Wuhan University
Journal of Natural Sciences

2024, Vol.29 No.2, 117-124

Article ID 1007-1202(2024)02-0117-08 DOI https://doi.org/10.1051/wujns/2024292117

Cite this article: XIAO Li, YU Peichao. Robust Estimation of Average Treatment Effects with Observational Studies[J]. *Wuhan Univ J of Nat Sci*, 2024, 29(2): 117-124.

Robust Estimation of Average Treatment Effects with Observational Studies

□ **XIAO Li[1], YU Peichao[2†]**

1. Department of Physical Education, Guilin University of Aerospace Technology, Guilin 541004, Guangxi, China;
2. School of Statistics and Mathematics, Zhongnan University of Economics and Law, Wuhan 430073, Hubei, China

Abstract: Estimating treatment effects has always been one of the hot issues in empirical research. It brings great challenges to estimating treatment effects because heterogeneity exists in the distribution of covariates between treated and controlled groups. Propensity score methods have been widely used to adjust for heterogeneity in observational studies. However, the propensity score is usually unknown and needs to be estimated. In this article, we propose a generalized single-index model to estimate the propensity score and use the propensity score residuals to reduce the estimation bias. The finite-sample performance of the proposed method is evaluated through simulation studies. We use the proposed method to evaluate the policy of "Sunshine Running" and find that the physical test scores of college students participating in the "Sunshine Running" can be improved by 3.72 points.

Key words: treatment effect; propensity score; generalized single-index model; partial linear model

CLC number: O212.1

0 Introduction

Many empirical issues in social sciences depend on evaluating treatment or policy effects. The estimation of causal effects is a hot issue in statistics and econometrics. Half of the 2021 Nobel Prize in Economics was awarded to Angrist and Imbens to recognize their contributions to the causal inference methodology[1,2]. Because the average treatment effect is commonly used to study causal effects in Ref. [3], we consider the average treatment effect to study causal effects in this article.

Randomized Controlled Trails (RCTs) have been considered as the "golden" method to evaluate the average treatment effect, where the treatment assignment is randomized. Hence, the average treatment effect can be obtained by comparing the difference in responses between treatment and control groups. However, it is often infeasible to conduct RCTs due to the ethics, morality, technology, or cost in practice. For example, it is unethical to randomize pregnant women to smoke for studying its effects on neonatal weight, and we should not randomly expose people to air pollution to evaluate the impact of PM2.5 on health.

With the rapid development of science and technol-

Received date: 2023-08-25

Foundation item: Supported by 2020 Guilin University of Aerospace Technology Teaching Group Construction Project (2020JXTD19), 2021 Guangxi Philosophy and Social Science Research Project (21FTY012), and 2022 Guangxi Higher Education Undergraduate Teaching Reform Project (2022JGA358)

Biography: XIAO Li, male, Master, Associate professor, research direction: school sports, sports statistics. E-mail: 16543871@qq.com

† Corresponding author. E-mail: yupeichaode123@163.com

ogy, observational studies are widely used to study treatment effects in which the subjects are observed and recorded in the natural state. Compared with RCTs, observational studies can provide richer information about markets, humans, and their behavioral characteristics. However, the treatment assignment in observational studies is far from random. Due to the lack of randomization, systematic differences in the distribution of baseline covariates between treated and controlled groups usually exist. If these differences cannot be appropriately adjusted, the estimator of treatment effects will be seriously biased. The propensity score (PS), proposed by Rosenbaum and Rubin[4], has been widely used to adjust for previously mentioned differences, which is defined as the conditional probability of receiving a particular treatment given the baseline covariates. Subjects with identical PS will have the same distribution of the baseline covariates regardless of which groups they come from.

There are usually four propensity score-based methods to adjust for heterogeneity in observational studies, including propensity score matching, propensity score stratification, propensity score inverse probability weighting, and propensity score covariate adjustment[5-9]. Propensity score covariate adjustment and partial least squares can estimate treatment effects in survival analysis[10,11]. Unfortunately, the propensity score is usually unknown and needs to be evaluated in observational studies. In many practical applications, PS is estimated by the Logistic or Probit models. However, the misspecification of the propensity score model will bring serious bias to the estimator of treatment effects. Lee[12] proposed a simple least squares estimator based on propensity score residuals to reduce the bias of the estimator of treatment effects, in which the propensity score was estimated by the Probit model. Inspired by Lee's method[12], we will study treatment effects with propensity score residuals and propose a more robust approach based on the generalized single-index model to estimate the propensity score.

The rest of the paper is organized as follows. We introduce the statistical inference procedures in Section 1. In Section 2, we evaluate the finite-sample performance of the proposed method through simulation studies. Section 3 applies the proposed method to evaluate the policy of "Sunshine Running" in the university. Conclusions are shown in Section 4.

1 Estimating Procedures of Average Treatment Effect

We adopted the counterfactual framework to study the average treatment effect[13,14]. In the counterfactual framework, two potential outcomes exist under the treatment and control, respectively. The foundation problem of causal inference is that only one of two potential outcomes can be observed for the same individual[15]. The average treatment effect is defined as the expectation of the difference between two potential outcomes.

To simplify the expression, we adopt the following notation: the binary variable D denotes the treatment assignment ($D=1$ for treatment, $D=0$ for control) and let $\boldsymbol{X}$ represent a p-vector of the baseline covariates; $Y(1)$, $Y(0)$ are potential outcomes under the active treatment ($D=1$) and control ($D=0$), respectively. Hence, the observed outcome is $Y=DY(1)+(1-D)Y(0)$. The observed data $\{(Y_i, D_i, \boldsymbol{X}_i), i=1,\cdots,n\}$ are assumed to be the independent copies of $(Y, D, \boldsymbol{X})$. In this paper, we consider the following average treatment effect (ATE) to study the causal effect:

$$\tau = E[Y(1)] - E[Y(0)] \tag{1}$$

The conditional average treatment effect (CATE) is:

$$\tau(\boldsymbol{X}) = E[Y(1)-Y(0)\,|\,\boldsymbol{X}] = E[Y(1)\,|\,\boldsymbol{X}] - E[Y(0)\,|\,\boldsymbol{X}] \tag{2}$$

According to the properties of the conditional expectation, we can have the following:

$$\tau = E[\tau(\boldsymbol{X})] \tag{3}$$

1.1 Identifiability of Average Treatment Effect

In order to identify the average treatment effect, we have to make the following assumptions.

