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Packing 4-Partite Tree into Complete 4-Partite Graph

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Abstract: For graphs G and H, an embedding of G into H is an injection $\phi: V(G) \to V(H)$ such that $\phi(a)\phi(b) \in E(H)$ whenever $ab \in E(G)$. A packing of p graphs $G_1, G_2, \cdots, G_p$ into H is a p-tuple $\Phi = (\phi_1, \phi_2, \cdots, \phi_p)$ such that, for $i = 1, 2, \cdots, p$, ϕ_i is an embedding of G_i into H and the p sets $\phi_i(E(G_i))$ are mutually disjoint. Motivated by the "Tree Packing Conjecture" made by Gyàrfàs and Lehel, Wang Hong conjectured that for each k-partite tree, there is a packing of two copies of $T(\mathbb{X})$ into a complete k-partite graph $B_{n+m}(\mathbb{Y})$, where $m = \lfloor k/2 \rfloor$. In this paper, we confirm this conjecture for $k = 4$.

Key words: packing of graph; tree packing conjecture; embedding of graph

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0 Introduction

We shall use standard graph theory notation. For any graph G, a neighbor of a vertex v is a vertex adjacent to v in G. $N_G(v)$ denotes the set of neighbors of a vertex v in G. The degree of v, denoted by $\deg_G(v)$, is $|N_G(v)|$. Given a subset A of $V(G)$, $N_G(v, A)$ is $N_G(v) \cap A$ for the vertex $v \in V(G)$. When the context is clear, the subscript G is omitted.

An endvertex is a vertex of degree 1 and non-endvertex is a vertex of degree >1. A node is a vertex adjacent to an endvertex. A supernode is a node x of G such that, with one exception, every neighbor of x is an endvertex. A star is a tree with only one non-endvertex.

An embedding of a graph G into H is an injection ϕ: $V(G) \to V(H)$ such that $\phi(a)\phi(b) \in E(H)$ whenever $ab \in E(G)$. A packing of p graphs $G_1, G_2, \cdots, G_p$ into H is a p-tuple $\Phi = (\phi_1, \phi_2, \cdots, \phi_p)$ such that, for $i = 1, 2, \cdots, p$, ϕ_i is an embedding of G_i into H and the p sets $\phi_i(E(G_i))$ are mutually disjoint. When all G_i are isomorphic to G, we call it a p-parking of G.

A k-partite graph G with the partition $\mathbb{X} = (X_1, X_2, \cdots, X_k)$ is denoted as $G(X_1, X_2, \cdots, X_k)$ or $G(\mathbb{X})$. In this case, it is said that G admits the partition $\mathbb{X}$ and $|\mathbb{X}| = k$. If G admits two distinct partitions $\mathbb{X}$ and $\mathbb{Y}$, then the notion that $G(\mathbb{X}) \neq G(\mathbb{Y})$ is adopted here. If G and H admit the k-partitions $\mathbb{X}$ and $\mathbb{Y}$, respectively, and ϕ is an embedding of G into H such that $\phi(X_i) \subset Y_i$, then ϕ is restrained and this is denoted as $\phi: G(\mathbb{X}) \to H(\mathbb{Y})$. A packing $\Phi = (\phi_1, \phi_2, \cdots, \phi_p)$ of $G_1(\mathbb{X}_1), G_2(\mathbb{X}_2), \cdots, G_p(\mathbb{X}_p)$ into $H(\mathbb{Y})$ is restrained if each embedding of Φ is restrained. A k-partite tree is a k-partite graph without cycles. Let $T(\mathbb{X})$ denote the k-partite tree with $\mathbb{X} =$

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$(X_1, X_2, \cdots, X_k)$ as its k-partition and $B_n(\mathbb{Y})$ denote the complete k-partite graph of order n with $\mathbb{Y}=(Y_1, Y_2, \cdots, Y_k)$ as its k-partition.

Packing problems are central to combinatorics. Many exciting results and elegant proofs of these results were obtained[1-4]. For a survey, see Refs. [5,6]. Among the best-known packing problems, the famous Tree Packing Conjecture of Gyárfás and Lehel[7] has driven a large amount of research in the area. Bollobás[8] confirmed the "Tree Packing Conjecture" for many small trees. Motivated by the "Tree Packing Conjecture", Wang Hong made the following conjecture:

Conjecture[9] For each k-partite tree $T(\mathbb{X})$ of order n, there is a restrained packing of two copies of $T(\mathbb{X})$ into a complete k-partite graph $B_{n+m}(\mathbb{Y})$, where $m=\lfloor k/2 \rfloor$.

This conjecture has been verified in Ref. [10] for $k=2$. Recently, Sapozhnikov[11] proved this conjecture with $k=3$, which is stated as the following proposition:

Proposition 1[11] Let $T(\mathbb{X})$ be a 3-partite tree of order n with partites X_1, X_2, X_3. Then there exists a restrained 2-packing of $T(\mathbb{X})$ into a complete 3-partite graph $B_{n+1}(Y_1, Y_2, Y_3)$.

In this paper, we prove that the conjecture is true for $k=4$.

1 Main Results

In the following lemmas, we assume that $T(\mathbb{X})$ is a counter-example of Theorem 1 of minimum order n.

Lemma 1[9] The endvertices of $T(\mathbb{X})$ are adjacent to the same node if they are in the same partite.

