Packing 4-Partite Tree into Complete 4-Partite Graph

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Abstract: For graphs $G$ and $H$, an embedding of $G$ into $H$ is an injection $\phi:V(G)\rightarrow V(H)$ such that $\phi(a)\phi(b)\in E(H)$ whenever $ab\in E(G)$. A packing of $p$ graphs $G_1,G_2,\cdots,G_p$ into $H$ is a $p$-tuple $\Phi=(\phi_1,\phi_2,\cdots,\phi_p)$ such that, for $i=1,2,\cdots,p$, $\phi_i$ is an embedding of $G_i$ into $H$ and the $p$ sets $\phi_i(E(G_i))$ are mutually disjoint. Motivated by the "Tree Packing Conjecture" made by Gyárfás and Lehel, Wang Hong conjectured that for each $k$-partite tree, there is a packing of two copies of $T(X)$ into a complete $k$-partite graph $B_{m,k}(Y)$, where $m=\lfloor k/2 \rfloor$. In this paper, we confirm this conjecture for $k=4$.

Key words: packing of graph; tree packing conjecture; embedding of graph

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(\(X_i, X_2, \ldots, X_r\)) as its \(k\)-partition and \(B_r(\mathcal{Y})\) denote the complete \(k\)-partite graph of order \(n\) with \(\mathcal{Y} = (Y_1, Y_2, \ldots, Y_r)\) as its \(k\)-partition.

Packing problems are central to combinatorics. Many exciting results and elegant proofs of these results were obtained\(^{[14]}\). For a survey, see Refs. \([5, 6]\). Among the best-known packing problems, the famous Tree Packing Conjecture of Gyárfás and Lehel\(^{[7]}\) has driven a large amount of research in the area. Bollobás\(^{[8]}\) confirmed the "Tree Packing Conjecture" for many small trees. Motivated by the "Tree Packing Conjecture", Wang Hong made the following conjecture:

**Conjecture\(^{[8]}\)** For each \(k\)-partite tree \(T(\mathcal{Y})\) of order \(n\), there is a restrained packing of two copies of \(T(\mathcal{Y})\) into a complete \(k\)-partite graph \(B_{k+1}(\mathcal{Y})\), where \(m = \lceil k/2 \rceil\).

This conjecture has been verified in Ref. \([10]\) for \(k = 2\). Recently, Sapozhnikov\(^{[11]}\) proved this conjecture with \(k = 3\), which is stated as the following proposition:

**Proposition 1\(^{[11]}\)** Let \(T(\mathcal{Y})\) be a 3-partite tree of order \(n\) with partites \(X_1, X_2, X_3\). Then there exists a restrained 2-packing of \(T(\mathcal{Y})\) into a complete 3-partite graph \(B_{3+1}(Y_1, Y_2, Y_3)\).

In this paper, we prove that the conjecture is true for \(k = 4\).

### 1 Main Results

In the following lemmas, we assume that \(T(\mathcal{Y})\) is a counter-example of Theorem 1 of minimum order \(n\).

**Lemma 1\(^{[9]}\)** The endvertices of \(T(\mathcal{Y})\) are adjacent to the same node if they are in the same partite.

**Lemma 2\(^{[8]}\)** If \(v\) is a node of \(T(\mathcal{Y})\) adjacent to endvertices in a partite \(X_i\), then \(|X_i|\) is odd and \(\deg(v, X_i) = (|X_i| + 1)/2\).

**Theorem 1** For each 4-partite tree \(T(\mathcal{Y})\) of order \(n\) with partition \(\mathcal{X} = (X_1, X_2, X_3, X_4)\), there is a restrained 2-packing of \(T(\mathcal{Y})\) into some complete 4-partite graph \(B_{4+1}(\mathcal{Y})\).

**Proof** We assume that \(T(\mathcal{Y})\) is a counter-example of Theorem 1 of minimum order \(n\) with partition \(\mathcal{X} = (X_1, X_2, X_3, X_4)\). Then \(|X_i| > 1\), \(i = 1, 2, 3, 4\), since Theorem 1 holds if \(|X_i| = 1\) for some \(i\) clearly.

If \(T(\mathcal{Y})\) has exactly one node \(v\), then \(T(\mathcal{Y})\) is a star. So, for some \(i\), \(\deg(v, X_i) = |X_i| > 1\). By Lemma 2, we have \(\deg(v, X_i) = (|X_i| + 1)/2 = |X_i|\), a contradiction. Therefore, there are at least two nodes in \(T(\mathcal{Y})\). By observing a longest path of \(T(\mathcal{Y})\), there exist at least two supernodes in \(T(\mathcal{Y})\).

