of $\mathbb{R}^n$ with C^1 boundary $\partial\Omega$ and $\boldsymbol{n}$ represents the unit outer normal vector to $\partial\Omega$. $\{\Gamma_0, \Gamma_1\}$ is a partition of the boundary $\partial\Omega$ satisfying $\partial\Omega = \Gamma_0 \cup \Gamma_1, \overline{\Gamma_0} \cap \overline{\Gamma_1} = \varnothing$.

Moreover, Γ_0 and Γ_1 are measurable over $\partial\Omega$, endowed with $(n-1)$-dimensional surface measure σ and $\sigma(\Gamma_0)>0$.

Problem (1) with $m=1$ can be regarded as a mathematical model to depict a heat reaction-diffusion process that occurs inside a solid body Ω surrounded by a fluid, with contact Γ_1 and having an internal cavity with a contact boundary Γ_0. In this physics background, the quantity of heat produced by the reaction is proportional to a superlinear power

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of the temperature. To avoid an internal explosion in Ω, a refrigeration system is installed in the fluid. The operational mechanism of this refrigeration system lies in the fact that the heat absorbed from the fluid is proportional to the power of the rate of change of the temperature, which can be expressed as: $\frac{\partial u}{\partial \boldsymbol{n}} = -|u_t|^{q-2}u_t, x \in \Gamma_1$, where $\frac{\partial u}{\partial \boldsymbol{n}}$ stands for the heat flux from Ω to the fluid.

Evolution equations [1-8] with boundary damping have attracted the attention of mathematicians in the past period. For instance, Fiscella and Vitillaro [9] studied the following problem with local nonlinear boundary dissipation

$$\begin{cases} u_t - \Delta u = |u|^{p-2}u, & x \in \Omega, t > 0, \\ u(x,t) = 0 & x \in \Gamma_0, t > 0, \\ \frac{\partial u}{\partial \boldsymbol{n}} = -|u_t|^{q-2}u_t, & x \in \Gamma_1, t > 0, \\ u(x,0) = u_0(x), & x \in \bar{\Omega}, \end{cases} \tag{2}$$

where $2 \leqslant p \leqslant 1 + \frac{2^*}{2}$ and 2^* denotes the Sobolev conjugate of 2. Using the monotonicity method of Lions[9] and a contraction argument, they proved the local well-posedness in the Hadamard sense. Moreover, in the case of a superlinear source, i.e. $p > 2$, under the condition

$$J(u_0) = \frac{1}{2}\int_\Omega |\nabla u_0(x)|^2 \mathrm{d}x - \frac{1}{p}\int_\Omega |u_0(x)|^p \mathrm{d}x < d := \inf_{u \in H^1_{\Gamma_1}(\Omega)\setminus\{0\}} \sup_{\lambda > 0} J(\lambda u),$$

they gave the global existence and finite time blow-up results. To be precise, if

$$K(u_0) = \int_\Omega |\nabla u_0(x)|^2 \mathrm{d}x - \int_\Omega |u_0(x)|^p \mathrm{d}x \geqslant 0,$$

then the weak solution is global, while if

$$K(u_0) = \int_\Omega |\nabla u_0(x)|^2 \mathrm{d}x - \int_\Omega |u_0(x)|^p \mathrm{d}x \leqslant 0$$

and $q < q_0(p) := \frac{2(n+1)p - 4(n-1)}{n(p-2)+4}$, then the weak solution will blow up in some finite time. In a recent work, the authors [10] considered problem (1) with $m = 1$, and obtained the finite time blow-up result for arbitrary high initial energy. Moreover, under some additional conditions, the authors also gave estimates of the blow-up time. In addition, using some differential inequality techniques, the authors [11, 12] considered the lower bounds for the blow-up time of blow-up solutions to some porous medium equations with null Dirichlet boundary conditions or homogeneous Neumann boundary conditions.

To the best of our knowledge, there is no previous work on the blow-up behavior of the solution to the problem (1). Building on the aforementioned work, we will analyze the effects of the nonlinear diffusion and the local linear boundary dissipation on the blow-up phenomenon of problem (1). In order to deal with the difficulties caused by the nonlinear diffusion term Δu^m better and more effectively, throughout this paper, we work with the following equivalent formulation of the problem (1) obtained by changing variables $u^m = v$,

$$\begin{cases} \left(v^{\frac{1}{m}}\right)_t = \Delta v + |v|^{\frac{p-2}{m}} v^{\frac{1}{m}}, & x \in \Omega, t > 0, \\ v(x,t) = 0, & x \in \Gamma_0, t > 0, \\ \frac{\partial v}{\partial \boldsymbol{n}} = -\left(v^{\frac{1}{m}}\right)_t, & x \in \Gamma_1, t > 0, \\ v(x,0) = v_0(x) = u_0^m(x), & x \in \Omega. \end{cases} \tag{3}$$

First, we obtain the finite time blow-up criterion of the solution to the problem (3) by using a modified concavity method (Theorem 1). Second, for any $a \in \mathbb{R}$, we prove that there exists a $v_0 \in H^1_{\Gamma_0}(\Omega)$ with initial energy $J(v_0) = a$ that leads to a finite time blow-up solution (Corollary 1). Finally, the lower bounds of the blow-up time are derived by combining the interpolation inequality for L^q-norms, the Sobolev embedding theorem, with some differential inequality techniques (Theorem 2).

The article is organized as follows: In Section 1, we introduce some notations and state some useful lemmas. In

Section 2, we give the finite time blow-up criterion and the lower and upper estimates of the blow-up time.