Lemma 2[9] If v is a node of $T(\mathbb{X})$ adjacent to endvertices in a partite X, then $|X|$ is odd and $\deg(v, X) = (|X|+1)/2$.

Theorem 1 For each 4-partite tree $T(\mathbb{X})$ of order n with partition $\mathbb{X}=(X_1, X_2, X_3, X_4)$, there is a restrained 2-packing of $T(\mathbb{X})$ into some complete 4-partite graph $B_{n+2}(\mathbb{Y})$.

Proof We assume that $T(\mathbb{X})$ is a counter-example of Theorem 1 of minimum order n with partition $\mathbb{X}=(X_1, X_2, X_3, X_4)$. Then $|X_i|>1, i=1,2,3,4$, since Theorem 1 holds if $|X_i|=1$ for some i clearly.

If $T(\mathbb{X})$ has exactly one node v, then $T(\mathbb{X})$ is a star. So, for some i, $\deg(v, X_i)=|X_i|>1$. By Lemma 2, we have $\deg(v, X_i)=(|X_i|+1)/2=|X_i|$, a contradiction. Therefore, there are at least two nodes in $T(\mathbb{X})$. By observing a longest path of $T(\mathbb{X})$, there exist at least two supernodes in $T(\mathbb{X})$.

Let v be a supernode of $T(\mathbb{X})$ and u be the only one non-endvertex adjacent to v. Without loss of generality, we may assume that $v \in X_1$. Let $V_i = N(v, X_i)$, $i=2,3,4$. Then there is at least one of V_i, $i=2,3,4$, which is nonempty. Without loss of generality, we may assume that $V_2 \neq \varnothing$. By Lemma 1, we can see that all the endvertices of X_2 are in the set V_2. Let $V=X_2 \backslash V_2$. Then, $X_2=V_2 \cup V$ and $|V_2|=|V|+1$ by Lemma 2. So we may assume that $V_2=\{v_1, v_2, \cdots, v_m\}$ and $V=\{v_{m+1}, v_{m+2}, \cdots, v_{2m-1}\}$.

Consider the graph $T(\mathbb{X}')=T(\mathbb{X})\backslash(X_2 \cup \{v\})$, where $\mathbb{X}'=(X_1', X_3', X_4')$ with $X_1'=X_1\backslash\{v\}, X_3'=X_3, X_4'=X_4$. So $T(\mathbb{X}')$ is a 3-partite tree with order n', where $n'=n-1-|X_2|$. By Proposition 1, there exists a restrained 2-packing $\Phi'=(\phi_1', \phi_2')$ of $T(\mathbb{X}')$ into a complete 3-partite graph $B_{n'+1}(Y_1, Y_3, Y_4)$ with $|X_i'| \leqslant |Y_i|, i=1,3,4$.

Suppose that there is $i \neq j$ such that $V_i \neq \varnothing, V_j \neq \varnothing$. Since u be the only one non-endvertex adjacent to v, without loss of generality, we may assume $u \notin X_j$ and $j=2$. Now add two vertices x_1, x_2 to Y_1 and a partite set Y_2 to $B_{n'+1}(Y_1, Y_3, Y_4)$ such that $|Y_2|=|X_2|$. Let $Y_2=\{y_1, y_2, \cdots, y_{2m-1}\}$. Then the packing Φ' can be extended to $T(\mathbb{X})$ as follows: define $\Phi(x)=\Phi'(x)$ for $x \in T(\mathbb{X}')$, $\phi_1(v)=x_1$, $\phi_2(v)=x_2$. Define $\Phi(X_2)$ as follows:

$$\phi_1(v_i)=y_i \quad \text{for} \quad i=1,2,\cdots,2m-1, \quad \phi_2(v_1)=y_1,$$

$\phi_2(v_i)=y_{2m-i+1}$ for $i=2,3,\cdots,2m-1$.

Thus, we extend the Φ' to $T(\mathbb{X})$ so that a restrained 2-packing Φ of $T(\mathbb{X})$ into the $B_{n+2}(\mathbb{Y})$ is obtained, where $\mathbb{Y}=(Y_1 \cup \{x_1, x_2\}, Y_2, Y_3, Y_4)$, a contradiction.

Therefore, there exists only one nonempty V_i, which implies that all the endvertices adjacent to v are in V_i and $u \in V_i$. Without loss of generality, we may assume that $i=2$ and $u=v_1$. Now add a vertex x to Y_1 and add a partite set Y_2 to $B_{n'+1}(Y_1, Y_3, Y_4)$ such that $|Y_2|=|X_2|+1$. Let $Y_2=\{y_1, y_2, \cdots, y_{2m}\}$. Then we extend the Φ' to $T(\mathbb{X})$ as follows:

Define $\Phi(x)=\Phi'(x)$ for $x \in T(\mathbb{X}')$ and $\phi_1(v)=\phi_2(v)=x$. Define $\Phi(X_2)$ as follows: $\phi_1(v_i)=y_i$ for $i=1,2,\cdots,2m-1, \phi_2(v_i)=y_{2m-i+1}$ for $i=1,2,\cdots,2m-1$. Thus, we extend Φ' to $T(\mathbb{X})$ so that a restrained 2-packing Φ of $T(\mathbb{X})$ into the $B_{n+2}(\mathbb{Y})$ is obtained, where $\mathbb{Y}=(Y_1 \cup \{x\}, Y_2, Y_3, Y_4)$, a contradiction.

The proof is completed.

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