Assumption 1: Stable Unit Treatment Value Assumption: The subject's potential outcomes are unrelated to the treatment assignment of other subjects.

Assumption 2: Ignorable: The treatment assignment and potential outcomes are independent given the baseline covariates, $(Y(1), Y(0)) \perp D\,|\,\boldsymbol{X}$.

Assumption 3: Overlap: $0 < c_1 \leqslant P(D=1\,|\,\boldsymbol{X}) \leqslant c_2 < 1$, where c_1, c_2 are constant.

Remarks: The above three assumptions are commonly used in the causal inference literature. Assumption 2 means that all the covariates that affect both the outcome and treatment assignment are measured; Assumption 3 ensures overlap exists in the distributions of covariates in treatment and control groups.

1.2 Models

There usually exist two types of models in the study of treatment effects: the outcome model, which links the response with the treatment assignment and baseline covariates, and the treatment assignment model, which defines the conditional probability of receiving a particular treatment given the baseline covariates. We generate the outcome through the following partial linear model:

$$Y_i = \alpha D_i + g(\boldsymbol{X}_i) + \varepsilon_i, i = 1, \cdots, n \tag{4}$$

where the parameter α denotes the average treatment effect, $g(\boldsymbol{X})$ denotes the unknown smoothed function, and ε is the random error with $E[\varepsilon | D, \boldsymbol{X}] = 0$. We consider the following generalized single-index model for the propensity score:

$$P(D = 1 | \boldsymbol{X}) = \frac{\exp(h(\boldsymbol{\gamma}_0^{\mathrm{T}} \boldsymbol{X}))}{1 + \exp(h(\boldsymbol{\gamma}_0^{\mathrm{T}} \boldsymbol{X}))} \tag{5}$$

where $h(\cdot)$ is an unknown smoothed function and $\boldsymbol{\gamma}_0$ denotes the unknown parameter. Furthermore, in order to identify model (5), $\boldsymbol{\gamma}_0$ needs to satisfy $\| \boldsymbol{\gamma}_0 \| = 1$ with $\gamma_{0,1} \geqslant 0$. The proposed model (5) includes the commonly used Logistic and Probit models as special cases.

1.3 Estimation Method of Average Treatment Effect

Inspired by Lee's method[12], we use the simple least squares estimator based on propensity score residuals to estimate the parameter α. Let $\pi(\boldsymbol{X})$ and $\hat{\pi}(\boldsymbol{X})$ denote the true and estimated propensity score and parameters $\hat{\boldsymbol{\gamma}}_n$, $\hat{\alpha}_n$ are estimators of $\boldsymbol{\gamma}$, α, respectively. Firstly, we estimate the propensity score by the maximum likelihood estimate (MLE). Since the function $h(\cdot)$ in model (5) is unknown, we use the sieve method to approximate the unknown function $h(\cdot)$. Define $\boldsymbol{H}_k(\boldsymbol{x}) = (h_0(\boldsymbol{x}), h_1(\boldsymbol{x}), \cdots, h_{k-1}(\boldsymbol{x}))^{\mathrm{T}}$ to denote the k-dimensional orthogonal basis $\boldsymbol{C}_k = (c_0, c_1, \cdots, c_{k-1})^{\mathrm{T}}$ denotes k-dimensional coefficients (A detailed description of the construction of orthogonal basis and the related properties can be found in Chen[16], Dong *et al*[17]). For model (5), we can use the following maximum likelihood function:

$$L_n(\boldsymbol{\gamma}, \boldsymbol{C}_k) = \frac{1}{n} \sum_{i=1}^{n} \{ D_i (\boldsymbol{H}_k(\boldsymbol{\gamma}^{\mathrm{T}} \boldsymbol{X}_i)^{\mathrm{T}} \boldsymbol{C}_k) - \log(1 + \exp(\boldsymbol{H}_k(\boldsymbol{\gamma}^{\mathrm{T}} \boldsymbol{X}_i)^{\mathrm{T}} \boldsymbol{C}_k)) \} \tag{6}$$

Then, $(\hat{\boldsymbol{\gamma}}_n, \hat{\boldsymbol{C}}_k)$, the estimator of $(\boldsymbol{\gamma}, \boldsymbol{C}_k)$, can be obtained by:

$$(\hat{\boldsymbol{\gamma}}_n, \hat{\boldsymbol{C}}_k) = \arg \min_{\boldsymbol{\gamma}, \boldsymbol{C}_k} [-L_n(\boldsymbol{\gamma}, \boldsymbol{C}_k) + \lambda(\| \boldsymbol{\gamma} \|_2^2 - 1)] \tag{7}$$

where the positive integer value k is a regulation parameter, λ is a Lagrange multiplier, and $\| \boldsymbol{\gamma} \|_2^2$ denotes the square of the Euclidean norm of $\boldsymbol{\gamma}$. Once obtained $(\hat{\boldsymbol{\gamma}}_n, \hat{\boldsymbol{C}}_k)$, the propensity score can be estimated as:

$$\hat{\pi}(\boldsymbol{X}) = \frac{\exp(\boldsymbol{H}_k(\hat{\boldsymbol{\gamma}}_n^{\mathrm{T}} \boldsymbol{X})^{\mathrm{T}} \hat{\boldsymbol{C}}_k)}{1 + \exp(\boldsymbol{H}_k(\hat{\boldsymbol{\gamma}}_n^{\mathrm{T}} \boldsymbol{X})^{\mathrm{T}} \hat{\boldsymbol{C}}_k)} \tag{8}$$

According to the simple least squares estimator based on the propensity score residuals, we can obtain the estimator of the average treatment effect:

$$\hat{\alpha}_n = \frac{\sum_{i=1}^{n} (D_i - \hat{\pi}(\boldsymbol{X}_i))(Y_i - \sum_{j=0}^{q} (\hat{\boldsymbol{\gamma}}_n^{\mathrm{T}} \boldsymbol{X}_i)^j)}{\sum_{i=1}^{n} (D_i - \hat{\pi}(\boldsymbol{X}_i))^2} \tag{9}$$

where q denotes the polynomial order, which is usually taken as 2 in practical applications. Furthermore, by similar arguments as Lee[12], we establish the asymptotic properties of $\hat{\alpha}_n$, presented in Theorem 1.