1 Preliminaries

In this section, we first introduce some notations, definitions, and some known results. Throughout this article, we denote $\|\cdot\|_q=\|\cdot\|_{L^q(\Omega)}, \|\cdot\|_{q,\Gamma_1}=\|\cdot\|_{L^q(\Gamma_1)}$ for some $q\in[1,+\infty]$, and the Hilbert space

$$H^1_{\Gamma_0}(\Omega)=\{v\in H^1(\Omega): v=0 \text{ for } x\in\Gamma_0\}.$$

$(\cdot,\cdot)$ and $(\cdot,\cdot)_{\Gamma_1}$ stand for the inner products on the Hilbert spaces $L^2(\Omega)$ and $L^2(\Gamma_1)$, respectively. From the trace theorem, one knows that there exists a continuous trace mapping $H^1_{\Gamma_0}(\Omega)\hookrightarrow L^2(\partial\Omega)$. Moreover, since $\sigma(\Gamma_0)>0$, then, Theorem 6.7-5 [13] tells us that a Poincaré-type inequality holds. Therefore, $\|\nabla v\|_2$ is equivalent to the norm $\|v\|_{H_{\Gamma_0}}^1=\left(\|v\|_2^2+\|\nabla v\|_2^2\right)^{\frac{1}{2}}$ in the space $H^1_{\Gamma_0}(\Omega)$. On the other hand, since $m>1$, one can define the following positive optimal constants

$$S_1=\sup_{\substack{v\in H^1_{\Gamma_0}(\Omega)\\ v\neq 0}}\frac{\left\|v^{\frac{m+1}{2m}}\right\|^2_{2,\Gamma_1}}{\|\nabla v\|_2^2},\quad S_2=\sup_{\substack{v\in H^1_{\Gamma_0}(\Omega)\\ v\neq 0}}\frac{\left\|v^{\frac{m+1}{2m}}\right\|^2_{2}}{\|\nabla v\|_2^2}. \tag{4}$$

The definition of the weak solution to the problem (3) is given as follows.

Definition 1 Suppose that $v_0\in H^1_{\Gamma_0}(\Omega)$, and $m+1<p<m(2^*-1)+1$, where 2^* represents the Sobolev conjugate of 2, namely

$$2^*=\begin{cases}\infty, & \text{if } n\leqslant 2;\\ \dfrac{2n}{n-2}, & \text{if } n>2.\end{cases}$$

A function $v:=v(x,t)\in L^\infty\left(0,T;H^1_{\Gamma_0}(\Omega)\right)$ with $\left(v^{\frac{1}{m}}\right)_t\in L^2\left((0,T)\times\Omega\right)$ is called a weak solution of problem (3) on $[0,T)\times\Omega$ if $v(x,0)=v_0=u_0^m$, and

$$\left(\left(v^{\frac{1}{m}}\right)_t,\phi\right)+\left(\nabla v(t),\nabla\phi\right)+\left(\left(v^{\frac{1}{m}}\right)_t,\phi\right)_{\Gamma_1}=\int_\Omega|v|^{\frac{p-2}{m}}v^{\frac{1}{m}}\phi\,\mathrm{d}x \tag{5}$$

holds for a.e. $t\in(0,T)$ and any $\phi\in H^1_{\Gamma_0}(\Omega)$. Moreover, the spatial trace of v has a distributional time derivative v_t on $(0,T)\times\partial\Omega$, belonging to $L^2\left((0,T)\times\partial\Omega\right)$.

Definition 2 Suppose that v is a weak solution of problem (3). We say that v blows up in finite time T if

$$\lim_{t\to T^-}\rho(t)=\lim_{t\to T^-}\frac{1}{m+1}\left(\left\|v^{\frac{m+1}{2m}}(t)\right\|_2^2+\left\|v^{\frac{m+1}{2m}}(t)\right\|_{2,\Gamma_1}^2\right)=+\infty.$$

In what follows, we introduce the energy-functional $J(v)=\dfrac{1}{2}\|\nabla v\|_2^2-\dfrac{m}{m+p-1}\|v\|_{\frac{m+p-1}{m}}^{\frac{m+p-1}{m}}$, and Nehari functional

$$K(v)=\|\nabla v\|_2^2-\|v\|_{\frac{m+p-1}{m}}^{\frac{m+p-1}{m}}$$

in $H^1_{\Gamma_0}(\Omega)$ related to problem (3). Evidently, one knows that $J(v)$ and $K(v)$ are continuous on $H^1_{\Gamma_0}(\Omega)$, and for a.e. $t\in(0,T)$,

$$\frac{\mathrm{d}}{\mathrm{d}t}J(v(t))=-\frac{4m}{(m+1)^2}\left(\left\|\left(v^{\frac{m+1}{2m}}\right)_t(t)\right\|_2^2+\left\|\left(v^{\frac{m+1}{2m}}\right)_t(t)\right\|_{2,\Gamma_1}^2\right)\leqslant 0 \tag{6}$$

which implies that

$$J(v(t))=J(v_0)-\frac{4m}{(m+1)^2}\int_0^t\left(\left\|\left(v^{\frac{m+1}{2m}}\right)_\tau(\tau)\right\|_2^2+\left\|\left(v^{\frac{m+1}{2m}}\right)_\tau(\tau)\right\|_{2,\Gamma_1}^2\right)\mathrm{d}\tau \tag{7}$$

Now, we introduce the definition of the potential well-depth $d=\inf_{v\in\mathcal{N}} J(v)=\inf_{v\in H^1_{\Gamma_0}(\Omega)\setminus\{0\}}\sup_{\lambda>0} J(\lambda v)$, where $\mathcal{N}$ is the Nehari manifold $\mathcal{N}=\left\{v\in H^1_{\Gamma_0}(\Omega)|K(v)=0\right\}\setminus\{0\}$. Indeed, d also can be characterized as

$$d=\frac{p-m-1}{2\left(m+p-1\right)}\left(\sup_{\substack{v\in H^1_{\Gamma_0}(\Omega)\\ v\neq 0}}\frac{\|v\|_{\frac{m+p-1}{m}}}{\|\nabla v\|_2}\right)^{-\frac{2(m+p-1)}{p-m-1}}.$$

We are now able to give some lemmas, which will play a key role in our proof of the main results.

Lemma 1 Suppose that $m+1<p<m\left(2^*-1\right)+1$. Then one has $\|v\|^{\frac{m+p-1}{m}}_{\frac{m+p-1}{m}}>\dfrac{2\left(m+p-1\right)}{\left(p-m-1\right)}d$ for any $v\in\mathcal{N}_-=\left\{v\in H^1_{\Gamma_0}(\Omega)|K\left(v\right)<0\right\}$.

