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Asymptotic Behavior of Singular Solution to the k-Hessian Equation with a Matukuma-Type Source

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Abstract: This paper is concerned with radially positive solutions of the k-Hessian equation involving a Matukuma-type source $S_k(D^2(-\varphi))=\frac{|\boldsymbol{x}|^{\lambda-2}}{(1+|\boldsymbol{x}|^2)^{\lambda/2}}\varphi^q,\quad \boldsymbol{x}\in\Omega$, where $S_k(D^2(-\varphi))$ is the k-Hessian operator, $q>k>1, \lambda>0, n>2k, k\in\mathbb{N}$, and Ω is a suitable bounded domain in $\mathbb{R}^n$. It turns out that there are two different types of radially positive solutions for $k>1$, i.e., M-solution (singular at $r=0$) and E-solution (regular at $r=0$), which is distinct from the case when $k=1$. For $1<q<[(n-2+\lambda)k]/(n-2k)$, we apply an iterative approach to improve accuracy of asymptotic expansions of M-solution step by step to the desired extend. In contrast to the case $k=1$, we require a more precise range of parameters due to repeated application of Taylor expansions, which also makes asymptotic expansions need more delicate investigation.

Key words: k-Hessian equation; singular solutions; asymptotic expansion

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0 Introduction

The purpose of this paper is to inquire about asymptotic behavior of radially positive solutions to the k-Hessian equation with a Matukuma-type source:

$$S_k(D^2(-\varphi))=\frac{|\boldsymbol{x}|^{\lambda-2}}{(1+|\boldsymbol{x}|^2)^{\lambda/2}}\varphi^q,\quad \boldsymbol{x}\in\Omega, \tag{1}$$

where $S_k(D^2(-\varphi))$ is the k-Hessian operator, $q>k>1, \lambda>0, n>2k, k\in\mathbb{N}$, and Ω is a suitable bounded domain in $\mathbb{R}^n$. The operators $\{S_k: k=1,\cdots,n\}$ are a family of operators including Laplace operator when $k=1$ and Monge-Ampère operator when $k=n$. The k-Hessian equation admits several significant applications in fluid mechanic, geometric problem and other applied subjects. For example, the k-Hessian equation is closely related to non-equilibrium phase transitions and statistical physics[1], the problem of prescribing the Gauss curvature of a hypersurface[2] and to the Monge-Ampère

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equation, which is of interest in complex geometry[3].

When $k=1$, equation (1) reduces to the classical Matukuma equation[4]

$$\Delta\varphi+\frac{|\boldsymbol{x}|^{\lambda-2}}{(1+|\boldsymbol{x}|^2)^{\lambda/2}}\varphi^q=0,\quad \boldsymbol{x}\in\mathbb{R}^n. \tag{2}$$

The existence and nonexistence of positive solutions to (2) could be referred to Refs. [5-9]. Batt *et al*[10] established a comprehensive theory of radially positive solutions to (2) in $\mathbb{R}^3$, and displayed that there exist three different types of solutions: M-solutions (singular at $r=0$), E-solutions (regular at $r=0$) and F-solutions (whose existence begins away from $r=0$). By applying an iterative method[11], Wang *et al*[12] generalized asymptotic expansions of M-solutions of (2) from $n=3$ to $n>3$. This iterative method also could be used for the Hénon equation $-\Delta_p\varphi=|\boldsymbol{x}|^\sigma\varphi^q$ with $p=2$, where the accurate asymptotic expansions of M-solutions was systematically derived in Ref. [13]. It is worth noting that the results obtained in Ref. [13] are more precise than those in Refs. [14,15]. Recently, Wang and Zhang[16] extended the work of Ref. [13] from $p=2$ to $1<p<N$. When $\lambda=2$, equation (2) reduces to

$$\Delta\varphi+\frac{1}{(1+|\boldsymbol{x}|^2)^{\lambda/2}}\varphi^q=0,\quad \boldsymbol{x}\in\mathbb{R}^n, \tag{3}$$

which was presented by astrophysicist Matukuma[17] for the description of certain stellar globular clusters in a steady state, where $q>1, n=3$, and $\varphi>0$ is the gravitational potential. Li[18] gave a nearly complete description of the structure of positive radial solutions to (3) when $1<q<5$ and proved a symmetry result for general nonlinear elliptic equations. Yanagida[19] established the uniqueness of positive radial entire solution with finite total mass and obtained its explicit structure for $n\geqslant 3$ and $1<q<(n+2)/(n-2)$. We refer to Refs. [20-22] about the Matukuma equation.