Theorem 1 $\sqrt{n}\,(\hat{\alpha}_n - \alpha)$ converges in distribution to a zero-mean normal distribution with variance $E^{-1}[(D - \pi(\boldsymbol{X}))^2] U E^{-1}[(D - \pi(\boldsymbol{X}))^2]$, where U is defined by (38) in the proof of Theorem 1.

To simply notations, we set

$$\pi(\boldsymbol{X}) = \frac{\exp(\boldsymbol{H}_k(\boldsymbol{\gamma}^{\mathrm{T}} \boldsymbol{X})^{\mathrm{T}} \boldsymbol{C}_k)}{1 + \exp(\boldsymbol{H}_k(\boldsymbol{\gamma}^{T} \boldsymbol{X})^{\mathrm{T}} \boldsymbol{C}_k)} \tag{10}$$

$$\Delta = D - \pi(\boldsymbol{X}) \tag{11}$$

$$V = Y - \sum_{j=0}^{q} (\boldsymbol{\gamma}^{\mathrm{T}} \boldsymbol{X})^j - \alpha \Delta \tag{12}$$

and its estimations are:

$$\hat{\pi}(\boldsymbol{X}_i) = \frac{\exp(\boldsymbol{H}_k(\hat{\boldsymbol{\gamma}}_n^{\mathrm{T}} \boldsymbol{X}_i)^{\mathrm{T}} \hat{\boldsymbol{C}}_k)}{1 + \exp(\boldsymbol{H}_k(\hat{\boldsymbol{\gamma}}_n^{\mathrm{T}} \boldsymbol{X}_i)^{\mathrm{T}} \hat{\boldsymbol{C}}_k)} \tag{13}$$

$$\hat{\Delta}_i = D_i - \hat{\pi}(\boldsymbol{X}_i) \tag{14}$$

$$\hat{V}_i = Y_i - \sum_{j=0}^{q} (\hat{\boldsymbol{\gamma}}_n^{\mathrm{T}} \boldsymbol{X}_i)^j - \hat{\alpha}_n \hat{\Delta}_i \tag{15}$$

In order to establish the asymptotic properties of the proposed estimators $\hat{\alpha}_n$, we need the following lemmas.

Lemma 1 Accounting for first-stage errors.

The moment conditions:

$$E[\{Y - \sum_{j=0}^{q} (\boldsymbol{\gamma}^{\mathrm{T}} \boldsymbol{X})^j - \alpha (D - \frac{\exp(\boldsymbol{H}_k(\boldsymbol{\gamma}^{\mathrm{T}} \boldsymbol{X})^{\mathrm{T}} \boldsymbol{C}_k)}{1 + \exp(\boldsymbol{H}_k(\boldsymbol{\gamma}^{\mathrm{T}} \boldsymbol{X})^{\mathrm{T}} \boldsymbol{C}_k)})\} \cdot \{D - \frac{\exp(\boldsymbol{H}_k(\boldsymbol{\gamma}^{\mathrm{T}} \boldsymbol{X})^{\mathrm{T}} \boldsymbol{C}_k)}{1 + \exp(\boldsymbol{H}_k(\boldsymbol{\gamma}^{\mathrm{T}} \boldsymbol{X})^{\mathrm{T}} \boldsymbol{C}_k)}\}] = 0 \tag{16}$$

for $\hat{\alpha}_n$, with $\boldsymbol{\gamma}$ and $\boldsymbol{C}_k$ replaced by $\boldsymbol{g}$ and $\boldsymbol{c}$, is

$$E[\{Y - \sum_{j=0}^{q} (\boldsymbol{g}^{\mathrm{T}} \boldsymbol{X})^j - \alpha (D - \frac{\exp(\boldsymbol{H}_k(\boldsymbol{g}^{\mathrm{T}} \boldsymbol{X})^{\mathrm{T}} \boldsymbol{c})}{1 + \exp(\boldsymbol{H}_k(\boldsymbol{g}^{\mathrm{T}} \boldsymbol{X})^{\mathrm{T}} \boldsymbol{c})})\}$$

$$\cdot\{D-\frac{\exp(\boldsymbol{H}_k(\boldsymbol{g}^{\mathrm{T}}\boldsymbol{X})^{\mathrm{T}}\boldsymbol{c})}{1+\exp(\boldsymbol{H}_k(\boldsymbol{g}^{\mathrm{T}}\boldsymbol{X})^{\mathrm{T}}\boldsymbol{c})}\}]=0 \tag{17}$$

The effect of $\hat{\boldsymbol{\gamma}}_n-\boldsymbol{\gamma}$ and $\hat{\boldsymbol{C}}_k-\boldsymbol{C}_k$ on the asymptotic distribution of $\hat{\alpha}_n-\alpha$ would be zero if the derivatives of the left-hand side were zero at $\boldsymbol{g}=\boldsymbol{\gamma}$ and $\boldsymbol{c}=\boldsymbol{C}_k$. The derivatives with respect to $\boldsymbol{g}$ at $\boldsymbol{g}=\boldsymbol{\gamma}$ and $\boldsymbol{c}=\boldsymbol{C}_k$ are

$$0-E[V\frac{\exp(\boldsymbol{H}_k(\boldsymbol{\gamma}^{\mathrm{T}}\boldsymbol{X})^{\mathrm{T}}\boldsymbol{C}_k)}{(1+\exp(\boldsymbol{H}_k(\boldsymbol{\gamma}^{\mathrm{T}}\boldsymbol{X})^{\mathrm{T}}\boldsymbol{C}_k))^2}\dot{\boldsymbol{H}}_k(\boldsymbol{\gamma}^{\mathrm{T}}\boldsymbol{X})^{\mathrm{T}}\boldsymbol{C}_k\boldsymbol{X}^{\mathrm{T}}] \tag{18}$$