Proof Since $\|\nabla v\|_2^2-\|v\|^{\frac{m+p-1}{m}}_{\frac{m+p-1}{m}}=K(v)<0,$ one knows that $v\neq 0$. Meanwhile, one can verify that $K(\lambda^* v)=0$ with

$$\lambda^*=\left(\frac{\|\nabla v\|_2^2}{\|v\|^{\frac{m+p-1}{m}}_{\frac{m+p-1}{m}}}\right)^{\frac{m}{p-m-1}}\in\left(0,1\right).$$

Thereupon, one has $\lambda^* v\in\mathcal{N}$. Furthermore, according to the definition of the potential well depth d, one can arrive at

$$\begin{aligned}d&=\inf_{v\in\mathcal{N}}J(v)\leqslant J(\lambda^* v)=\frac{p-m-1}{2(p+m-1)}\|\lambda^* v\|^{\frac{m+p-1}{m}}_{\frac{m+p-1}{m}}+\frac{1}{2}K\left(\lambda^* v\right)\\&=\frac{p-m-1}{2(p+m-1)}(\lambda^*)^{\frac{m+p-1}{m}}\|v\|^{\frac{m+p-1}{m}}_{\frac{m+p-1}{m}}<\frac{p-m-1}{2\left(m+p-1\right)}\|v\|^{\frac{m+p-1}{m}}_{\frac{m+p-1}{m}}\end{aligned}\tag{8}$$

which results in $\|v\|^{\frac{m+p-1}{m}}_{\frac{m+p-1}{m}}>\dfrac{2\left(m+p-1\right)}{\left(p-1-m\right)}d.$

The proof of the Lemma 1 is completed.

Lemma 2 Suppose that $m+1<p<m\left(2^*-1\right)+1$, and the weak solution v of problem (3) blows up in finite time T. Then there is a $t^*\in\left[0,T\right)$ such that $v\left(t^*\right)\in\mathcal{N}_-$.

Proof Suppose that $K\left(v\left(t\right)\right)\geqslant 0$ for any $t\in[0,T)$. Then, from (6), it follows that

$$J\left(v_0\right)\geqslant J\left(v\left(t\right)\right)=\frac{p-m-1}{2(m+p-1)}\|\nabla v\left(t\right)\|_2^2+\frac{m}{m+p-1}K\left(v\left(t\right)\right)\geqslant\frac{p-m-1}{2(m+p-1)}\|\nabla v\left(t\right)\|_2^2,$$

which contradicts the assumption that v is a finite-time blow-up weak solution. The proof of Lemma 2 is completed.

Lemma 3[14] Suppose that a positive function F on $\left[0,T\right]$ satisfies the following conditions: (i) F is differentiable on $\left[0,T\right]$ and F' is absolutely continuous on $\left[0,T\right]$ with $F'(0)>0$; (ii) there exists a positive constant $\alpha>0$ such that

$$F\left(t\right)F''\left(t\right)-\left(1+\alpha\right)\left(F'\left(t\right)\right)^2\geqslant 0$$

holds for a.e. $t\in\left[0,T\right]$. Then $T\leqslant\dfrac{F\left(0\right)}{\alpha F'\left(0\right)}$.

2 The Finite Time Blow-up Results

Recalling that $\rho\left(t\right)=\dfrac{1}{m+1}\left(\left\|v^{\frac{m+1}{2m}}\left(t\right)\right\|_2^2+\left\|v^{\frac{m+1}{2m}}\left(t\right)\right\|_{2,\Gamma_1}^2\right)$, one can easily check that

$$\frac{\mathrm{d}}{\mathrm{d}t}\rho\left(t\right)=\frac{1}{m}\left[\left(v^{\frac{1}{m}}\left(t\right),v_t\left(t\right)\right)+\left(v^{\frac{1}{m}}\left(t\right),v_t\left(t\right)\right)_{\Gamma_1}\right]=-K\left(v\left(t\right)\right)\tag{9}$$

For any $T_1\in\left(0,T\right)$, we define an auxiliary function as the form

$$F(t)=\int_0^t \rho(\tau)\mathrm{d}\tau+\left(T_1-t\right)\rho(0)+\frac{m\beta}{m+1}(t+\sigma)^2,\ t\in\left[0,T_1\right] \tag{10}$$

where β and σ are two positive parameters which will be determined later. It is clear that F is positive on $\left[0,T_1\right]$. By a simple calculation, one has

$$F'(t)=\rho(t)-\rho(0)+\frac{2m\beta}{m+1}(t+\sigma)=\int_0^t\frac{\mathrm{d}}{\mathrm{d}t}\rho(\tau)\mathrm{d}\tau+\frac{2m\beta}{m+1}(t+\sigma)=\frac{1}{m}\int_0^t\left[\left(v^{\frac{1}{m}},v_\tau\right)+\left(v^{\frac{1}{m}},v_\tau\right)_{\Gamma_1}\right]\mathrm{d}\tau+\frac{2m\beta}{m+1}(t+\sigma) \tag{11}$$

and $F'(0)=\dfrac{2m\beta\sigma}{m+1}>0$. Moreover, from (6) and (7), it follows that

$$\begin{aligned}F''(t)&=\frac{\mathrm{d}}{\mathrm{d}t}\rho(t)+\frac{2m\beta}{m+1}=-K\left(v(t)\right)+\frac{2m\beta}{m+1}=-\left(\frac{m+p-1}{m}J(v(t)-\frac{p-m-1}{2m}\left\|\nabla v(t)\right\|_2^2\right)+\frac{2m\beta}{m+1}\\&=\frac{p-m-1}{2m}\left\|\nabla v(t)\right\|_2^2-\frac{m+p-1}{m}J(v_0)+\frac{2m\beta}{m+1}+\frac{4(m+p-1)}{(m+1)^2}\int_0^t\left(\left\|\left(v^{\frac{m+1}{2m}}\right)_\tau(\tau)\right\|_2^2+\left\|\left(v^{\frac{m+1}{2m}}\right)_\tau(\tau)\right\|_{2,\Gamma_1}^2\right)\mathrm{d}\tau\end{aligned} \tag{12}$$