When $k>1$, Sánchez and Vergara[23] considered the problem

$$\begin{cases} S_k(D^2\varphi)=\lambda|\boldsymbol{x}|^\sigma\left(1-\varphi\right)^q, & \boldsymbol{x}\in B\\ \varphi<0, & \boldsymbol{x}\in B\\ \varphi=0, & \boldsymbol{x}\in\partial B\end{cases} \tag{4}$$

where B is the unit ball in $\mathbb{R}^n, n>2k, k\in\mathbb{N}, \lambda>0, q>k$, and $\sigma\geqslant 0$. The existence, multiplicity and uniqueness of radially symmetric bounded solutions to (4) were investigated by a dynamical systems approach. Lately, Miyamoto *et al*[24] extended the problem (4) into

$$\begin{cases} S_k(D^2\varphi)=\mu\dfrac{|\boldsymbol{x}|^{\lambda-2}}{(1+|\boldsymbol{x}|^2)^{\lambda/2}}\left(1-\varphi\right)^q, & \boldsymbol{x}\in B\\ \varphi<0, & \boldsymbol{x}\in B\\ \varphi=0, & \boldsymbol{x}\in\partial B\end{cases}$$

where B denotes the unit ball in $\mathbb{R}^n, n>2k, k\in\mathbb{N}, \mu>0, q>k$, and $\lambda\geqslant 2$. Combining dynamical-systems tools, the intersection number between a singular and a regular solution and the super/subsolution method, the existence and multiplicity of solutions for the above problem were obtained. The problems with k-Hessian operator have attracted lots of attention, see e.g., Refs. [25-32].

It is known from Refs. [10,12] that the equation (1) with $k=1$ admits three different types of radially positive solutions: the F-, E- and M-solutions. Furthermore, the E- and F-solutions are regular, and the M-solutions are singular. However, when $k>1$, it turns out that the equation (1) only has the E- and M-solutions, see Section 1.1. From the above literatures, the study of M-solutions to (1) is quite scarce. Hence, we shall pay our attention to the existence and asymptotic behavior of the singular solution (i.e., the M-solutions). To this end, let $p=n-2+\lambda$.

When $p>[(n-2k)q]/k$, i.e., $1<q<[(n-2+\lambda)k]/(n-2k)$, we firstly give some a priori estimates in Theorem 1. Similar to Refs. [10,13], we find the M-solution admits a splitting form: $\varphi=S+\Theta$, where S is the singular term and Θ is the regular one. To derive more accurate asymptotic expansions of S and Θ, we introduce a new parameter $\omega:=p-[(n-2k)q]/k$, and choose $k_0\in\mathbb{N}$ such that $2k_0<(n-2k)/k\leqslant 2(k_0+1)$ in Theorem 1, 2 and Theorem 3. Furthermore, we separate the range $p>[(n-2k)q]/k$ into three subcases: (i) $[(n-2k)q]/k<p<[(n-2k)(q+1)]/k$; (ii) $p=[(n-2k)(q+1)]/k$; (iii) $p>[(n-2k)(q+1)]/k$ in Section 2. It is worth noting that we require more precise ranges of $(n-2k)/k$ and ω for the subcase (i), which is the most complicated and difficult case in these three subcases. Combining a priori estimates with an iterative method of Refs. [10,11], we could obtain the precise asymptotic expansions of S and Θ near the origin.

The case with Laplace operator (i.e., $k=1$) and weight term $K(r)= r^{\lambda-2}/(1+r^2)^{\lambda/2}$ has been examined in Ref. [12]; the case with p-Laplace operator and weight term $K(r)=r^{\sigma}$ has been discussed in Ref. [16]. These provided us the significant references to solve problems for the case with k-Hessian operator (i.e., $k>1$) and weight term $K(r)= r^{\lambda-2}/(1+r^2)^{\lambda/2}$. The schemes we used in current paper are as follows.

First, motivated by Ref. [12], we replace Laplace operator (i.e., $k=1$) with k-Hessian operator (i.e., $k>1$), which leads to some computational challenges as follows. Since $\Delta\varphi=r^{1-n}(r^{n-1}\varphi')'$, we find that the exponent of φ' is 1. A straightforward ordinary differential equation (ODE) analysis implies that $\varphi'=-\frac{1}{r^{n-1}}\left[\hat{c}+\int_0^r s^{n-1}K(s)\varphi^q(s)\mathrm{d}s\right]$, where $K(r)= r^{\lambda-2}/(1+r^2)^{\lambda/2}$. Based on the Taylor expansion for $K(r)$, the asymptotic expansion of φ' could be obtained. However, since $S_k(D^2(-\varphi))=c_{n,k}r^{1-n}(r^{n-k}(-\varphi')^k)'$, $c_{n,k}=\binom{n}{k}\Big/n$, we have that the exponent of φ' is k, and then deduce

$$\varphi'=-\frac{1}{r^{(n-k)/k}}\left[\bar{c}+\int_0^r c_{n,k}^{-1}s^{n-1}K(s)\varphi^q(s)\mathrm{d}s\right]^{1/k}:=-\frac{1}{r^{(n-k)/k}}\bar{K}^{1/k}(r). \tag{5}$$