where $\dot{\boldsymbol{H}}_k(\cdot)$ denotes the derivative of $\boldsymbol{H}_k(\cdot)$. The first term is zero because $E[(D-\pi(\boldsymbol{X}))\,|\,\boldsymbol{X}]=0$, we denote:

$$q_{\alpha\gamma}=-E[V\frac{\exp(\boldsymbol{H}_k(\boldsymbol{\gamma}^{\mathrm{T}}\boldsymbol{X})^{\mathrm{T}}\boldsymbol{C}_k)}{(1+\exp(\boldsymbol{H}_k(\boldsymbol{\gamma}^{\mathrm{T}}\boldsymbol{X})^{\mathrm{T}}\boldsymbol{C}_k))^2}\dot{\boldsymbol{H}}_k(\boldsymbol{\gamma}^{\mathrm{T}}\boldsymbol{X})^{\mathrm{T}}\boldsymbol{C}_k\boldsymbol{X}^{\mathrm{T}}] \tag{19}$$

and $q_{\alpha\gamma}$ can be estimated consistently by:

$$\hat{q}_{\alpha\gamma}=-\frac{1}{n}\sum_{i=1}^{n}\hat{V}_i\frac{\exp(\boldsymbol{H}_k(\hat{\boldsymbol{\gamma}}_n^{\mathrm{T}}\boldsymbol{X}_i)^{\mathrm{T}}\hat{\boldsymbol{C}}_k)}{(1+\exp(\boldsymbol{H}_k(\hat{\boldsymbol{\gamma}}_n^{\mathrm{T}}\boldsymbol{X}_i)^{\mathrm{T}}\hat{\boldsymbol{C}}_k))^2}\dot{\boldsymbol{H}}_k(\hat{\boldsymbol{\gamma}}_n^{\mathrm{T}}\boldsymbol{X}_i)^{\mathrm{T}}\hat{\boldsymbol{C}}_k\boldsymbol{X}_i^{\mathrm{T}} \tag{20}$$

The derivatives with respect to $\boldsymbol{c}$ at $\boldsymbol{g}=\boldsymbol{\gamma}$ and $\boldsymbol{c}=\boldsymbol{C}_k$ are

$$0-E[V\frac{\exp(\boldsymbol{H}_k(\boldsymbol{\gamma}^{\mathrm{T}}\boldsymbol{X})^{\mathrm{T}}\boldsymbol{C}_k)}{(1+\exp(\boldsymbol{H}_k(\boldsymbol{\gamma}^{\mathrm{T}}\boldsymbol{X})^{\mathrm{T}}\boldsymbol{C}_k))^2}\boldsymbol{H}_k(\boldsymbol{\gamma}^{\mathrm{T}}\boldsymbol{X})^{\mathrm{T}}] \tag{21}$$

The first term is zero because $E[(D-\pi(\boldsymbol{X}))\,|\,\boldsymbol{X}]=0$, we denote

$$q_{\alpha C_k}=-E[V\frac{\exp(\boldsymbol{H}_k(\boldsymbol{\gamma}^{\mathrm{T}}\boldsymbol{X})^{\mathrm{T}}\boldsymbol{C}_k)}{(1+\exp(\boldsymbol{H}_k(\boldsymbol{\gamma}^{\mathrm{T}}\boldsymbol{X})^{\mathrm{T}}\boldsymbol{C}_k))^2}\boldsymbol{H}_k(\boldsymbol{\gamma}^{\mathrm{T}}\boldsymbol{X})^{\mathrm{T}}] \tag{22}$$

and $q_{\alpha C_k}$ can be estimated consistently by:

$$\hat{q}_{\alpha C_k}=-\frac{1}{n}\sum_{i=1}^{n}\hat{V}_i\frac{\exp(\boldsymbol{H}_k(\hat{\boldsymbol{\gamma}}_n^{\mathrm{T}}\boldsymbol{X}_i)^{\mathrm{T}}\hat{\boldsymbol{C}}_k)}{(1+\exp(\boldsymbol{H}_k(\hat{\boldsymbol{\gamma}}_n^{\mathrm{T}}\boldsymbol{X}_i)^{\mathrm{T}}\hat{\boldsymbol{C}}_k))^2}\boldsymbol{H}_k(\hat{\boldsymbol{\gamma}}_n^{\mathrm{T}}\boldsymbol{X}_i)^{\mathrm{T}} \tag{23}$$

Lemma 2 The asymptotic properties of $\hat{\boldsymbol{\gamma}}_n$ and $\hat{\boldsymbol{C}}_k$.

We denote the log-likelihood function for model (5) is

$$F_n(\boldsymbol{\gamma},\boldsymbol{C}_k)=\sum_{i=1}^{n}\{D_i(\boldsymbol{H}_k(\boldsymbol{\gamma}^{\mathrm{T}}\boldsymbol{X}_i)^{\mathrm{T}}\boldsymbol{C}_k)-\log(1+\exp(\boldsymbol{H}_k(\boldsymbol{\gamma}^{\mathrm{T}}\boldsymbol{X}_i)^{\mathrm{T}}\boldsymbol{C}_k))\} \tag{24}$$

then we have the score function with respect to $\boldsymbol{\gamma}$

$$\boldsymbol{s}_{\gamma i}=\frac{\partial F(\boldsymbol{\gamma},\boldsymbol{C}_k)}{\partial\boldsymbol{\gamma}}=D_i\dot{\boldsymbol{H}}_k(\boldsymbol{\gamma}^{\mathrm{T}}\boldsymbol{X}_i)^{\mathrm{T}}\boldsymbol{C}_k\boldsymbol{X}_i^{\mathrm{T}}-\pi(\boldsymbol{X}_i)\dot{\boldsymbol{H}}_k(\boldsymbol{\gamma}^{\mathrm{T}}\boldsymbol{X}_i)^{\mathrm{T}}\boldsymbol{C}_k\boldsymbol{X}_i^{\mathrm{T}} \tag{25}$$

