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Canonical Transformations and Poisson Theory for Dynamics with Non-Standard Lagrangians

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Abstract: The canonical transformation and Poisson theory of dynamical systems with exponential, power-law, and logarithmic non-standard Lagrangians are studied, respectively. The criterion equations of canonical transformation are established, and four basic forms of canonical transformations are given. The dynamic equations with non-standard Lagrangians admit Lie algebraic structure. From this, we establish the Poisson theory, which makes it possible to find new conservation laws through known conserved quantities. Some examples are put forward to demonstrate the use of the theory and verify its effectiveness.

Key words: non-standard Lagrangians; dynamical equations; canonical transformation; Poisson theory
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0 Introduction

As commonly understood, for a system of particles, the kinetic energy minus the potential energy is defined as a Lagrangian, which can be called the standard Lagrangian or the natural Lagrangian. The so-called "non-standard Lagrangian", in which, in general, there is neither kinetic energy item nor potential energy item can be traced back to the "non-natural Lagrangians" mentioned by Arnold in his monograph [1]. In recent years, it has been found that many nonconservative systems or nonlinear dynamical equations, such as the nonlinear Emden equation, Riccati equation, Lienard nonlinear oscillation equation, Duffing-van der Pol equation, etc., can be derived from variational principles whose action functional takes non-standard Lagrangians as the integrand. To address dissipative and nonlinear dynamics problems, El-Nabulsi [2] introduced two actions with non-standard Lagrangians, namely exponential form, and power-law form, and derived their corresponding Euler-Lagrange equations. Saha and Talukdar[3,4] researched the variational inverse problem, and constructed several important non-standard Lagrangians for dissipative and nonlinear dynamics, such as oscillators moving in viscous media, the nonlinear Emden equation, the Lotka-Volterra model, etc. Musielak[5] constructed Lagrangians for

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variable-coefficient dissipative dynamics systems, including standard and non-standard Lagrangians. Symmetry and conserved quantity have always been a hot spot in analytical mechanics [6-8]. Recently, Zhang and his collaborators have studied Noether theorems for systems with non-standard Lagrangians [9,10], non-standard Hamiltonians[11], non-standard Birkhoffians[12], and Lie symmetry [13,14], Mei symmetry [15], first integral and method of reduction [16,17]. There have been some results about nonlinear dynamical equations and their symmetries with non-standard Lagrangians[18-26], but canonical transformation and Poisson theory for non-standard Lagrangian dynamics have not been involved.

Transformation is an important technique to study problems in analytical mechanics. Through variable transformation, the original complex differential equations become more accessible to solve. Canonical transformation is an essential kind of transformation, and the Hamilton equation retains its canonical form under canonical transformation, thus laying the foundation for Hamilton-Jacobi theory [27-29]. In recent years, classical canonical transformation theory has been extended to fractional mechanical systems [30], Birkhoff systems [31,32], and time scale cases [33]. However, the transformation theory of the non-standard Lagrangian system is still an open subject. The first integral of a dynamic system provides excellent convenience for solving the differential equations of motion. Poisson theory is an important tool for many scholars to study dynamic systems. In the classical mechanics, the Poisson theory encompasses the definition of Poisson parentheses, the establishment of Poisson conditions for the first integral, and the derivation of a new first integral based on a known one. Mei [34-36] studied the algebraic structure and Poisson theory for holonomic and nonholonomic systems, specifically Birkhoff systems. Subsequently, research on Poisson theory has yielded a series of significant achievements [37-41]. Since dynamical systems constructed from non-standard Lagrangians are usually nonlinear and can be simplified using canonical transformations, it is possible to find more first integrals using Poisson's method. In this article, we will explore the canonical transformation and Poisson theory with exponential, power-law, and logarithmic non-standard Lagrangians and use the two tools of canonical transformation and Poisson theory to solve the motion and first integrals of nonlinear system.

1 Dynamic Systems with Exponential Lagrangians

1.1 Canonical Transformations

Let the configuration of the mechanical system be determined by N generalized coordinates $q_k(k=1,2,\cdots,N)$, the action with exponential Lagrangians is:

$$S_{\mathrm{E}}=\int_{t_1}^{t_2}\exp\left[L\left(t,q_k,\dot{q}_k\right)\right]\mathrm{d}t \tag{1}$$

Corresponding to action (1), the Hamilton principle is expressed as:

$$\delta S_{\mathrm{E}}=0 \tag{2}$$

And the dynamic equations are [6]

$$\exp(L)\left(\frac{\partial L}{\partial q_k}-\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}_k}-\frac{\partial L}{\partial \dot{q}_k}\frac{\mathrm{d}L}{\mathrm{d}t}\right)=0(k=1,2,\cdots,N) \tag{3}$$

Define the generalized momentum corresponding to exponential Lagrangians as:

$$p_k=\frac{\partial \exp L}{\partial \dot{q}_k}=\exp L\frac{\partial L}{\partial \dot{q}_k} \tag{4}$$

and the exponential Hamiltonian as:

$$\exp\left[H\left(t,q_k,p_k\right)\right]=p_k\dot{q}_k-\exp\left[L\left(t,q_k,\dot{q}_k\right)\right] \tag{5}$$

After canonicalization, the equations (3) read:

$$\dot{q}_k=\exp(H)\frac{\partial H}{\partial p_k},\ \dot{p}_k=-\exp(H)\frac{\partial H}{\partial q_k} \tag{6}$$

We study the transformation from variables q_k and p_k to new variables Q_k and P_k, namely:

$$Q_k=Q_k\left(t,q_s,p_s\right),\ P_k=P_k\left(t,q_s,p_s\right) \tag{7}$$

Let $\Delta=\dfrac{\partial\left(Q_k,P_k\right)}{\partial\left(q_k,p_k\right)}\neq 0$, then, the transformation is invertible. If under transformation (7), the form of equations (6) remains unchanged, i.e.,

$$\dot{Q}_k=\exp(K)\frac{\partial K}{\partial P_k},\ \dot{P}_k=-\exp(K)\frac{\partial K}{\partial Q_k} \tag{8}$$

where $K=K\left(t,Q_k,P_k\right)$ is a new function. The transformation (7) is the canonical transformation.