Not only do we need to use Taylor expansion for $K(r)$, but we also need to use Taylor expansion for $\bar{K}(r)$. The repeated application of Taylor expansion makes the calculation more complex. Inspired by Ref. [16], we replace weight term $K(r)=r^{\sigma}$ with $K(r)= r^{\lambda-2}/(1+r^2)^{\lambda/2}$. Since $\Delta_p\varphi=r^{1-n}(r^{n-1}|\varphi'|^{p-2}\varphi')'$, we have the exponent of φ' is $p-1$, and then derive

$$\varphi'=-\frac{1}{r^{(n-1)/(p-1)}}\left[\tilde{c}+\int_0^r s^{n-1}K(s)\varphi^q(s)\mathrm{d}s\right]^{1/(p-1)}:=-\frac{1}{r^{(n-1)/(p-1)}}\bar{\bar{K}}^{1/(p-1)}(r). \tag{6}$$

In a similar manner with (5), we also need to use Taylor expansion for $\bar{\bar{K}}$ corresponding to $\bar{K}$ in (5), however, in this paper we require to use Taylor expansion for $K(r)$ once more. Second, we shall state that the precise ranges for $(n-2k)/k$ and ω are necessary in Theorem 1. Wang and Zhang[16] obtained that φ is in the form of

$$\varphi=\frac{c}{r^{(n-p)/(p-1)}}\left[1+\sum_{i=1}^{n_0+1}\hat{a}_i r^{i\mu}+o(r^{(n_0+1)\mu})\right]:=\frac{c}{r^{(n-p)/(p-1)}}\left[1+D_1+o(r^{(n_0+1)\mu})\right],$$

where $\mu=N+\sigma-[(N-p)q]/(p-1)$, $\sigma>p$, and n_0 is a positive integer.

When $n_0\mu<(n-p)/(p-1)<(n_0+1)\mu$, they split the term $r^{-(n-p)/(p-1)}D_1$ into singular term (i.e., $\sum_{i=1}^{n_0}\hat{a}_i r^{i\mu-(n-p)/(p-1)}$) and regular term (i.e., $\hat{a}_{n_0+1}r^{(n_0+1)\mu-(n-p)/(p-1)}$). In this paper, we obtain for $(n-2k)/k<(n_0+1)\omega$,

$$\varphi=\frac{c}{r^{(n-2k)/k}}\left[1+\sum_{i=1}^{n_0+1}\hat{a}_{i0}r^{i\omega}+\sum_{i=1}^{n_0}\sum_{j=1}^{\eta_i}\hat{a}_{ij}r^{i\omega+2j}+o(r^{(n_0+1)\omega})\right]:=\frac{c}{r^{(n-2k)/k}}\left[1+D_2+D_3+o(r^{(n_0+1)\omega})\right],$$

where $\eta_i=(n_0-i+1)k_0+n_0-i$ and $\hat{a}_{ij}$ are some constants depending upon $c,\lambda,q,k,k_0,n,$ and n_0. The presence of the term D_3 is due to the repeated use of Taylor expansion. In a similar manner with $r^{-(n-p)/(p-1)}D_1$, we shall split the term $r^{-(n-2k)/k}D_2$ into singular and regular terms when $n_0\omega<(n-2k)/k<(n_0+1)\omega$. But this range is no longer sufficient to divide the term $r^{-(n-2k)/k}D_3$ into singular and regular terms. To solve this problem, we require the following precise range of $(n-2k)/k$:

$$2(n_0+1)k_0+2n_0<(n-2k)/k<(n_0+1)\omega.$$

On the other hand, when $(n-2k)/k=(n_0+1)\omega$, since the fact that the size of the exponents $(n-2k)/k$ and $i\omega+2j$ could not be determined, we introduce a precise range on ω, i.e., $2k_0+2n_0/(n_0+1)<\omega\leqslant 2(k_0+1)$.

1 Preliminaries

1.1 Classification of Positive Solutions

In this subsection, we will separate radially positive solutions of a more general problem including (1) into two distinct types. Firstly, we state the definitions of the k-Hessian operator and maximal solution.