On the other hand, employing the Young's inequality and Cauchy-Schwartz inequality, we obtain:

$$\begin{aligned}\xi(t)=&\left[\frac{1}{m}\int_0^t\left(\left\|v^{\frac{m+1}{2m}}(\tau)\right\|_2^2+\left\|v^{\frac{m+1}{2m}}(\tau)\right\|_{2,\Gamma_1}^2\right)\mathrm{d}\tau+\beta(t+\sigma)^2\right]\cdot\left[\frac{4m}{(m+1)^2}\int_0^t\left(\left\|\left(v^{\frac{m+1}{2m}}\right)_\tau(\tau)\right\|_2^2+\left\|\left(v^{\frac{m+1}{2m}}\right)_\tau(\tau)\right\|_{2,\Gamma_1}^2\right)\mathrm{d}\tau+\frac{4m^2\beta}{(m+1)^2}\right]\\&-\left[\frac{1}{m}\int_0^t\left[\left(v^{\frac{1}{m}},v_\tau\right)+\left(v^{\frac{1}{m}},v_\tau\right)_{\Gamma_1}\right]\mathrm{d}\tau+\frac{2m\beta}{m+1}(t+\sigma)\right]^2\\\geqslant&\left\{\frac{2}{m+1}\left(\int_0^t\left[\left(v^{\frac{m+1}{2m}},\left(v^{\frac{m+1}{2m}}\right)_\tau\right)+\left(v^{\frac{m+1}{2m}},\left(v^{\frac{m+1}{2m}}\right)_\tau\right)_{\Gamma_1}\right]\mathrm{d}\tau\right)+\frac{2m\beta}{m+1}(t+\sigma)\right\}^2-\left[\frac{1}{m}\int_0^t\left[\left(v^{\frac{1}{m}},v_\tau\right)+\left(v^{\frac{1}{m}},v_\tau\right)_{\Gamma_1}\right]\mathrm{d}\tau+\frac{2m\beta}{m+1}(t+\sigma)\right]^2\\=&\left[\frac{1}{m}\int_0^t\left[\left(v^{\frac{1}{m}},v_\tau\right)+\left(v^{\frac{1}{m}},v_\tau\right)_{\Gamma_1}\right]\mathrm{d}\tau+\frac{2m\beta}{m+1}(t+\sigma)\right]^2-\left[\frac{1}{m}\int_0^t\left[\left(v^{\frac{1}{m}},v_\tau\right)+\left(v^{\frac{1}{m}},v_\tau\right)_{\Gamma_1}\right]\mathrm{d}\tau+\frac{2m\beta}{m+1}(t+\sigma)\right]^2=0\end{aligned} \tag{13}$$

Now, we are in the position to estimate $FF''-\lambda(F')^2$ with $\lambda=\dfrac{m+p-1}{m+1}>1$. Combining (10), (11), and (12), one can arrive at:

$$\begin{aligned}FF''-\lambda(F')^2&=FF''-\lambda\left[\frac{1}{m}\int_0^t\left[\left(v^{\frac{1}{m}},v_\tau\right)+\left(v^{\frac{1}{m}},v_\tau\right)_{\Gamma_1}\right]\mathrm{d}\tau+\frac{2m\beta}{m+1}(t+\sigma)\right]^2\\&=FF''+\lambda\left[\xi(t)-\frac{m+1}{m}\left(F(t)-(T-t)\rho(0)\right)\cdot\left(\frac{4m}{(m+1)^2}\int_0^t\left(\left\|\left(v^{\frac{m+1}{2m}}\right)_\tau(\tau)\right\|_2^2+\left\|\left(v^{\frac{m+1}{2m}}\right)_\tau(\tau)\right\|_{2,\Gamma_1}^2\right)\mathrm{d}\tau+\frac{4m^2\beta}{(m+1)^2}\right)\right]\\&\geqslant FF''-\frac{4m\lambda\beta}{m+1}F(t)-\frac{4\lambda}{m+1}F(t)\int_0^t\left(\left\|\left(v^{\frac{m+1}{2m}}\right)_\tau(\tau)\right\|_2^2+\left\|\left(v^{\frac{m+1}{2m}}\right)_\tau(\tau)\right\|_{2,\Gamma_1}^2\right)\mathrm{d}\tau\\&=F\left[\frac{p-m-1}{2m}\left\|\nabla v(t)\right\|_2^2-\frac{m+p-1}{m}J(v_0)+\frac{2m\beta}{m+1}+\frac{4(m+p-1)}{(m+1)^2}\int_0^t\left(\left\|\left(v^{\frac{m+1}{2m}}\right)_\tau(\tau)\right\|_2^2+\left\|\left(v^{\frac{m+1}{2m}}\right)_\tau(\tau)\right\|_{2,\Gamma_1}^2\right)\mathrm{d}\tau\right.\\&\quad\left.-\frac{4\lambda}{m+1}\int_0^t\left(\left\|\left(v^{\frac{m+1}{2m}}\right)_\tau(\tau)\right\|_2^2+\left\|\left(v^{\frac{m+1}{2m}}\right)_\tau(\tau)\right\|_{2,\Gamma_1}^2\right)\mathrm{d}\tau-\frac{4m\lambda\beta}{m+1}\right]\\&=F\left[\left(\frac{4(m+p-1)}{(m+1)^2}-\frac{4\lambda}{m+1}\right)\int_0^t\left(\left\|\left(v^{\frac{m+1}{2m}}\right)_\tau(\tau)\right\|_2^2+\left\|\left(v^{\frac{m+1}{2m}}\right)_\tau(\tau)\right\|_{2,\Gamma_1}^2\right)\mathrm{d}\tau\right.\end{aligned} \tag{14}$$

Noticing that, $\lambda=\frac{m+p-1}{m+1}$, then (14) leads to

$$FF''-\frac{m+p-1}{m+1}\left(F'\right)^2\geqslant F\cdot\left[\frac{p-m-1}{2m}\left\|\nabla v(t)\right\|_2^2-\frac{m+p-1}{m}J\left(v_0\right)-\frac{2m\beta\left(2p+m-3\right)}{\left(m+1\right)^2}\right] \tag{15}$$

Up to now, from the above discussion, one can summarize the following lemma.

Lemma 4 Suppose that $\rho(0)>0$ and there is a positive constant such that

$$\frac{p-m-1}{2m}\left\|\nabla v(t)\right\|_2^2-\frac{m+p-1}{m}J\left(v_0\right)-\frac{2m\beta\left(2p+m-3\right)}{\left(m+1\right)^2}\geqslant 0$$

holds for any $t\in\left(0,T_1\right]$. Then $0<T_1\leqslant\frac{(m+1)^3\rho(0)}{m\left(p-2\right)^2\beta}$.