Let \(v\) be a supernode of \(T(\mathcal{Y})\) and \(u\) be the only one non-endvertex adjacent to \(v\). Without loss of generality, we may assume that \(v \in X_1\). Let \(V_i = N(v, X_i), i = 1, 2, 3, 4\). Then there is at least one of \(V_i\), \(i = 2, 3, 4\), which is non-empty. Without loss of generality, we may assume that \(V_2 \neq \emptyset\). By Lemma 1, we can see that all the endvertices of \(X_2\) are in the set \(V_2\). Let \(V = X_2 \setminus V_2\). Then, \(X_1 = V_2 \cup V\) and \(|V_2| = |V| + 1\) by Lemma 2. So we may assume that \(V_2 = \{v_1, v_2, \ldots, v_n\}\) and \(V = \{v_{n+1}, v_{n+2}, \ldots, v_{2n-3}\}\).

Consider the graph \(T(\mathcal{X'}) = T(\mathcal{Y}) \triangle \{v, v_i\}\), where \(\mathcal{X'} = (X_1', X_2', X_3, X_4)\) with \(X_1' = X_1 \setminus \{v\}, X_2' = X_2 \setminus \{v_i\}\).

So \(T(\mathcal{X'})\) is a 3-partite tree with order \(n'\), where \(n' = n - 1 - |X_2|\). By Proposition 1, there exists a restrained 2-packing \(\Phi' = (\phi'_1, \phi'_2)\) of \(T(\mathcal{X'})\) into a complete 3-partite graph \(B_{3+1}(Y_1', Y_2', Y_3)\) with \(|X_i'| \leq |Y_i|, i = 1, 2, 3\).

Suppose that there is \(i \neq j\) such that \(V_i \neq \emptyset, V_j \neq \emptyset\). Since \(u\) is the only one non-endvertex adjacent to \(v\), without loss of generality, we may assume \(u \notin X_1\) and \(j = 1\). Now add two vertices \(x_1, x_2\) to \(V_1\) and a partite set \(Y_2\) to \(B_{3+1}(Y_1, Y_2, Y_3)\) such that \(|Y_2| = |X_2|\). Let \(Y_1 = \{v_1, v_2, \ldots, v_{2n-3}\}\). Then the packing \(\Phi'\) can be extended to \(T(\mathcal{X'})\) as follows: define \(\Phi(x) = \Phi'(x)\) for \(x \in T(\mathcal{X'})\), \(\phi'_1(v) = v_i\), \(\phi'_2(v) = x_2\). Define \(\Phi(X_i)\) as follows:

\[
\phi'_1(v) = y_i, \quad \phi'_2(v) = y_1, \quad \phi'_1(v) = y_{2n-1}, \quad \phi'_2(v) = y_{2n-1}, \quad \text{for } i = 2, 3, \ldots, 2m-1.
\]

Thus, we extend the \(\Phi'\) to \(T(\mathcal{X'})\) so that a restrained 2-packing \(\Phi\) of \(T(\mathcal{Y})\) into the \(B_{4+1}(\mathcal{Y})\) is obtained, where \(\mathcal{Y} = (Y_1 \cup \{x_1, x_2\}, Y_2, Y_3)\), a contradiction.

Therefore, there exists only one nonempty \(V_i\), which implies that all the endvertices adjacent to \(v\) are in \(V_i\) and \(u \notin \mathcal{V}_i\). Without loss of generality, we may assume that \(i = 2\) and \(u \notin \mathcal{V}_i\). Now add a vertex \(x\) to \(V_i\) and add a partite set \(Y_2\) to \(B_{3+1}(Y_1, Y_2, Y_3)\) such that \(|Y_1| = |X_1| + 1\). Let \(Y_2 = \{v_1, v_2, \ldots, v_{2m-3}\}\). Then we extend the \(\Phi'\) to \(T(\mathcal{X'})\) as follows:

Define \(\Phi(x) = \Phi'(x)\) for \(x \in T(\mathcal{X'})\) and \(\phi'_1(v) = \phi'_1(v) = x\). Define \(\Phi(X_i)\) as follows: \(\phi'_1(v_i) = y_i\), \(\phi'_2(v_i) = y_1\), \(\phi'_1(v_i) = y_{2n-1}\), \(\phi'_2(v_i) = y_{2n-1}\), for \(i = 1, 2, \ldots, 2m-1\). Thus, we extend \(\Phi'\) to \(T(\mathcal{X'})\) so that a restrained 2-packing \(\Phi\) of \(T(\mathcal{Y})\) into the \(B_{4+1}(\mathcal{Y})\) is obtained, where \(\mathcal{Y} = (Y_1 \cup \{x\}, Y_2, Y_3)\), a contradiction.

The proof is completed.
References