Definition 1 Let $\vartheta \in C^2(\Omega)$, $1 \leqslant k \leqslant n$, $k \in \mathbb{N}$ and $\Lambda=(\lambda_1,\lambda_2,\cdots,\lambda_n)$ be the eigenvalues of the Hessian matrix $(D^2\vartheta)$. Then the k-Hessian operator is given by the formula $S_k(D^2\vartheta)=P_k(\Lambda)=\sum\limits_{1\leqslant i_1<\cdots<i_k\leqslant n}\lambda_{i_1}\cdots\lambda_{i_k}$, where $P_k(\Lambda)$ is the k-th elementary symmetric polynomial in the eigenvalues Λ, see Ref. [33].

Note that the k-Hessian operators are fully nonlinear for $k\neq 1$. Furthermore, they are not elliptic in general, unless they are restricted to the class

$$\Phi_\vartheta^k(\Omega)=\left\{\vartheta\in C^2(\Omega)\cap C^1(\bar{\Omega}): S_i(D^2\vartheta)\geqslant 0 \text{ in } \Omega, i=1,\cdots,k\right\}. \tag{7}$$

Observe that $\Phi_\vartheta^k(\Omega)$ belongs to the class of subharmonic functions. Moreover, it follows from the maximum principle[33] that the functions in $\Phi_\vartheta^k(\Omega)$ are negative in Ω.

To investigate positive solutions of (1), under the change of variable $\varphi=-\vartheta$, it is not hard to obtain $S_k(D^2\vartheta)=(-1)^k S_k(D^2\varphi)$ by the k-homogeneity of the k-Hessian operator [23,34].

Definition 2 A function $\varphi=-\vartheta\in\Phi_\vartheta^k(\Omega)$ is called a supersolution (resp. subsolution) of (1) if

$$S_k(D^2(-\varphi))\geqslant(\text{resp.}\leqslant)\frac{|\boldsymbol{x}|^{\lambda-2}}{(1+|\boldsymbol{x}|^2)^{\lambda/2}}\varphi^q \text{ in } \Omega.$$

Observe that the trivial function $\varphi\equiv 0$ is always a subsolution.

Definition 3 We say that a function φ is a maximal solution of (1) if φ is a solution of (1) and, for every subsolution ψ of (1), we have $\psi\leqslant\varphi$.

Remark 1 Introduction of functional space $\Phi_\vartheta^k(\Omega)$ is to ensure that the k-Hessian operators are elliptic. Then the maximum principle and the super/subsolutions method could be applied to investigate existence of the solutions to (1).

Let K be a positive function in $C^1(\mathbb{R}^+)$ such that $r^{(n-k)/k}K(r)$ is bounded as $r\to+\infty$. Suppose that $\varphi:(R_-,R)\to(0,+\infty)$ is a maximal solution of the problem

$$c_{n,k}r^{1-n}(r^{n-k}(-\varphi')^k)'=K(r)\varphi^q,\ \ q>k>1, n>2k, \tag{8}$$

where $0\leqslant R_-<R<+\infty$. Now we introduce the space of functions Φ^k defined on $\varphi=-\vartheta$ and $\Omega=(R_-,R)$ as in (7), for problem (8):

$$\Phi^k=\left\{\varphi\in C^2((R_-,R))\cap C^1([R_-,R]): (r^{n-i}(-\varphi')^i)'\geqslant 0 \text{ in } (R_-,R), i=1,\cdots,k\right\}.$$

Note that the functions in Φ^k are non-negative on $[R_-,R]$. If $(r^{n-i}(-\varphi')^i)'>0$ for every $i=1,\cdots,k$, then any function in Φ^k is positive and strictly decreasing on $[R_-,R]$. Let $r_0\in(R_-,R)$ and

$$G(r):=-r_0^{n-k}(-\varphi'(r_0))^k-\int_{r_0}^r c_{n,k}^{-1}s^{n-1}K(s)\varphi^q(s)\mathrm{d}s \text{ in } (R_-,R) \tag{9}$$

It follows that $G(r)=-r^{n-k}(-\varphi')^k$, i.e., $(-\varphi')^k=-\dfrac{G(r)}{r^{n-k}}$ and $G'(r)=-c_{n,k}^{-1}r^{n-1}K(r)\varphi^q<0$ in (R_-,R). Hence the limit $G_0:=\lim\limits_{r\to R_-}G(r)\in(-\infty,+\infty]$ exists.