By the definition of the influence function, we have the influence function with respect to $\boldsymbol{\gamma}$,

$$\boldsymbol{\eta}_{\gamma i}=E^{-1}[\boldsymbol{s}_{\gamma}\boldsymbol{s}_{\gamma}^{\mathrm{T}}]\boldsymbol{s}_{\gamma i} \tag{26}$$

then

$$\sqrt{n}\,(\hat{\boldsymbol{\gamma}}_n-\boldsymbol{\gamma})=\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\boldsymbol{\eta}_{\gamma i}+o_p(1) \tag{27}$$

Similarly, we also have the score function with respect to $\boldsymbol{C}_k$,

$$\boldsymbol{s}_{C_k i}=\frac{\partial F(\boldsymbol{\gamma},\boldsymbol{C}_k)}{\partial\boldsymbol{C}_k}=D_i\boldsymbol{H}_k(\boldsymbol{\gamma}^{\mathrm{T}}\boldsymbol{X}_i)^{\mathrm{T}}-\pi(\boldsymbol{X}_i)\boldsymbol{H}_k(\boldsymbol{\gamma}^{\mathrm{T}}\boldsymbol{X}_i)^{\mathrm{T}} \tag{28}$$

By the definition of the influence function, we have the influence function with respect to $\boldsymbol{C}_k$,

$$\boldsymbol{\eta}_{C_k i}=E^{-1}[\boldsymbol{s}_{C_k}\boldsymbol{s}_{C_k}^{\mathrm{T}}]\boldsymbol{s}_{C_k i} \tag{29}$$

then

$$\sqrt{n}\,(\hat{\boldsymbol{C}}_k-\boldsymbol{C}_k)=\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\boldsymbol{\eta}_{C_k i}+o_p(1) \tag{30}$$

Proof of Theorem 1

Following the first-order condition

$$\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\{D_i-\hat{\pi}(\boldsymbol{X}_i)\}\{Y_i-\sum_{j=0}^{q}(\hat{\boldsymbol{\gamma}}_n^{\mathrm{T}}\boldsymbol{X}_i)^j-\hat{\alpha}_n(D_i-\hat{\pi}(\boldsymbol{X}_i)\}=0 \tag{31}$$

then apply the mean value theorem to $\hat{\alpha}_n$ around α to get

$$\frac{1}{\sqrt{n}}\sum_{i=1}^{n}[\{D_i-\hat{\pi}(\boldsymbol{X}_i)\}\{Y_i-\sum_{j=0}^{q}(\hat{\boldsymbol{\gamma}}_n^{\mathrm{T}}\boldsymbol{X}_i)^j-\alpha(D_i-\hat{\pi}(\boldsymbol{X}_i)\}-\{D_i-\hat{\pi}(\boldsymbol{X}_i)\}^2\{\hat{\alpha}_n-\alpha\}]=0 \tag{32}$$

So, the term $\sqrt{n}\,(\hat{\alpha}_n-\alpha)$ can be written as

$$\sqrt{n}\,(\hat{\alpha}_n-\alpha)=\frac{\frac{1}{\sqrt{n}}\sum_{i=1}^{n}(D_i-\hat{\pi}(\boldsymbol{X}_i))(Y_i-\sum_{j=0}^{q}(\hat{\boldsymbol{\gamma}}_n^{\mathrm{T}}\boldsymbol{X}_i)^j-\alpha\,(D_i-\hat{\pi}(\boldsymbol{X}_i)))}{\frac{1}{n}\sum_{i=1}^{n}(D_i-\hat{\pi}(\boldsymbol{X}_i))^2} \tag{33}$$

Due to $\hat{\boldsymbol{\gamma}}_n$ and $\hat{\boldsymbol{C}}_k$, we should go one step further applying the mean value theorem to $\hat{\boldsymbol{\gamma}}_n$ and $\hat{\boldsymbol{C}}_k$ as follows. Expand the numerator of (33) to $(\hat{\boldsymbol{\gamma}}_n,\hat{\boldsymbol{C}}_k)$ around $(\boldsymbol{\gamma},\boldsymbol{C}_k)$ to get

$$\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\hat{\Delta}_i\hat{V}_i=\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\Delta_i V_i+\hat{q}_{\alpha\gamma}\sqrt{n}\,(\hat{\boldsymbol{\gamma}}_n-\boldsymbol{\gamma})+\hat{q}_{\alpha C_k}\sqrt{n}\,(\hat{\boldsymbol{C}}_k-\boldsymbol{C}_k) \tag{34}$$

Invoking the uniform LLN (Law of Large Numbers), the denominator of (33), $\hat{q}_{\alpha\gamma}$ and $\hat{q}_{\alpha C_k}$ can be replaced with $E[(D-\pi(\boldsymbol{X}))^2]$, $q_{\alpha\gamma}$ and $q_{\alpha C_k}$, respectively. $q_{\alpha\gamma}$, $\hat{q}_{\alpha\gamma}$, $q_{\alpha C_k}$ and $\hat{q}_{\alpha C_k}$ are defined by (19), (20), (22) and (23) in Lemma 1.

Then we have,

$$\sqrt{n}\,(\hat{\alpha}_n-\alpha)=\frac{\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\Delta_i V_i+q_{\alpha\gamma}\sqrt{n}\,(\hat{\boldsymbol{\gamma}}_n-\boldsymbol{\gamma})+q_{\alpha C_k}\sqrt{n}\,(\hat{\boldsymbol{C}}_k-\boldsymbol{C}_k)}{E[(D-\pi(\boldsymbol{X}))^2]}+o_p(1) \tag{35}$$

By (27) and (30) in Lemma 2, we have

$$\sqrt{n}\,(\hat{\alpha}_n-\alpha)=\frac{\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\{\Delta_i V_i+q_{\alpha\gamma}\boldsymbol{\eta}_{\gamma i}+q_{\alpha C_k}\boldsymbol{\eta}_{C_k i}\}}{E[(D-\pi(\boldsymbol{X}))^2]}+o_p(1) \tag{36}$$

Hence,

$$\sqrt{n}\,(\hat{\alpha}_n-\alpha)\xrightarrow{d}N(0,E^{-1}[(D-\pi(\boldsymbol{X}))^2]UE^{-1}\cdot[(D-\pi(\boldsymbol{X}))^2]) \tag{37}$$

where

$$U=E[\{\Delta V+q_{\alpha\gamma}\boldsymbol{\eta}_{\gamma}+q_{\alpha C_k}\boldsymbol{\eta}_{C_k}\}\{\Delta V+q_{\alpha\gamma}\boldsymbol{\eta}_{\gamma}+q_{\alpha C_k}\boldsymbol{\eta}_{C_k}\}^{\mathrm{T}}] \tag{38}$$

The proof is completed.