Since equation (6) is deduced from principle (2), the expression in equation (8) must align with that of (6). Therefore, we have:

$$\delta\int_{t_1}^{t_2}\left[p_k\dot{q}_k-\exp(H)\right]\mathrm{d}t=0 \tag{9}$$

$$\delta\int_{t_1}^{t_2}\left[P_k\dot{Q}_k-\exp(K)\right]\mathrm{d}t=0 \tag{10}$$

Therefore, the integrand functions can be written as the following relation:

$$\left[p_k\dot{q}_k-\exp(H)\right]-\left[P_k\dot{Q}_k-\exp(K)\right]=\frac{\mathrm{d}F}{\mathrm{d}t} \tag{11}$$

where an arbitrary function F is called a generating function. Equation (11) can be written as:

$$p_k \mathrm{d}q_k - P_k \mathrm{d}Q_k + \left[\exp(K) - \exp(H)\right]\mathrm{d}t = \mathrm{d}F \quad (12)$$

Equation (12) is the criterion equation for the canonical transformation. Since $4N$ canonical variables q_k, p_k and Q_k, P_k are related by $2N$ transformation relations (7), only $2N$ variables are independent. We can select the generating function F as a function of $2N$ variables and thus obtain the following four basic types of canonical transformations.

The first type of generating function, denoted by F_1, has the form:

$$F = F_1(t, q_k, Q_k) \quad (13)$$

Substituting formula (13) into equation (12), we obtain:

$$\left(p_k - \frac{\partial F_1}{\partial q_k}\right)\mathrm{d}q_k + \left(-P_k - \frac{\partial F_1}{\partial Q_k}\right)\mathrm{d}Q_k + \left(\exp(K) - \exp(H) - \frac{\partial F_1}{\partial t}\right)\mathrm{d}t = 0 \quad (14)$$

Let the coefficients of $\mathrm{d}q_k$, $\mathrm{d}Q_k$, and $\mathrm{d}t$ be zero, respectively, and we get:

$$p_k = \frac{\partial F_1}{\partial q_k},\ P_k = -\frac{\partial F_1}{\partial Q_k},\ \exp(K) = \exp(H) + \frac{\partial F_1}{\partial t} \quad (15)$$

The second type of generating function F_2 is:

$$F_2(t, q_k, P_k) = F_1(t, q_k, Q_k) + Q_k P_k \quad (16)$$

Substituting formula (16) into equation (12), we obtain:

$$\left(p_k - \frac{\partial F_2}{\partial q_k}\right)\mathrm{d}q_k + \left(Q_k - \frac{\partial F_2}{\partial P_k}\right)\mathrm{d}P_k + \left(\exp(K) - \exp(H) - \frac{\partial F_2}{\partial t}\right)\mathrm{d}t = 0 \quad (17)$$

So we have:

$$p_k = \frac{\partial F_2}{\partial q_k},\ Q_k = \frac{\partial F_2}{\partial P_k},\ \exp(K) = \exp(H) + \frac{\partial F_2}{\partial t} \quad (18)$$

The third type of generating function F_3 is:

$$F_3(t, p_k, Q_k) = F_1(t, q_k, Q_k) - q_k p_k \quad (19)$$

Then we get:

$$q_k = -\frac{\partial F_3}{\partial p_k},\ P_k = -\frac{\partial F_3}{\partial Q_k},\ \exp(K) = \exp(H) + \frac{\partial F_3}{\partial t} \quad (20)$$

The fourth type of generating function F_4 is:

$$F_4(t, p_k, P_k) = F_1(t, q_k, Q_k) - q_k p_k + Q_k P_k \quad (21)$$

Then we get:

$$q_k = -\frac{\partial F_4}{\partial p_k},\ Q_k = \frac{\partial F_4}{\partial P_k},\ \exp(K) = \exp(H) + \frac{\partial F_4}{\partial t} \quad (22)$$

1.2 Poisson Theory

Let $a^k = q_k$, $a^{n+k} = p_k$ $(k = 1, 2, \cdots, N)$, then equations (6) can be transformed into contravariant algebraic form:

$$\dot{a}^\mu = \exp(H)\,\omega^{\mu\nu}\frac{\partial H}{\partial a^\nu}(\mu, \nu = 1, 2, \cdots, 2n) \quad (23)$$

where $\left(\omega^{\mu\nu}\right) = \begin{bmatrix} 0_{n\times n} & I_{n\times n} \\ -I_{n\times n} & 0_{n\times n} \end{bmatrix}$ is a contravariant tensor.

Let $A(a)$ be some functions. According to equation (23), we define a product

$$\dot{A}(a) = \frac{\partial A}{\partial a^\mu}\omega^{\mu\nu}\frac{\partial \exp(H)}{\partial a^\nu} = A \circ \exp(H) \quad (24)$$

It is easy to verify that the product (24) satisfies the left distributive law, right distributive law and scalar law. Furthermore, it adheres to antisymmetry and Jacobi identity. Thus, we have:

Theorem 1 The system (23), which is determined by exponential Lagrangians, admits not only a compatible algebraic structure but also a Lie algebraic structure.

Given that Eq. (23) possesses a Lie algebraic structure, we can establish Poisson theory as follows:

Theorem 2 The sufficient and necessary condition for $I(t, a^\mu) = \text{const.}$ to be a first integral of the system (23) is that:

$$\frac{\partial I}{\partial t} + I \circ \exp(H) = 0 \quad (25)$$

Formula (25) is referred to as the generalized Poisson condition with exponential Lagrangians.

Theorem 3 If the exponential Hamiltonian does not depend on t, then $\exp(H) = h$ is the first integral of the system (23).

Theorem 4 If $I_1(t, a^\mu)$ and $I_2(t, a^\mu)$ are two first integrals of the system (23), which are not involute, then $I_3 = I_1 \circ I_2$ is also a first integral.

Theorem 5 If $I(t, a^\mu)$ is a first integral of the system (23) which involves t, but the exponential Hamiltonian does not depend on t, then $\frac{\partial I}{\partial t}, \frac{\partial^2 I}{\partial t^2}, \cdots$ are all first integrals.