Proof Since $\frac{m+p-1}{m+1}=1+\frac{p-2}{m+1}>1$, then, a direct application of Lemma 3 and (15) yields that

$$T_1<\frac{F(0)}{\frac{p-2}{m+1}F'(0)}=\frac{T_1\rho(0)+\frac{m\beta\sigma^2}{m+1}}{\frac{p-2}{m+1}\cdot\frac{2m\beta\sigma}{m+1}}=\frac{\rho(0)(m+1)^2}{2m\left(p-2\right)\beta\sigma}T_1+\frac{m+1}{2\left(p-2\right)}\sigma \tag{16}$$

Choosing $\sigma\in\left(\frac{\rho(0)(m+1)^2}{2m\left(p-2\right)\beta},+\infty\right)$ to guarantee $\frac{\rho(0)(m+1)^2}{2m\left(p-2\right)\beta\sigma}<1$, then (16) tells us that,

$$T_1\leqslant\left(1-\frac{\rho(0)(m+1)^2}{2m\left(p-2\right)\beta\sigma}\right)^{-1}\frac{m+1}{2\left(p-2\right)}\sigma \tag{17}$$

By a series of calculations, one can verify that the right side of (17) takes its minimum at the point

$$\sigma=\sigma_\beta=\frac{\rho(0)(m+1)^2}{m\left(p-2\right)\beta}\in\left(\frac{\rho(0)(m+1)^2}{2m\left(p-2\right)\beta},+\infty\right).$$

In other words, one has $T_1\leqslant\frac{(m+1)^3\rho(0)}{m\left(p-2\right)^2\beta}$.

The proof of Lemma 4 is completed.

Now, we give the finite time blow-up criteria as follows.

Theorem 1 Suppose that $m+1<p<m\left(2^*-1\right)+1$ and the initial data v_0 belongs to one of the following sets:

$$\mathcal{B}_1=\left\{v\in H^1_{\Gamma_0}:J(v)<d,K(v)<0\right\},\mathcal{B}_2=\left\{v\in H^1_{\Gamma_0}:J(v)<\frac{\left(p-m-1\right)\left(\left\|v^{\frac{m+1}{2m}}\right\|_2^2+\left\|v^{\frac{m+1}{2m}}\right\|_{2,\Gamma_1}^2\right)}{2\left(m+p-1\right)\left(S_1+S_2\right)}\right\}.$$

Then, the weak solution v to the problem (3) blows up in a finite time. Moreover, one has

$$T\leqslant\frac{2m\left(2p+m-3\right)}{\left(p-2\right)^2\left(m+p-1\right)}\cdot\frac{\left\|v_0^{\frac{m+1}{2m}}\right\|_2^2+\left\|v_0^{\frac{m+1}{2m}}\right\|_{2,\Gamma_1}^2}{d-J\left(v_0\right)}$$

for $v_0\in\mathcal{B}_1$ and

$$T\leqslant\frac{2m\left(2p+m-3\right)}{\left(p-2\right)^2\left(m+p-1\right)}\cdot\frac{\left\|v_0^{\frac{m+1}{2m}}\right\|_2^2+\left\|v_0^{\frac{m+1}{2m}}\right\|_{2,\Gamma_1}^2}{\frac{p-m-1}{2\left(m+p-1\right)\left(S_1+S_2\right)}\left(\left\|v_0^{\frac{m+1}{2m}}\right\|_2^2+\left\|v_0^{\frac{m+1}{2m}}\right\|_{2,\Gamma_1}^2\right)-J\left(v_0\right)}$$

for $v_0\in\mathcal{B}_2$.

Proof If $v_0 \in \mathcal{B}_1$. Then, with the help of (7), one has, for any $t \in [0, T)$,

$$J(v(t)) = J(v_0) - \frac{4m}{(m+1)^2}\int_0^t \left(\left\| \left(v^{\frac{m+1}{2m}} \right)_\tau (\tau) \right\|_2^2 + \left\| \left(v^{\frac{m+1}{2m}} \right)_\tau (\tau) \right\|_{2,\Gamma_1}^2 \right) \mathrm{d}\tau \leqslant J(v_0) < d \tag{18}$$

Suppose that there exists a $t_1 \in [0, T)$ such that $K(v(t_1)) = 0$ and $K(v(t)) < 0$ for any $t \in [0, t_1)$. Thereupon, Lemma 1 and the continuity of the mapping $t \mapsto \|\nabla v(t)\|_2$ are applicable to produce

$$\|\nabla v(t_1)\|_2^2 = \lim_{t \to t_1^-} \|\nabla v(t)\|_2^2 \geqslant \frac{2(m+p-1)}{p-m-1} d > 0,$$

which implies that $v(t_1) \in \mathcal{N}$. And hence, one has $J(v(t_1)) \geqslant \inf_{v \in N} J(v) = d$, which contradicts (18). That is to say, for any $t \in [0, T)$, one can claim that $v(t) \in \mathcal{B}_1$ provided that $v_0 \in \mathcal{B}_1$.

Selecting

$$\beta \in \left(0, \frac{(m+p-1)(m+1)^2}{2m^2(2p+m-3)}\left(d - J(v_0)\right)\right],$$

and keeping Lemma 1 in mind, one has, for any $t \in (0, T)$,

$$\begin{aligned} &\frac{p-m-1}{2m}\|\nabla v(t)\|_2^2 - \frac{m+p-1}{m} J(v_0) - \frac{2m\beta(2p+m-3)}{(m+1)^2} \\ &> \frac{p-m-1}{2m} \cdot \frac{2(m+p-1)}{p-1-m} d - \frac{m+p-1}{m} J(v_0) - \frac{2m\beta(2p+m-3)}{(m+1)^2} \\ &= \frac{m+p-1}{m} d - \frac{m+p-1}{m} J(v_0) - \frac{2m\beta(2p+m-3)}{(m+1)^2} \geqslant 0. \end{aligned}$$

A direct application of Lemma 4 tells us that,

$$0 < T \leqslant \frac{2m(m+1)(2p+m-3)\rho(0)}{(p-2)^2(m+p-1)(d - J(v_0))} = \frac{2m(2p+m-3)}{(p-2)^2(m+p-1)} \cdot \frac{\left\| v_0^{\frac{m+1}{2m}} \right\|_2^2 + \left\| v_0^{\frac{m+1}{2m}} \right\|_{2,\Gamma_1}^2}{d - J(v_0)}.$$