2 Simulation Studies

In this section, extensive simulation studies are conducted to evaluate the finite-sample performance of the proposed method. We consider different scenarios to simulate the observational studies in reality.

2.1 Simulation 1: Idealized Scenario

The outcome is generated by the following linear regression model:

$$Y=\alpha D+\beta_1 X_1+\beta_2 X_2+\beta_3 X_3+\varepsilon \tag{39}$$

where, D denotes the treatment assignment, the covariates X_1, X_2 and X_3 are independent of each other, X_1 and X_2 follow the standard normal distribution, X_3 follows Bernoulli distribution with successful probability of 0.4, the error term ε follows the standard normal distribution, $(\beta_1,\beta_2,\beta_3)=(0.5, 0, 1)$ and α is set to 0 or 0.5. The propensity score is generated by the Logistic model:

$$P(D=1\,|\,X_1,X_2,X_3)=\frac{\exp(\gamma_1 X_1+\gamma_2 X_2+\gamma_3 X_3)}{1+\exp(\gamma_1 X_1+\gamma_2 X_2+\gamma_3 X_3)} \tag{40}$$

where $(\gamma_1,\gamma_2,\gamma_3)=(0.8, 0, -0.6)$, then the proportion of subjects in the treatment group is about 45%. We compare the proposed estimator $\hat{\alpha}_P$ with two estimators: $\hat{\alpha}_R$, the estimator based on the covariate adjustment with propensity score[18,19]; $\hat{\alpha}_L$, and the estimator based on propensity score residuals[12].

We generate 500 simulated data sets with the total sample size n being 150 or 300. The sample mean, sample standard deviation and sample root mean square error of 500 estimators are given in the columns "Mean", "SD" and "RMSE", respectively. Furthermore, the column "ESD" shows a mean of the estimated standard deviation, and "CP" gives the nominal 95% confidence interval coverage rate using the estimated standard deviation. The bootstrap method is adopted to obtain the estimated variance, and the number of bootstraps is 100. The simulation results are summarized in Table 1.

Table 1 Simulation result for the idealized scenario

n	Method	Mean	SD	ESD	CP	RMSE
			$\alpha=0$			
150	$\hat{\alpha}_R$	0.004 5	0.178 6	0.178 5	0.936 0	0.178 5
	$\hat{\alpha}_L$	0.004 5	0.179 3	0.179 3	0.932 0	0.179 2
	$\hat{\alpha}_P$	0.004 5	0.179 3	0.179 4	0.932 0	0.179 2
300	$\hat{\alpha}_R$	−0.003 1	0.125 0	0.125 0	0.950 0	0.124 9
	$\hat{\alpha}_L$	−0.003 2	0.125 4	0.125 3	0.950 0	0.125 3
	$\hat{\alpha}_P$	−0.003 2	0.125 4	0.125 3	0.950 0	0.125 3
			$\alpha=0.5$			
150	$\hat{\alpha}_R$	0.504 5	0.178 6	0.178 5	0.936 0	0.178 5
	$\hat{\alpha}_L$	0.504 4	0.179 3	0.179 4	0.932 0	0.179 2
	$\hat{\alpha}_P$	0.504 4	0.179 3	0.179 4	0.932 0	0.179 2
300	$\hat{\alpha}_R$	0.496 9	0.125 0	0.125 0	0.950 0	0.124 9
	$\hat{\alpha}_L$	0.496 8	0.125 5	0.125 3	0.950 0	0.125 4
	$\hat{\alpha}_P$	0.496 8	0.125 5	0.125 3	0.950 0	0.125 4

From Table 1, we obtain that the three estimators $\hat{\alpha}_R$, $\hat{\alpha}_L$ and $\hat{\alpha}_P$ are all approximately unbiased. The corresponding estimated standard error agrees well with the sampling standard error, and the coverage probability of a 95% confidence interval is around the nominal level of the three estimators.

2.2 Simulation 2: Single-Index Model Scenario

The outcome is generated by the following partial single-index model:

$$Y=\alpha D+\exp(\beta_1 X_1+\beta_2 X_2+\beta_3 X_3)+\varepsilon \tag{41}$$

The treatment assignment model is:

$$P(D=1\,|\,X_1,X_2,X_3)=\frac{\exp((\gamma_1 X_1+\gamma_2 X_2+\gamma_3 X_3)^2+4(\gamma_1 X_1+\gamma_2 X_2+\gamma_3 X_3))}{1+\exp((\gamma_1 X_1+\gamma_2 X_2+\gamma_3 X_3)^2+4(\gamma_1 X_1+\gamma_2 X_2+\gamma_3 X_3))} \tag{42}$$

The parameter settings are the same as in Simulation 1. The simulation results are summarized in Table 2.