Theorem 6 If $I(t, a^\mu)$ is a first integral of the system (23), which involves the variable a^ρ, but the exponential Hamiltonian does not depend on a^ρ, then $\frac{\partial I}{\partial a^\rho}, \frac{\partial^2 I}{\partial a^{\rho 2}}, \cdots$ are all first integrals.

1.3 Examples

Example 1 Consider a nonlinear dynamical system with exponential Lagrangians. The action is [2]

$$S_{\mathrm{E}} = \int_{t_1}^{t_2} \exp(tq\dot{q})\,\mathrm{d}t \quad (26)$$

Try to find the canonical transformation of the system and solve its motion.

From action (26), we get $L=tq\dot{q}$. Substituting $L=tq\dot{q}$ into equation (3), we get:

$$tq\ddot{q}+q\dot{q}+t\dot{q}^2+\frac{1}{t}=0 \tag{27}$$

According to Eqs. (4) and (5), we get:

$$p=tq\exp(tq\dot{q}) \tag{28}$$

$$H=\frac{p}{tq}\left(\ln\frac{p}{q}-\ln t-1\right) \tag{29}$$

Select the generating function:

$$F=F_1(t,q,Q)=\frac{1}{2}q^2\mathrm{e}^Q \tag{30}$$

From Eq. (15), we get:

$$p=\frac{\partial F_1}{\partial q}=q\mathrm{e}^Q, P=-\frac{\partial F_1}{\partial Q}=-\frac{1}{2}q^2\mathrm{e}^Q \tag{31}$$

And the new Hamiltonian K is:

$$K=\frac{1}{t}\mathrm{e}^Q(Q-\ln t-1) \tag{32}$$

The canonical equation with Q,P as the new variables is:

$$\dot{Q}=\frac{\partial K}{\partial P}=0, \dot{P}=-\frac{\partial K}{\partial Q}=\frac{1}{t}\mathrm{e}^Q(\ln t-Q) \tag{33}$$

By integrating the above equation, we get:

$$Q=c_1, P=\frac{1}{2}\ln^2 t-c_1\ln t+c_2 \tag{34}$$

Substituting Eq. (34) into Eq. (31), we get:

$$q=\pm\sqrt{2c_1\ln t-\ln^2 t+2c_3} \tag{35}$$

Example 2 On the basis of $F_1=\frac{1}{2}q^2\mathrm{e}^Q$ in Example 1, we try to find canonical transformations of the other three basic forms.

Take the generating function as:

$$F_2(t,q,P)=F_1+QP=-P+P\ln\frac{-2P}{q^2} \tag{36}$$

Then the transformation (18) gives:

$$p=\frac{\partial F_2}{\partial q}=-\frac{2P}{q}, Q=\frac{\partial F_2}{\partial P}=\ln\frac{-2P}{q^2} \tag{37}$$

Take

$$F_3(t,Q,p)=F_1-qp=-\frac{p^2}{2\mathrm{e}^Q} \tag{38}$$

Then the transformation (20) gives:

$$q=-\frac{\partial F_3}{\partial p}=\frac{p}{\mathrm{e}^Q}, P=-\frac{\partial F_3}{\partial Q}=-\frac{p^2}{2\mathrm{e}^Q} \tag{39}$$

Take

$$F_4(t,P,p)=F_1-qp+QP=P+P\ln\frac{p^2}{-2P} \tag{40}$$

Then the transformation (22) gives:

$$q=-\frac{\partial F_4}{\partial p}=\frac{p}{\mathrm{e}^Q}, Q=\frac{\partial F_4}{\partial P}=\ln\frac{p^2}{-2P} \tag{41}$$

Example 3 Try to study the first integral of the system in Example 1.

Let

$$q=a^1, p=a^2 \tag{42}$$

The Hamiltonian (29) can be written as

$$H=\frac{a^2}{ta^1}\left(\ln\frac{a^2}{a^1}-\ln t-1\right) \tag{43}$$

The equation exhibits a Lie algebraic structure. Utilizing the generalized Poisson condition (25), we can readily verify that:

$$I_1=\frac{a^2}{a^1}+\frac{a^2}{a^1}\ln\frac{a^2}{a^1}=c_1 \tag{44}$$

$$I_2=\mathrm{e}^{\frac{a^2}{a^1}}=c_2 \tag{45}$$

are the first integrals of the system.

Assuming the initial conditions $q=1$ and $\dot{q}=1$, the trajectory of motion q, the Hamiltonian H, and the conserved quantities I_1 and I_2 can be easily calculated, as shown in Fig. 1 and Fig. 2.

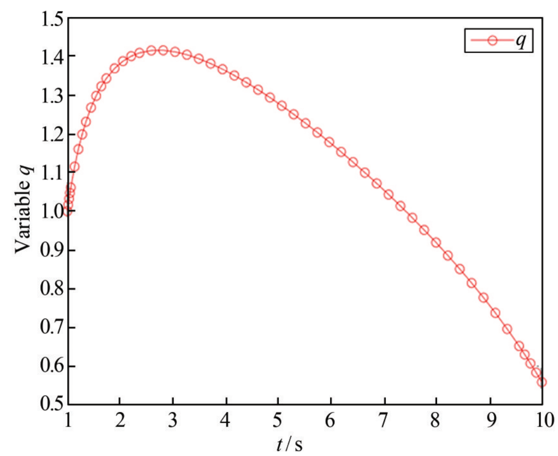

Fig. 1 Simulation of the trajectory of motion $q(t)$ in Eq. (35)

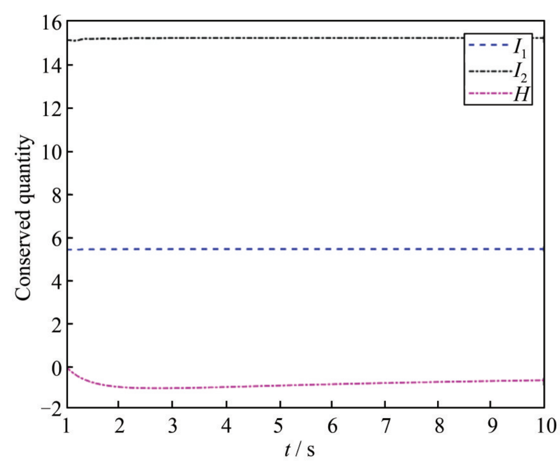

Fig. 2 The values of the Hamiltonian in Eq. (43) and the conserved quantities in Eqs. (44)-(45)