If $v_0 \in \mathcal{B}_2$. Then from (4) and (9), it follows that,

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\rho(t) &= -K(v(t)) = \frac{p-m-1}{2m}\|\nabla v(t)\|_2^2 - \frac{m+p-1}{m} J(v(t)) \\ &\geqslant \frac{p-m-1}{2m} \cdot \frac{(m+1)\rho(t)}{S_1+S_2} - \frac{m+p-1}{m} J(v(t)) = \frac{m+p-1}{Am}\left(\rho(t) - AJ(v(t))\right) \end{aligned} \tag{19}$$

where,

$$A = \frac{2(m+p-1)(S_1+S_2)}{(m+1)(p-m-1)} > 0.$$

Putting $H(t) = \rho(t) - AJ(v(t))$ and combining (6) with (19), one has, for a.e. $t \in (0, T)$,

$$\frac{\mathrm{d}}{\mathrm{d}t} H(t) = \frac{\mathrm{d}}{\mathrm{d}t}\rho(t) - A\frac{\mathrm{d}}{\mathrm{d}t} J(v(t)) \geqslant \frac{\mathrm{d}}{\mathrm{d}t}\rho(t) \geqslant \frac{m+p-1}{Am} H(t).$$

Integrating the above inequality from 0 to t results in:

$$H(t) \geqslant \mathrm{e}^{\frac{m+p-1}{Am}} H(0). \tag{20}$$

On the other hand, the assumption $v_0 \in \mathcal{B}_2$ implies that,

$$H(0) = \rho(0) - AJ(v_0) = \frac{1}{m+1}\left(\left\| v_0^{\frac{m+1}{2m}} \right\|_2^2 + \left\| v_0^{\frac{m+1}{2m}} \right\|_{2,\Gamma_1}^2 \right) - \frac{2(m+p-1)(S_1+S_2)}{(m+1)(p-m-1)} J(v_0) > 0 \tag{21}$$

Combining (19), (20) with (21) yields that, for a.e. $t \in (0, T)$,

$$\frac{\mathrm{d}}{\mathrm{d}t}\rho(t) \geqslant \frac{m+p-1}{Am} H(t) \geqslant \frac{m+p-1}{Am} \mathrm{e}^{\frac{m+p-1}{Am}} H(0) > 0,$$

which means that $\rho(t)$ is increasing in $[0,T)$. Therefore, one can see that,

$$\begin{aligned}&\frac{p-m-1}{2m}\|\nabla v(t)\|_2^2-\frac{m+p-1}{m}J(v_0)-\frac{2m\beta(2p+m-3)}{(m+1)^2}\\ &\geqslant\frac{p-m-1}{2m}\cdot\frac{(m+1)\rho(t)}{S_1+S_2}-\frac{m+p-1}{m}J(v_0)-\frac{2m\beta(2p+m-3)}{(m+1)^2}\\ &\geqslant\frac{p-m-1}{2m}\cdot\frac{(m+1)\rho(0)}{S_1+S_2}-\frac{m+p-1}{m}J(v_0)-\frac{2m\beta(2p+m-3)}{(m+1)^2}=\frac{m+p-1}{Am}H(0)-\frac{2m\beta(2p+m-3)}{(m+1)^2}>0\end{aligned}\tag{22}$$

provided that

$$\beta\in\left(0,\frac{(m+p-1)(m+1)^2H(0)}{2Am^2(2p+m-3)}\right].$$

According to Lemma 4, one can obtain the following estimate,

$$0<T\leqslant\frac{2Am(m+1)(2p+m-3)\rho(0)}{(p-2)^2(m+p-1)H(0)},$$

namely,

$$0<T\leqslant\frac{2m(2p+m-3)}{(p-2)^2(m+p-1)}\cdot\frac{\left\|v_0^{\frac{m+1}{2m}}\right\|_2^2+\left\|v_0^{\frac{m+1}{2m}}\right\|_{2,\Gamma_1}^2}{\frac{p-m-1}{2(m+p-1)(S_1+S_2)}\left(\left\|v_0^{\frac{m+1}{2m}}\right\|_2^2+\left\|v_0^{\frac{m+1}{2m}}\right\|_{2,\Gamma_1}^2\right)-J(v_0)}.$$

The proof of Theorem 1 is completed.

In fact, Theorem 1 tells us the sets $\mathcal{B}_1$ and $\mathcal{B}_2$ are invariant under the semi-flow associated with problem (3). Namely, if $v_0\in\mathcal{B}_1$, then $v(t)\in\mathcal{B}_1$, while $v_0\in\mathcal{B}_2$, then $v(t)\in\mathcal{B}_2$. On the other hand, with the help of (4), for any $v\in\mathcal{B}_2$, one has

$$\begin{aligned}K(v)&=\|\nabla v\|_2^2-\|v\|_{\frac{m+p-1}{m}}^{\frac{m+p-1}{m}}=\frac{m+p-1}{m}J(v)-\frac{p-m-1}{2m}\|\nabla v\|_2^2\\ &\leqslant\frac{m+p-1}{m}\left(J(v)-\frac{(p-m-1)\left(\left\|v^{\frac{m+1}{2m}}\right\|_2^2+\left\|v^{\frac{m+1}{2m}}\right\|_{2,\Gamma_1}^2\right)}{2(m+p-1)(S_1+S_2)}\right)<0.\end{aligned}$$

Therefore, one can claim that $v(t)\in\mathcal{N}_-=\{v\in H^1_{\Gamma_0}|K(v)<0\}$ for any $t\in[0,T)$ provided initial data $v_0\in\mathcal{B}_1\cup\mathcal{B}_2$. Based on the above arguments, it is natural to ask whether or not the condition $v_0\in\mathcal{N}_-$ is sufficient enough for a finite-time blow-up. This is a difficult task, and the authors [15] conducted a similar study.

In addition, one can know that both $\mathcal{B}_1$ and $\mathcal{B}_2$ are non-empty sets. Moreover, Corollary 1 implies that for any $a\in\mathbb{R}$, there exists a $v_0\in H^1_{\Gamma_0}(\Omega)$ with initial energy $J(v_0)=a$, which leads to finite time blow-up solution.