From Table 2, the three estimators $\hat{\alpha}_R$, $\hat{\alpha}_L$ and $\hat{\alpha}_P$ are all approximately unbiased. When the sample size is 150, we can know that the proposed estimator $\hat{\alpha}_P$ has the smallest sample standard deviation and sample root mean square error. The coverage proportions of the proposed estimator $\hat{\alpha}_P$ is apparently closer to the nominal 95% than that of the estimators $\hat{\alpha}_R$ and $\hat{\alpha}_L$. When the sample size is increased to 300, all indicators show that

Table 2 Simulation result for the single-index model scenario

n	Method	Mean	SD	ESD	CP	RMSE
			$\alpha=0$			
150	$\hat{\alpha}_R$	−0.005 6	0.274 6	0.269 7	0.940 0	0.274 4
	$\hat{\alpha}_L$	0.001 1	0.273 7	0.269 3	0.936 0	0.273 4
	$\hat{\alpha}_P$	0.007 9	0.268 9	0.272 6	0.946 0	0.268 8
300	$\hat{\alpha}_R$	−0.009 4	0.195 5	0.186 2	0.926 0	0.195 5
	$\hat{\alpha}_L$	−0.003 0	0.193 9	0.186 1	0.928 0	0.193 7
	$\hat{\alpha}_P$	0.001 8	0.188 3	0.183 8	0.952 0	0.188 1
			$\alpha=0.5$			
150	$\hat{\alpha}_R$	0.494 4	0.274 6	0.269 7	0.940 0	0.274 4
	$\hat{\alpha}_L$	0.505 2	0.273 5	0.269 2	0.936 0	0.273 2
	$\hat{\alpha}_P$	0.507 1	0.268 9	0.272 8	0.946 0	0.268 8
300	$\hat{\alpha}_R$	0.490 6	0.195 5	0.186 2	0.926 0	0.195 5
	$\hat{\alpha}_L$	0.500 7	0.193 7	0.186 1	0.928 0	0.193 6
	$\hat{\alpha}_P$	0.501 1	0.188 3	0.183 9	0.950 0	0.188 1

Table 3 Simulation result for a more complex scenario

n	Method	Mean	SD	ESD	CP	RMSE
			$\alpha=0$			
150	$\hat{\alpha}_R$	0.258 4	0.199 1	0.197 7	0.742 0	0.326 1
	$\hat{\alpha}_L$	0.240 4	0.225 9	0.217 1	0.794 0	0.329 7
	$\hat{\alpha}_P$	−0.069 4	0.201 2	0.204 5	0.946 0	0.212 6
300	$\hat{\alpha}_R$	0.247 4	0.144 1	0.137 2	0.550 0	0.286 2
	$\hat{\alpha}_L$	0.239 0	0.162 0	0.155 8	0.650 0	0.288 7
	$\hat{\alpha}_P$	−0.070 8	0.136 3	0.139 5	0.928 0	0.153 4
			$\alpha=0.5$			
150	$\hat{\alpha}_R$	0.758 4	0.199 1	0.197 7	0.742 0	0.326 1
	$\hat{\alpha}_L$	0.740 8	0.225 7	0.217 0	0.794 0	0.329 9
	$\hat{\alpha}_P$	0.429 9	0.201 3	0.204 6	0.948 0	0.213 0
300	$\hat{\alpha}_R$	0.747 4	0.144 1	0.137 2	0.550 0	0.286 2
	$\hat{\alpha}_L$	0.739 4	0.162 0	0.155 8	0.650 0	0.288 9
	$\hat{\alpha}_P$	0.428 5	0.136 3	0.139 5	0.928 0	0.153 8

the proposed estimator $\hat{\alpha}_P$ is the most effective.

2.3 Simulation 3: More Complex Scenario

The outcome is generated by the following model:

$$Y=\alpha D+(\beta_1 X_1+\beta_1 X_2+\beta_1 X_3)^2+\varepsilon \tag{43}$$

The treatment assignment model is:

$$P(D=1\mid X_1,X_2,X_3)=\frac{\exp(\gamma_1 X_1+\gamma_2 X_2+\gamma_3 X_3+X_1^2)}{1+\exp(\gamma_1 X_1+\gamma_2 X_2+\gamma_3 X_3+X_1^2)} \tag{44}$$

The parameter settings are the same as in Simulation 1. The simulation results are summarized in Table 3.

From Table 3, when the treatment assignment model is no longer the single-index model, the three estimators $\hat{\alpha}_R$, $\hat{\alpha}_L$ and $\hat{\alpha}_P$ are all biased. However, the biases of the estimators $\hat{\alpha}_R$ and $\hat{\alpha}_L$ are relatively large, while the bias of the proposed estimator $\hat{\alpha}_P$ is relatively small. The average of the estimated standard deviation of $\hat{\alpha}_P$ is close to the sample standard deviation. The coverage proportions of the proposed estimator $\hat{\alpha}_P$ is much more relative to the nominal level than the estimators $\hat{\alpha}_R$ and $\hat{\alpha}_L$. Therefore, from the simulation results of three different scenarios, the performance of our proposed estimator $\hat{\alpha}_P$ is optimal.

3 Real Data Analysis

Nowadays, due to unhealthy work and rest habits, many college students have the problem of poor physical quality. In order to continuously improve the overall level of physical health of college students, relevant government documents put forward mandatory requirements for the physical health level of college students; many universities also organize "Sunshine Running" to improve the physical health of students[20,21]. In this section, we apply the proposed method to evaluate the effectiveness of the "Sunshine Running" policy offered by the university to improve college students ' physical test scores. The college where we collected the data implemented the "Sunshine Running" policy among the freshmen admitted in 2018 but did not implement the policy in other grades. We collected physical test data from the class of grad 2017 at this university in 2018 as the control group and from the class of grad 2018 in 2019 as the treatment group. In order to make the collected data independent of each other, we randomly selected one male and one female in each major of the university to record their physical test scores, gender, age, and Body Mass Index (BMI, weight divided by height squared). We collected 82 students in the treatment group and 69 students in the control group, for a total of 151 students.

To simplify the expression, we use the following notation: Y denotes the physical test score, D represents the treatment assignment, Sex denotes gender (male=1, female=2), Age denotes age, and BMI denotes Body Mass Index. We consider the following outcome model:

$$Y=\alpha D+g(\text{Sex},\text{Age},\text{BMI})+\varepsilon \tag{45}$$

where the error term ε follows the standard normal distribution, $g(\cdot,\cdot,\cdot)$ is an unknown smoothed function.