2 Dynamic Systems with Power-Law Lagrangians

2.1 Canonical Transformations

Let the configuration of the mechanical system be determined by N generalized coordinates $q_s(s=1,2,\cdots,N)$, the action with power-law Lagrangians is:

$$S_{\mathrm{P}}=\int_{t_1}^{t_2} L^{1+\gamma}(t,q_s,\dot{q}_s)\,\mathrm{d}t \tag{46}$$

Corresponding to action (46), the Hamilton principle is expressed as:

$$\delta S_{\mathrm{P}}=0 \tag{47}$$

And the dynamic equations are [6]

$$(1+\gamma)L^{\gamma}\left[\frac{\partial L}{\partial q_s}-\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}_s}-\frac{\gamma}{L}\frac{\partial L}{\partial \dot{q}_s}\frac{\mathrm{d}L}{\mathrm{d}t}\right]=0\;(s=1,2,\cdots,N) \tag{48}$$

where $\gamma\neq-1$.

Define the generalized momentum corresponding to power-law Lagrangians as:

$$p_s=\frac{\partial L^{1+\gamma}}{\partial \dot{q}_s}=(1+\gamma)L^{\gamma}\frac{\partial L}{\partial \dot{q}_s} \tag{49}$$

And the power-law Hamiltonian as:

$$H^{1+\gamma}(t,q_s,p_s)=p_s\dot{q}_s-L^{1+\gamma}(t,q_s,\dot{q}_s) \tag{50}$$

After canonicalization, the equations (48) read:

$$\dot{q}_s=(1+\gamma)H^{\gamma}\frac{\partial H}{\partial p_s},\; \dot{p}_s=-(1+\gamma)H^{\gamma}\frac{\partial H}{\partial q_s} \tag{51}$$

We study the transformation from variables q_k and p_k to new variables Q_k and P_k, namely:

$$Q_s=Q_s(t,q_k,\dot{q}_k),\quad P_s=P_s(t,q_k,\dot{q}_k) \tag{52}$$

Let $\Delta=\dfrac{\partial(Q_s,P_s)}{\partial(q_s,p_s)}\neq0$, then, the transformation is invertible. If under transformation (52), the form of equation (51) remains unchanged, i.e.,

$$\dot{Q}_s=(1+\gamma)K^{\gamma}\frac{\partial K}{\partial P_s},\quad \dot{P}_s=-(1+\gamma)K^{\gamma}\frac{\partial K}{\partial Q_s} \tag{53}$$

where $K=K(t,Q_s,P_s)$ is a new function. The transformation (52) is the canonical transformation.

Since equation (51) is derived from principle (47), the expression in equation (53) must align with (51). Thus, we have:

$$\delta\int_{t_1}^{t_2}(p_s\dot{q}_s-H^{1+\gamma})\,\mathrm{d}t=0 \tag{54}$$

$$\delta\int_{t_1}^{t_2}(P_s\dot{Q}_s-K^{1+\gamma})\,\mathrm{d}t=0 \tag{55}$$

Therefore, the criterion equation for the canonical transformation is:

$$p_s\mathrm{d}q_s-P_s\mathrm{d}Q_s+(K^{1+\gamma}-H^{1+\gamma})\mathrm{d}t=\mathrm{d}F \tag{56}$$

According to the case that the generating function contains old and new variables, we get the following four basic types of canonical transformation.

The first type of generating function, denoted by F_1, has the form

$$F=F_1(t,q_s,Q_s) \tag{57}$$

Substituting formula (57) into equation (56), we obtain

$$\left(p_s-\frac{\partial F_1}{\partial q_s}\right)\mathrm{d}q_s+\left(-P_s-\frac{\partial F_1}{\partial Q_s}\right)\mathrm{d}Q_s+\left(K^{1+\gamma}-H^{1+\gamma}-\frac{\partial F_1}{\partial t}\right)\mathrm{d}t=0 \tag{58}$$

Let the coefficients of $\mathrm{d}q_k$, $\mathrm{d}Q_k$, and $\mathrm{d}t$ be zero, respectively, we have:

$$p_s=\frac{\partial F_1}{\partial q_s},P_s=-\frac{\partial F_1}{\partial Q_s},K^{1+\gamma}=H^{1+\gamma}+\frac{\partial F_1}{\partial t}\;(s=1,2,\cdots,N) \tag{59}$$

The second type of generating function F_2 is:

$$F_2(t,q_s,P_s)=F_1(t,q_s,Q_s)+Q_sP_s \tag{60}$$

Substituting formula (60) into equation (56), we obtain:

$$\left(p_s-\frac{\partial F_2}{\partial q_s}\right)\mathrm{d}q_s+\left(Q_s-\frac{\partial F_2}{\partial P_s}\right)\mathrm{d}P_s+\left(K^{1+\gamma}-H^{1+\gamma}-\frac{\partial F_2}{\partial t}\right)\mathrm{d}t=0 \tag{61}$$

So we have:

$$p_s=\frac{\partial F_2}{\partial q_s},Q_s=\frac{\partial F_2}{\partial P_s},K^{1+\gamma}=H^{1+\gamma}+\frac{\partial F_2}{\partial t}\;(s=1,2,\cdots,N) \tag{62}$$

The third type of generating function is:

$$F_3(t,p_s,Q_s)=F_1(t,q_s,Q_s)-q_sp_s \tag{63}$$

Then we have:

$$q_s=-\frac{\partial F_3}{\partial p_s},P_s=-\frac{\partial F_3}{\partial Q_s},\;K^{1+\gamma}=H^{1+\gamma}+\frac{\partial F_3}{\partial t}\;(s=1,2,\cdots,N) \tag{64}$$