Corollary 1 For any $a\in\mathbb{R}$, denote the energy level set by:

$$J^a=\{v\in H^1_{\Gamma_0}(\Omega)|J(v)=a\}.$$

Then $J^a\cap\mathcal{B}_2\neq\varnothing$.

Proof Suppose that Ω_1 and Ω_2 are two disjoint open subdomains of Ω, and

$$\operatorname{dist}(\overline{\Omega_1},\partial\Omega)>0,\ \operatorname{dist}(\overline{\Omega_2},\partial\Omega)>0,\ \operatorname{dist}(\overline{\Omega_1},\overline{\Omega_2})>0.$$

From the proof of Theorem 3.7 [16], one knows that there is a sequence $\{v_k\}\subset H^1_0(\Omega_1)$ such that,

$$\frac{1}{2}\int_{\Omega_1}|\nabla v_k(x)|^2\mathrm{d}x-\frac{m}{m+p-1}\int_{\Omega_1}|v_k(x)|^{\frac{m+p-1}{m}}\mathrm{d}x\to+\infty\text{ as }k\to\infty.\tag{23}$$

On the other hand, choosing an arbitrary nonzero function $\omega\in C_0^\infty(\Omega)$ with supp $\omega\subset\Omega_2$, then,

$$a-\left(\frac{1}{2}\int_{\Omega_2}\left|\nabla\left(r\omega(x)\right)\right|^2\mathrm{d}x-\frac{m}{m+p-1}\int_{\Omega_2}\left|r\omega(x)\right|^{\frac{m+p-1}{m}}\mathrm{d}x\right)\to+\infty \text{ as } r\to\infty \tag{24}$$

and there exists a $r_0>0$ such that

$$\frac{p-m-1}{2(p+m-1)(S_1+S_2)}\int_{\Omega_2}\left|r^{\frac{m+1}{2m}}\cdot\omega^{\frac{m+1}{2m}}(x)\right|^2\mathrm{d}x=\frac{r^{\frac{m+1}{m}}(p-m-1)}{2(p+m-1)(S_1+S_2)}\int_{\Omega_2}\left|\omega(x)\right|^{\frac{m+1}{m}}\mathrm{d}x>a \tag{25}$$

holds for any $r>r_0$. By (23) and (24), there are $k_0\in\mathbb{Z}^+$ and $r_1>r_0$ such that

$$\frac{1}{2}\int_{\Omega_1}\left|\nabla v_{k_0}(x)\right|^2\mathrm{d}x-\frac{m}{m+p-1}\int_{\Omega_1}\left|v_{k_0}(x)\right|^{\frac{m+p-1}{m}}\mathrm{d}x=a-\left(\frac{1}{2}\int_{\Omega_2}\left|\nabla\left(r_1\omega(x)\right)\right|^2\mathrm{d}x-\frac{m}{m+p-1}\int_{\Omega_2}\left|r_1\omega(x)\right|^{\frac{m+p-1}{m}}\mathrm{d}x\right). \tag{26}$$

Let $v_0=\tilde{v}+r_1\omega$, where,

$$\tilde{v}=\begin{cases}0, & x\in\Omega\backslash\Omega_1\\ v_{k_0}(x), & x\in\Omega_1.\end{cases}$$

It is not difficult to show that $v_0\in H^1_{\Gamma_0}(\Omega)$ and $v_0(x)=0$ in $\Omega\backslash(\Omega_1\cup\Omega_2)$. From (25) and (26), it follows that,

$$\begin{aligned}J(v_0)&=\frac{1}{2}\left(\int_{\Omega_1}+\int_{\Omega_2}\right)\left|\nabla v_0(x)\right|^2\mathrm{d}x-\frac{m}{m+p-1}\left(\int_{\Omega_1}+\int_{\Omega_2}\right)\left|v_0(x)\right|^{\frac{m+p-1}{m}}\mathrm{d}x\\&=\left(\frac{1}{2}\int_{\Omega_1}\left|\nabla v_{k_0}(x)\right|^2\mathrm{d}x-\frac{m}{m+p-1}\int_{\Omega_1}\left|v_{k_0}(x)\right|^{\frac{m+p-1}{m}}\mathrm{d}x\right)+\left(\frac{1}{2}\int_{\Omega_2}\left|\nabla\left(r_1\omega(x)\right)\right|^2\mathrm{d}x-\frac{m}{m+p-1}\int_{\Omega_2}\left|r_1\omega(x)\right|^{\frac{m+p-1}{m}}\mathrm{d}x\right)\overset{(26)}{=}a\\&\overset{(25)}{<}\frac{p-m-1}{2(p+m-1)(S_1+S_2)}\int_{\Omega_2}\left|r_1^{\frac{m+1}{2m}}\cdot\omega^{\frac{m+1}{2m}}(x)\right|^2\mathrm{d}x=\frac{p-m-1}{2(p+m-1)(S_1+S_2)}\int_{\Omega_2}\left|v_0^{\frac{m+1}{2m}}(x)\right|^2\mathrm{d}x\\&\leqslant\frac{p-m-1}{2(m+p-1)(S_1+S_2)}\left(\left\|v_0^{\frac{m+1}{2m}}\right\|_2^2+\left\|v_0^{\frac{m+1}{2m}}\right\|_{2,\Gamma_1}^2\right),\end{aligned} \tag{27}$$

which means that $v_0\in J^a\cap\mathcal{B}_2$. The proof of Corollary 1 is completed.

Theorem 2 Suppose that $n>2, m+1<p<(m+1)\left(1+\frac{2}{n}\right)$, the weak solution v of problem (3) blows up in finite time T and $v(t)\in\mathcal{N}_-$ for any $t\in[0,T)$. Then

$$T\geqslant\tilde{S}\left(\left\|v_0^{\frac{m+1}{2m}}\right\|_2^2+\left\|v_0^{\frac{m+1}{2m}}\right\|_{2,\Gamma_1}^2\right)^{1-\frac{2m\sigma(m+p-1)}{(m+1)\left[2m-(m+p-1)(1-\sigma)\right]}},$$

where $\sigma\in(0,1)$ and $\tilde{S}>0$ are two constants given by (28) and (33), respectively.