We consider the following generalized single-index model to generate the treatment assignment:

$$P(D=1\,|\,\text{Sex, Age, BMI}) = \frac{\exp(h(\gamma_1\text{Sex}+\gamma_2\text{Age}+\gamma_3\text{BMI}))}{1+\exp(h(\gamma_1\text{Sex}+\gamma_2\text{Age}+\gamma_3\text{BMI}))} \quad (46)$$

where $h(\cdot)$ is an unknown smoothed function, $\gamma_1,\gamma_2,\gamma_3$ are unknown parameters.

Likewise, we considered three estimators $\hat{\alpha}_R$,$\hat{\alpha}_L$ and $\hat{\alpha}_P$. We considered them in the simulation studies to study the effect of the "Sunshine Running" policy. We use the bootstrap to derive the variance of the three estimators, and the number of bootstraps is 100. The estimators, the standard deviations, and p-value of $\hat{\alpha}_R$, $\hat{\alpha}_L$ and $\hat{\alpha}_P$ are summarized in Table 4.

Table 4 Results for the "Sunshine Running" policy

Method	$\hat{\alpha}$	SD	p - value
$\hat{\alpha}_R$	3.550 3	1.354 6	0.004 4
$\hat{\alpha}_L$	3.528 3	1.306 9	0.003 5
$\hat{\alpha}_P$	3.719 6	1.249 0	0.001 5

From Table 4, we can conclude that all the estimators confirm that the "Sunshine Running" policy has a significant impact on improving college students' physical test scores. From the standard deviations of the three estimators, the proposed estimator $\hat{\alpha}_P$ is the most effective, and the estimator $\hat{\alpha}_R$ is the worst.

4 Conclusion

Estimating treatment effects or evaluating policy effects is an essential issue in empirical science, which can provide us with the basis for quantitative decision-making. With the development of science and technology, a large number of high-quality observational data have been collected, and how to use high-quality and information-rich observational data to estimate treatment effects is one of the hot issues in current research. The propensity score has been widely used to adjust for heterogeneity in observational studies. However, the misspecification of the propensity score will cause serious bias. In this paper, we proposed the generalized single-index model to estimate the propensity score and used the propensity score residuals to reduce the estimation bias. From the results of simulation studies and real data analysis, our proposed method is more effective than other competitive methods. Next, we will consider more robust methods to estimate the treatment effect, such as deep learning, in the future work.

References

[1] Imbens G W, Angrist J D. Identification and estimation of local average treatment effects [J]. *Econometrica*, 1994, **62**(2): 467-475.

[2] Angrist J D, Imbens G W, Rubin D B. Identification of causal effects using instrumental variables [J]. *Journal of the American Statistical Association*, 1996, **91**(434): 444-455.

[3] Imbens G W. Nonparametric estimation of average treatment effects under exogeneity: A review [J]. *Review of Economics and Statistics*, 2004, **86**(1): 4-29.

[4] Rosenbaum P R, Rubin D B. The central role of the propensity score in observational studies for causal effects [J]. *Biometrika*, 1983, **70**(1): 41-55.

[5] Austin P C. An introduction to propensity score methods for reducing the effects of confounding in observational studies [J]. *Multivariate Behavioral Research*, 2011, **46**(3): 399-424.

[6] Austin P C, Schuster T. The performance of different propensity score methods for estimating absolute effects of treatments on survival outcomes: A simulation study [J]. *Statistical Methods in Medical Research*, 2016, **25**(5): 2214-2237.

[7] Herman M, Robins J M. *Causal Inference*: *What if* [M]. Boca Raton: Chapman & Hall/CRC, 2020.

[8] Imbens G W, Rubin D B. *Causal Inference in Statistics*, *Social*, *and Biomedical Sciences* [M]. Cambridge: Cambridge University Press, 2015.

[9] Lunceford J K, Davidian M. Stratification and weighting via the propensity score in estimation of causal treatment effects: A comparative study [J]. *Statistics in Medicine*, 2004, **23**(19): 2937-2960.

[10] Cao Y X, Zhang C X, Yu C J. Estimating survival treatment effects with covariate adjustment using propensity score[J]. *Acta Mathematica Sinica*, *English Series*, 2022, **38**(11): 2057-2068.

[11] Cao Y X, Yu J C. Partial least squares method for treatment effect in observational studies with censored outcomes[J]. *Wuhan University Journal of Natural Sciences*, 2018, **23**(6): 487-492.

[12] Lee M J. Simple least squares estimator for treatment effects using propensity score residuals [J]. *Biometrika*, 2018, **105** (1): 149-164.

[13] Rubin D B. Estimating causal effects of treatments in ran-

domized and nonrandomized studies [J]. *Journal of Educational Psychology*, 1974, **66**(5): 688.

[14] Rubin D B. Inference and missing data [J]. *Biometrika*, 1976, **63**(3): 581-592.

[15] Holland P W. Statistics and causal inference [J]. *Journal of the American Statistical Association*, 1986, **81**(396): 945-960.

[16] Chen X. Large sample sieve estimation of semi-nonparametric models [J]. *Handbook of Econometrics*, 2007, **6**: 5549-5632.

[17] Dong C H, Gao J T, Peng B. Series estimation for single-index models under constraints [J]. *Australian & New Zealand Journal of Statistics*, 2019, **61**(3): 299-335.

[18] Vansteelandt S, Daniel R M. On regression adjustment for the propensity score [J]. *Statistics in Medicine*, 2014, **33**(23): 4053-4072.

[19] Zou B M, Zou F, Shuster J J, *et al*. On variance estimate for covariate adjustment by propensity score analysis [J]. *Statistics in Medicine*, 2016, **35**(20): 3537-3548.

[20] Ministry of Education. Opinions on deepening the teaching reform of undergraduate education and comprehensively improving the quality of talent cultivation [EB/OL]. [2019-10-08]. http://www. moe. gov. cn/srcsite/A08/s7056/201910/t20191011_402759.html.

[21] The General Office of the CPC Central Committee, the State Council. Opinions on comprehensively strengthening and improving school sports work in the new era [EB/OL]. [2020-10-15]. http://www. moe. gov. cn/jyb_xxgk/moe_1777/moe_1778/202010/t20201015_494794.html.

□