The fourth type of generating function is:

$$F_4(t,p_s,P_s)=F_1(t,q_s,Q_s)-q_sp_s+Q_sP_s \tag{65}$$

Then we have:

$$q_s=-\frac{\partial F_4}{\partial p_s},Q_s=\frac{\partial F_4}{\partial P_s},\;K^{1+\gamma}=H^{1+\gamma}+\frac{\partial F_4}{\partial t}\;(s=1,2,\cdots,N) \tag{66}$$

2.2 Poisson Theory

Let $a^s=q_s$, $a^{n+s}=p_s$ $(s=1,2,\cdots,N)$, then equation (51) can be transformed into a contravariant algebraic form

$$\dot{a}^{\mu}=\omega^{\mu\nu}\frac{\partial H^{1+\gamma}}{\partial a^{\nu}}\;(\mu,\nu=1,2,\cdots,2n) \tag{67}$$

where $\left(\omega^{\mu\nu}\right)=\begin{bmatrix}0_{n\times n} & I_{n\times n}\\ -I_{n\times n} & 0_{n\times n}\end{bmatrix}$ is a contravariant tensor.

Let $A(a)$ be some function. According to equation (67), we define a product

$$\dot{A}(a)=\frac{\partial A}{\partial a^{\mu}}\omega^{\mu\nu}\frac{\partial H^{1+\gamma}}{\partial a^{\nu}}=A\circ H^{1+\gamma} \tag{68}$$

The product (68) conforms to Lie algebra axioms and thus has Theorem 7.

Theorem 7 The system (67), which is determined by power-law Lagrangians, admits not only a compatible algebraic structure but also a Lie algebraic structure.

Since Eq. (67) has a Lie algebraic structure, we can establish Poisson theory as follows:

Theorem 8 The sufficient and necessary condition for $I\left(t,a^{\mu}\right)=\text{const.}$ to be a first integral of the system (67) is that:

$$\frac{\partial I}{\partial t}+I\circ H^{1+\gamma}=0 \tag{69}$$

Formula (69) is termed the generalized Poisson condition for power-law Lagrangians.

Theorem 9 If the power-law Hamiltonian does not depend on t, $H^{1+\gamma}=h$ is a first integral of the system (67).

Theorem 10 If $I_1\left(t,a^{\mu}\right)$ and $I_2\left(t,a^{\mu}\right)$ are two first integrals of the system (67), which are not involute, then $I_3=I_1\circ I_2$ is a first integral.

Theorem 11 If $I\left(t,a^{\mu}\right)$ is a first integral of the system (67) which involves t, but the power-law Hamiltonian does not depend on t, then $\frac{\partial I}{\partial t},\frac{\partial^2 I}{\partial t^2},\cdots$ are all first integrals.

Theorem 12 If $I\left(t,a^{\mu}\right)$ is a first integral of the system (67) which involves the variable a^{ρ}, but the power-law Hamiltonian does not depend on the variable a^{ρ}, then $\frac{\partial I}{\partial a^{\rho}},\frac{\partial^2 I}{\partial a^{\rho 2}},\cdots$ all are first integrals.

2.3 Examples

Example 4 The action with power-law Lagrangians is [3]

$$S_{\mathrm{P}}=\int_{t_1}^{t_2}\left(\dot{q}+kq^2\right)^{-1}\mathrm{d}t \tag{70}$$

Try to find the canonical transformation of the system and solve its motion.

In this problem, $L=\dot{q}+kq^2, \gamma=-2$. Let $k=1$, then the dynamical equation reads:

$$\ddot{q}+3q\dot{q}+q^3=0 \tag{71}$$

This is an over-damped system. According to Eqs. (49) and (50), we get:

$$p=-\frac{1}{\left(\dot{q}+q^2\right)^2} \tag{72}$$

$$H=-2\sqrt{-p}-pq^2 \tag{73}$$

Select the generating function:

$$F=F_1\left(q,Q,t\right)=-qQ^2 \tag{74}$$

From Eq. (59), we get:

$$p=\frac{\partial F_1}{\partial q}=-Q^2, P=-\frac{\partial F_1}{\partial Q}=2qQ \tag{75}$$

and

$$K=-2Q+\frac{P^2}{4} \tag{76}$$

The canonical equation with Q,P as the new variables is:

$$\dot{Q}=\frac{\partial K}{\partial P}=\frac{1}{2}P, \dot{P}=-\frac{\partial K}{\partial Q}=2 \tag{77}$$

By integrating the above equation, we get:

$$Q=4t^2+c_1, P=2t+c_2 \tag{78}$$

By substituting Eqs. (78) into Eqs. (75), it can be concluded that the damping harmonic vibration law of the overdamped system is:

$$q=\frac{2t+c_2}{2\left(4t^2+c_1\right)} \tag{79}$$

Example 5 On the basis of $F_1=-qQ^2$ in Example 4, try to find the generating functions and canonical transformation of the other three basic forms.

Take the generating function as:

$$F_2\left(t,q,P\right)=F_1+QP=\frac{P^2}{4q} \tag{80}$$

Then the transformation (62) gives:

$$p=\frac{\partial F_2}{\partial q}=-\frac{p^2}{4q^2}, Q=\frac{\partial F_2}{\partial P}=\frac{P}{2q} \tag{81}$$

In this case, the generating function F_3 is:

$$F_3\left(t,p,Q\right)=F_1-qp=0 \tag{82}$$

Take

$$F_4\left(t,P,p\right)=F_1-qp+QP=P\sqrt{-p} \tag{83}$$

Then the transformation (66) gives:

$$q=-\frac{\partial F_4}{\partial p}=\frac{P}{2\sqrt{-p}}, Q=\frac{\partial F_4}{\partial P}=\sqrt{-p} \tag{84}$$

Example 6 Try to study the first integral of the system in Example 1.

Let

$$q=a^1, p=a^2 \tag{85}$$

The Hamiltonian (73) can be written as:

$$H=-2\sqrt{-a^2}-a^2\left(a^1\right)^2 \tag{86}$$

The equation has a Lie algebraic structure.

Since H does not explicitly contain t, the system has an integral $H=h$ by Theorem 9.