Proof Since $m>1$ and $n>2$, one can infer that,

$$(m+1)\left(1+\frac{2}{n}\right)<\frac{2mn}{n-2}-m+1.$$

Furthermore, from the assumption $p<(m+1)\left(1+\frac{2}{n}\right)$, it follows that,

$$1<\frac{m+1}{m}<\frac{m+p-1}{m}<\frac{2n}{n-2},$$

and

$$\sigma=\left(\frac{m}{m+p-1}-\frac{n-2}{2n}\right)\left(\frac{m}{m+1}-\frac{n-2}{2n}\right)^{-1}\in(0,1). \tag{28}$$

Noticing that $v(t)\in\mathcal{N}_-$ for any $t\in[0,T)$, the interpolation inequality for L^q-norms and the embedding $H^1(\Omega)\hookrightarrow L^{\frac{2n}{n-2}}(\Omega)$ can be used to obtain:

$$\left\|\nabla v(t)\right\|_2^2<\left\|v(t)\right\|_{\frac{m+p-1}{m}}^{\frac{m+p-1}{m}}\leqslant\left\|v(t)\right\|_{\frac{2n}{n-2}}^{\frac{(m+p-1)(1-\sigma)}{m}}\left\|v^{\frac{m+1}{2m}}(t)\right\|_2^{\frac{2(m+p-1)\sigma}{m+1}}\leqslant C^{\frac{(m+p-1)(1-\sigma)}{m}}\left\|\nabla v(t)\right\|_2^{\frac{(m+p-1)(1-\sigma)}{m}}\left\|v^{\frac{m+1}{2m}}(t)\right\|_2^{\frac{2(m+p-1)\sigma}{m+1}}, \tag{29}$$

where C denotes the optimal constant of the embedding $H^1(\Omega)\hookrightarrow L^{\frac{2n}{n-2}}(\Omega)$. Keeping $p<(m+1)\left(1+\frac{2}{n}\right)$ in mind, one can show that $2-\frac{(m+p-1)(1-\sigma)}{m}>0$. And hence, (29) implies that

$$\left\|\nabla v(t)\right\|_2\leqslant C^{\frac{(m+p-1)(1-\sigma)}{2m-(m+p-1)(1-\sigma)}}\left\|v^{\frac{m+1}{2m}}(t)\right\|_2^{\frac{2m\sigma(m+p-1)}{(m+1)[2m-(m+p-1)(1-\sigma)]}}. \tag{30}$$

From (9), (29) and (30), it follows that,

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\rho(t)&=-K(v(t))=\left\|v(t)\right\|_{\frac{m+p-1}{m}}^{\frac{m+p-1}{m}}-\left\|\nabla v(t)\right\|_2^2<\left\|v(t)\right\|_{\frac{m+p-1}{m}}^{\frac{m+p-1}{m}}\leqslant C^{\frac{(m+p-1)(1-\sigma)}{m}}\left\|\nabla v(t)\right\|_2^{\frac{(m+p-1)(1-\sigma)}{m}}\left\|v^{\frac{m+1}{2m}}(t)\right\|_2^{\frac{2(m+p-1)\sigma}{m+1}}\\&\leqslant C^{\frac{2(m+p-1)(1-\sigma)}{2m-(1-\sigma)(m+p-1)}}\left\|v^{\frac{m+1}{2m}}(t)\right\|_2^{\frac{4m\sigma(m+p-1)}{(m+1)[2m-(m+p-1)(1-\sigma)]}}<C^{\frac{2(m+p-1)(1-\sigma)}{2m-(1-\sigma)(m+p-1)}}\left(\left\|v^{\frac{m+1}{2m}}(t)\right\|_2^2+\left\|v^{\frac{m+1}{2m}}(t)\right\|_{2,\Gamma_1}^2\right)^{\frac{2m\sigma(m+p-1)}{(m+1)[2m-(m+p-1)(1-\sigma)]}}\\&=S_3\left[\rho(t)\right]^{\frac{2m\sigma(m+p-1)}{(m+1)[2m-(m+p-1)(1-\sigma)]}},\end{aligned} \tag{31}$$

where,

$$S_3=C^{\frac{2(m+p-1)(1-\sigma)}{2m-(1-\sigma)(m+p-1)}}(m+1)^{\frac{2m\sigma(m+p-1)}{(m+1)[2m-(m+p-1)(1-\sigma)]}}.$$

It is not difficult to verify that,

$$\frac{2m\sigma(m+p-1)}{(m+1)\left[2m-(m+p-1)(1-\sigma)\right]}>1.$$

From (31), it follows that,

$$\left[\rho(t)\right]^{1-\frac{2m\sigma(m+p-1)}{(m+1)[2m-(m+p-1)(1-\sigma)]}}-\left[\rho(0)\right]^{1-\frac{2m\sigma(m+p-1)}{(m+1)[2m-(m+p-1)(1-\sigma)]}}>S_3\left[1-\frac{2m\sigma(m+p-1)}{(m+1)\left[2m-(m+p-1)(1-\sigma)\right]}\right]t. \tag{32}$$

Letting $t\to T$, then (32) results in:

$$-\left[\rho(0)\right]^{1-\frac{2m\sigma(m+p-1)}{(m+1)[2m-(m+p-1)(1-\sigma)]}}>S_3\left[1-\frac{2m\sigma(m+p-1)}{(m+1)\left[2m-(m+p-1)(1-\sigma)\right]}\right]T,$$

which means that,

$$T>\frac{\left[\rho(0)\right]^{1-\frac{2m\sigma(m+p-1)}{(m+1)[2m-(m+p-1)(1-\sigma)]}}}{S_3\left[\frac{2m\sigma(m+p-1)}{(m+1)\left[2m-(m+p-1)(1-\sigma)\right]}-1\right]}=\tilde{S}\left(\left\|v_0^{\frac{m+1}{2m}}\right\|_2^2+\left\|v_0^{\frac{m+1}{2m}}\right\|_{2,\Gamma_1}^2\right)^{1-\frac{2m\sigma(m+p-1)}{(m+1)[2m-(m+p-1)(1-\sigma)]}},$$

where,

$$\tilde{S}=S_3^{-1}\left[\frac{2m\sigma(m+p-1)}{(m+1)\left[2m-(m+p-1)(1-\sigma)\right]}-1\right]^{-1}(m+1)^{\frac{2m\sigma(m+p-1)}{(m+1)[2m-(m+p-1)(1-\sigma)]}-1}. \tag{33}$$

The proof of Theorem 2 is completed.

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