By using the generalized Poisson condition (69), we can easily verify that:

$$I_1=\sqrt{-a^2}+\frac{a^2\left(a^1\right)^2}{2}=c_1 \tag{87}$$

$$I_2=\frac{1}{2}t^2+\left(1-ta^1\right)\sqrt{-a^2}=c_2 \tag{88}$$

$$I_3=-\left(\left(-a^2\right)^{\frac{1}{2}}+\frac{1}{2}\left(a^1\right)^2\left(-a^2\right)\right)^{-\frac{1}{2}}=c_3 \tag{89}$$

are the first integrals of the system.

Calculate the Poisson parentheses:

$$I_4=I_1\circ I_2=\frac{1}{2}a^1\left(-a^2\right)^{\frac{1}{2}}-\frac{1}{2}t=c_4 \tag{90}$$

According to Theorem 10, I_4 is also a first integral.

Since I_2 involves time t, while the power-law Hamiltonian $H^{1+\gamma}$ does not explicitly involve time t, according to Theorem 11, we get

$$I_5=\frac{\partial I_2}{\partial t}=t-a^1\sqrt{-a^2}=c_5 \tag{91}$$

Formula (91) is the first integral.

Assuming the initial conditions $q=1$ and $\dot{q}=1$, the trajectory of motion q, the Hamiltonian H, and the conserved quantities I_1,I_2,I_3,I_4 and I_5 can be easily calculated, as shown in Fig. 3 and Fig. 4.

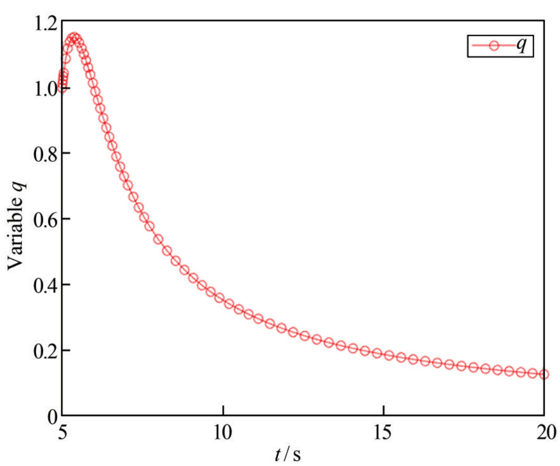

Fig. 3 Simulation of the trajectory of motion $q(t)$ in Eq. (79)

3 Dynamic Systems with Logarithmic Lagrangians

3.1 Canonical Transformations

Let the configuration of the mechanical system be determined by N generalized coordinates $q_k(k=$

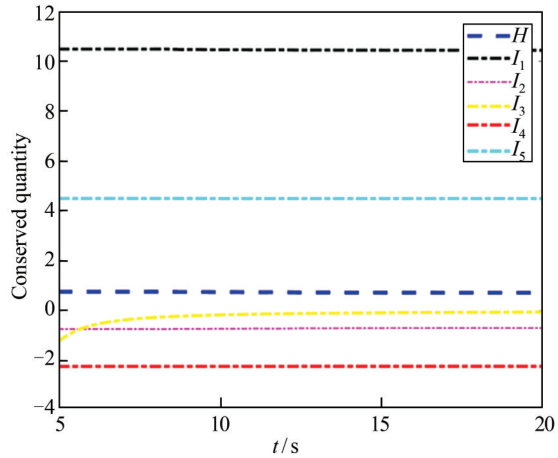

Fig. 4 The values of the Hamiltonian in Eq. (86) and the conserved quantities in Eqs. (87)-(91)

$1,2,\cdots,N)$, the action with logarithmic Lagrangians is:

$$S_L=\int_{t_1}^{t_2}\log_b L\left(t,q_k,\dot{q}_k\right)\mathrm{d}t,\ b>0 \tag{92}$$

Corresponding to action (92), the Hamilton principle is expressed as:

$$\delta S_L=0 \tag{93}$$

And the dynamic equations are:

$$\frac{1}{L\ln b}\left(\frac{\partial L}{\partial q_k}+\frac{1}{L}\frac{\partial L}{\partial \dot{q}_k}-\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}_k}\right)=0,\ k=1,2,\cdots,N \tag{94}$$

Define the generalized momentum corresponding to logarithmic Lagrangians as:

$$p_k=\frac{\partial \log_b L}{\partial \dot{q}_k}=\frac{1}{L\ln b}\frac{\partial L}{\partial \dot{q}_k} \tag{95}$$

And the logarithmic Hamiltonian as:

$$\log_b H\left(t,q_k,p_k\right)=p_k\dot{q}_k-\log_b L\left(t,q_k,\dot{q}_k\right) \tag{96}$$

Then equation (94) can be reduced to canonical equations as:

$$\dot{q}_k=\frac{1}{H\ln b}\frac{\partial H}{\partial p_k},\dot{q}_k=-\frac{1}{H\ln b}\frac{\partial H}{\partial q_k} \tag{97}$$

We study the transformation from variables q_k and p_k to new variables Q_k and P_k, namely:

$$Q_k=Q_k\left(t,q_s,p_s\right),P_k=P_k\left(t,q_s,p_s\right) \tag{98}$$

Let $\Delta=\frac{\partial\left(Q_k,P_k\right)}{\partial\left(q_k,p_k\right)}\neq 0$, then, the transformation is invertible. If under transformation (98), the form of equations (97) remains unchanged, i.e.,

$$\dot{Q}_k=\frac{1}{K\ln b}\frac{\partial K}{\partial P_k},\dot{P}_k=-\frac{1}{K\ln b}\frac{\partial K}{\partial Q_k} \tag{99}$$

where $K=K\left(t,Q_k,P_k\right)$ is a new function. The transformation (98) is the canonical transformation.

Since equation (97) is derived from principle (93), the form of equation (99) must agree with (97), so we have:

$$\delta\int_{t_1}^{t_2}\left(p_k\dot{q}_k-\log_b H\right)\mathrm{d}t=0 \tag{100}$$

$$\delta\int_{t_1}^{t_2}\left(P_k\dot{Q}_k-\log_b K\right)\mathrm{d}t=0 \tag{101}$$

Therefore, the integrand functions should satisfy the following criterion equation:

$$p_k\mathrm{d}q_k-P_k\mathrm{d}Q_k+\left(\log_b K-\log_b H\right)\mathrm{d}t=\mathrm{d}F \tag{102}$$

Four basic forms of canonical transformation are given below.

The first type of generating function, denoted as F_1, has the form:

$$F=F_1\left(t,q_k,Q_k\right) \tag{103}$$

By substituting Eq. (103) into Eq. (102), we get:

$$\left(p_k-\frac{\partial F_1}{\partial q_k}\right)\mathrm{d}q_k+\left(-P_k-\frac{\partial F_1}{\partial Q_k}\right)\mathrm{d}Q_k+\left(\log_b K-\log_b H-\frac{\partial F_1}{\partial t}\right)\mathrm{d}t=0 \tag{104}$$

Let the coefficients of $\mathrm{d}q_k$, $\mathrm{d}Q_k$, and $\mathrm{d}t$ be zero, respectively, we have:

$$p_k=\frac{\partial F_1}{\partial q_k},P_k=-\frac{\partial F_1}{\partial Q_k},$$
$$\log_b K=\log_b H+\frac{\partial F_1}{\partial t}\left(k=1,2,\cdots,N\right) \tag{105}$$

The second type of generating function F_2 is:

$$F_2\left(t,q_k,P_k\right)=F_1\left(t,q_k,Q_k\right)+Q_kP_k \tag{106}$$

By substituting equation (106) into equation (102), we get:

$$\left(p_k-\frac{\partial F_2}{\partial q_k}\right)\mathrm{d}q_i+\left(Q_k-\frac{\partial F_2}{\partial P_k}\right)\mathrm{d}P_i+\left(\log_b K-\log_b H-\frac{\partial F_2}{\partial t}\right)\mathrm{d}t=0 \tag{107}$$

So we have:

$$p_k=\frac{\partial F_2}{\partial q_k},Q_k=\frac{\partial F_2}{\partial P_k},\log_b K=\log_b H+\frac{\partial F_2}{\partial t}\left(k=1,2,\cdots,N\right) \tag{108}$$

The third type of generating function F_3 is:

$$F_3\left(t,p_k,Q_k\right)=F_1\left(t,q_k,Q_k\right)-q_kp_k \tag{109}$$

Then we have:

$$q_k=-\frac{\partial F_3}{\partial p_k},P_k=-\frac{\partial F_3}{\partial Q_k},\ \log_b K=\log_b H+\frac{\partial F_3}{\partial t} \tag{110}$$

The fourth type of generating function F_4 is:

$$F_4\left(t,p_k,P_k\right)=F_1\left(t,q_k,Q_k\right)-q_kp_k+Q_kP_k \tag{111}$$

Then we have:

$$q_k=-\frac{\partial F_4}{\partial p_k},Q_k=\frac{\partial F_4}{\partial P_k},\ \log_b K=\log_b H+\frac{\partial F_4}{\partial t} \tag{112}$$

3.2 Poisson Theory

Let $a^k=q_k$, $a^{n+k}=p_k\left(k=1,2,\cdots,N\right)$, then equation (97) can be transformed into a contravariant algebraic form:

$$\dot{a}^\mu=\omega^{\mu\nu}\frac{\partial\log_b H}{\partial a^\nu},\ \mu,\nu=1,2,\cdots,2n \tag{113}$$

where $\left(\omega^{\mu\nu}\right)=\begin{bmatrix}0_{n\times n} & I_{n\times n}\\ -I_{n\times n} & 0_{n\times n}\end{bmatrix}$ is a contravariant tensor.

Let $A\left(a\right)$ be some function. According to equation (113), we define a product

$$\dot{A}\left(a\right)=\frac{\partial A}{\partial a^\mu}\omega^{\mu\nu}\frac{\partial\log_b H}{\partial a^\nu}=A\circ\log_b H \tag{114}$$

The product (114) conforms to Lie algebra axioms and thus has Theorem 13.

Theorem 13 The system (113), which is determined by logarithmic Lagrangians, admits not only a compatible algebraic structure but also a Lie algebraic structure.

Since Eq. (113) has a Lie algebraic structure, we can establish Poisson theory as follows:

Theorem 14 The sufficient and necessary condition for $I\left(t,a^\mu\right)=\mathrm{const.}$ to be a first integral of the system (113) is that:

$$\frac{\partial I}{\partial t}+I\circ\log_b H=0 \tag{115}$$

Formula (115) is called the generalized Poisson condition with logarithmic Lagrangians.

Theorem 15 If the logarithmic Hamiltonian does not depend on t, $\log_b H=h$ is a first integral of the system (113).

Theorem 16 If $I_1\left(t,a^\mu\right)$ and $I_2\left(t,a^\mu\right)$ are two first integrals of the system (113), which are not in involution, then $I_3=I_1\circ I_2$ is a first integral.

Theorem 17 If $I\left(t,a^\mu\right)$ is a first integral of the system (113) which involves t, but the logarithmic Hamiltonian does not depend on t, then $\frac{\partial I}{\partial t},\frac{\partial^2 I}{\partial t^2},\cdots$ are all first integrals.

Theorem 18 If $I\left(t,a^\mu\right)$ is a first integral of the system (113) which involves the variable a^ρ, but the logarithmic Hamiltonian does not depend on the variable a^ρ, then $\frac{\partial I}{\partial a^\rho},\frac{\partial^2 I}{\partial a^{\rho 2}},\cdots$ are all first integrals.

3.3 Examples

Example 7 The action with logarithmic Lagrangians is [4]

$$S_L=\int_{t_1}^{t_2}\ln\left(\dot{q}+q^2\right)\mathrm{d}t \tag{116}$$

In this problem, we have $L=\dot{q}+q^2$. The differential equation of motion reads

$$\ddot{q}+4q\dot{q}+2q^3=0 \tag{117}$$

According to Eqs. (95) and (96), we get:

$$p=\frac{1}{\dot{q}+q^2} \tag{118}$$

$$H=1-pq^2+\ln p \tag{119}$$

Select the generating function:

$$F=F_1(t,q,Q)=qQ^2 \tag{120}$$

From Eq. (105), we get:

$$p=\frac{\partial F_1}{\partial q}=Q^2, P=-\frac{\partial F_1}{\partial Q}=-2qQ \tag{121}$$

and

$$K=1-\frac{P^2}{4}+2\ln Q \tag{122}$$

The canonical equation with Q,P as the new variables is:

$$\dot{Q}=\frac{\partial K}{\partial P}=-\frac{P}{2}, \dot{P}=-\frac{\partial K}{\partial Q}=-\frac{2}{Q} \tag{123}$$

From equation (123), we get:

$$\ddot{Q}=-\frac{\dot{P}}{2}, \ddot{Q}=\frac{1}{Q} \tag{124}$$

Solving the differential equation (124) gives:

$$\ln|Q|=\left(\operatorname{erf}^{-1}\frac{(t-c_2)e^{\frac{c_1}{2}}\sqrt{2}}{\sqrt{\pi}}\right)^2+c_3-\frac{c_1}{2} \tag{125}$$

From Eqs. (121) and (125), we get:

$$q=-\frac{P}{2Q}=\frac{1}{Q}\frac{\mathrm{d}Q}{\mathrm{d}t}=\frac{\mathrm{d}\ln Q}{\mathrm{d}t} \tag{126}$$

It can be concluded that the damped harmonic vibration law of the critical damped system is:

$$q=\sqrt{2}\,e^{(\operatorname{erf}^{-1}(z))^2-\frac{c_1}{2}}\operatorname{erf}^{-1}(z) \tag{127}$$

where,

$$z=\frac{(t-c_2)\sqrt{2}\,e^{\frac{c_1}{2}}}{\sqrt{\pi}} \tag{128}$$

Example 8 On the basis of $F_1=qQ^2$ in Example 7, try to find the generating function and canonical transformation of the other three basic forms.

Take the generating function as

$$F_2(t,q,P)=F_1+QP=-\frac{P^2}{4q} \tag{129}$$

Then, the transformation (108) gives

$$p=\frac{\partial F_2}{\partial q}=\frac{P^2}{4q^2}, Q=\frac{\partial F_2}{\partial P}=-\frac{P}{2q} \tag{130}$$

In this case, the generating function F_3 is:

$$F_3(t,p,Q)=F_1-qp=0 \tag{131}$$

Take

$$F_4(t,P,p)=F_1-qp+QP=P\sqrt{p} \tag{132}$$

Then the transformation (112) gives:

$$q=-\frac{\partial F_4}{\partial p}=-\frac{P}{2\sqrt{p}}, Q=\frac{\partial F_4}{\partial P}=\sqrt{p} \tag{133}$$

Example 9 Try to study the first integral in Example 7.

Let

$$q=a^1, p=a^2 \tag{134}$$

The Hamiltonian (119) can be expressed as

$$H=1-a^2(a^1)^2+\ln a^2 \tag{135}$$

The equation has a Lie algebraic structure.

Since H does not explicitly contain t, the system has the generalized energy integral $H=h$ by Theorem 15.

By using the generalized Poisson condition (115), we can easily verify that

$$I_1=(a^1)^2a^2-\ln a^2+\cosh t+\sinh t-e^t=c_1 \tag{136}$$

is a first integral.

Assuming the initial conditions $q=1$ and $\dot{q}=1$, the trajectory of motion q, the Hamiltonian H, and the conserved quantity I_1 can be easily calculated, as shown in Fig. 5 and Fig. 6.

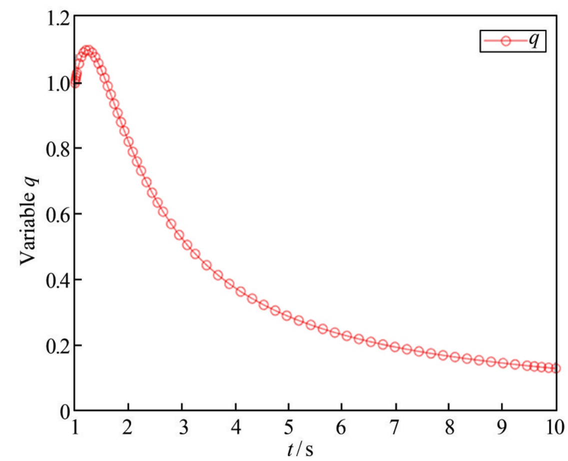

Fig. 5 Simulation of the trajectory of motion $q(t)$ in Eq. (127)

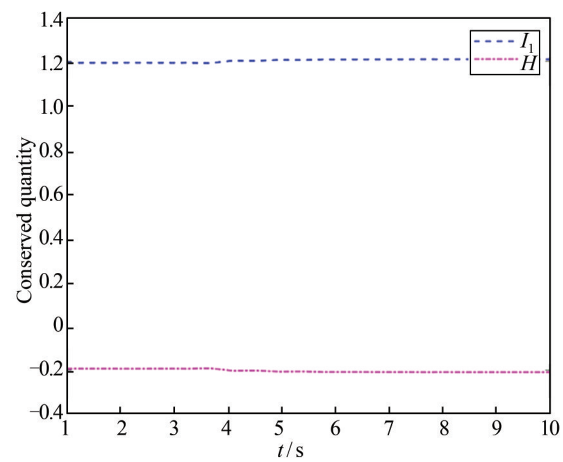

Fig. 6 The values of the Hamiltonian in Eq. (135) and the conserved quantity in Eq. (136)

4 Conclusion

In summary, we investigated the canonical transformation and Poisson theory of three kinds of dynamical equations featuring non-standard Lagrangians. We presented the criterion equations for the canonical transformation of the systems under consideration. According to different generating function choices, we gave four basic forms of canonical transformation. We verified that dynamic equations with non-standard Lagrangians have the Lie algebraic structure. Building upon this foundation, we formulated Poisson's theory, comprising five theorems that offer an approach for identifying new conservation laws from known conserved quantities. Several examples were presented to illustrate the application of the results and verify their effectiveness.

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