We claim that $G_0\leqslant 0$. If not, there exists some $r^*\in(R_-,r_0)$ such that $G(r)>G(r^*)>0$ in (R_-,r^*). Hence, $(-\varphi')^k=-\dfrac{G(r)}{r^{n-k}}<0$ in (R_-,r^*), which is impossible.

For $G_0\leqslant 0$, we claim that $R_-=0$. We argue by contradiction. Suppose that $R_->0$. Then there exists $r_*=(R_-,r^*)$ such that $G(r_*)<0$, $\varphi'(r_*)<0$, $G(r)$ and $\varphi'(r)$ are bounded in (R_-,r_*). Therefore, $\varphi(r)$ could be extended beyond R_-, which is a contradiction. Thus $R_-=0$ and $\varphi'(r)<0$ in $(0,R)$. Therefore, the limit $\lim\limits_{r\to 0}\varphi(r)\in(0,+\infty]$ exists. In this case, we define $R_0:=0$ and have $R_0=\inf\left\{r\in(0,R)|\varphi'(r)<0\right\}$.

The solutions of (8) could be classified as follows:

(i) if $\lim\limits_{r\to 0}\varphi(r)=+\infty$, then we call φ an M-solution;

(ii) if $\lim\limits_{r\to 0}\varphi(r)<+\infty$, then we call φ an E-solution.

Remark 2 It is known Refs. [10, 12] that when $G_0>0\Rightarrow R_0>R_->0$, there exists an F-solution. Because the functions are restricted to Φ^k, they are positive and strictly decreasing on $[R_-,R]$. It follows that $G_0\leqslant 0$. Hence, the k-Hessian equation (8) has no F-solution.

In this paper, we set $K(r)=r^{\lambda-2}/(1+r^2)^{\lambda/2}$ with $\lambda>0$, the equation (8) reduces to the radial form of (1) , i.e.,

$$c_{n,k} r^{1-n}(r^{n-k}(-\varphi')^k)' = \frac{r^{\lambda-2}}{(1+r^2)^{\lambda/2}}\varphi^q, \quad c_{n,k} = \binom{n}{k}\Big/ n, \tag{10}$$

where $q > k > 1, n > 2k$.

In the following, we introduce a lemma that will be frequently used to examine the existence and uniqueness of regular term of the M-solutions when $p > [(n-2k)q]/k$. The proof can be established by similar argument as in Ref. [4].

Lemma 1 Assume that $\beta \in \mathbb{R}$ and $f(r,\Theta):(0,\infty)\times\mathbb{R}\to\mathbb{R}$ satisfies the following conditions:

(i) $f(r,\Theta) \in C\left((0,\infty)\times\mathbb{R}\right)$;

(ii) $f(r,\beta) \in L^1_{\text{loc}}[0,\infty)$;

(iii) there exist a number $\delta > 0$ and a function

$$L_\beta:(0,\delta)\to[0,\infty] \text{ with } L_\beta(r)\in L^1[0,\delta],$$

such that for every $r \in (0,\delta)$ and $\Theta_1, \Theta_2 \in [\beta-\delta, \beta+\delta]$,

$$|f(r,\Theta_1) - f(r,\Theta_2)| \leqslant L_\beta(r)|\Theta_1 - \Theta_2|.$$

Then the initial value problem:

$$\Theta' = f(r,\Theta), \quad \Theta(0) = \beta$$

admits a unique solution Θ on $(0,R)$ with $\Theta(0) := \lim_{r\to 0}\Theta = \beta$.

1.2 Transformation to Lotka-Volterra System

In this subsection we discuss the solutions φ of (8) when $r \in (0,R)$. Inspired by Refs. [23,24], we adopt a more general transformation

$$u(t) := r^k \frac{K(r)\varphi^q}{c_{n,k}(-\varphi')^k}, \quad v(t) := r\frac{-\varphi'}{\varphi}, \quad r := \mathrm{e}^t, \tag{11}$$

where $\varphi' = \mathrm{d}\varphi/\mathrm{d}r$. Set $J_\varphi := (0,R)$ and $I_\phi := \ln J_\varphi$. We find that $\phi := (u,v): I_\phi \to \mathbb{R}^+\times\mathbb{R}^+$ is a maximal solution of a non-autonomous Lotka-Volterra system

$$\begin{cases} \dot{u} = u\left[\, p(t) - u - qv \,\right], \\ \dot{v} = v\left[-\dfrac{n-2k}{k} + \dfrac{u}{k} + v \right], \end{cases} \tag{12}$$

where " · " denotes differentiation with respect to t, $\mathbb{R}^+\times\mathbb{R}^+$ is an invariant set of system (12), i.e., the positive u- and v-axes are invariant, and $p(t) = n + r\dfrac{K'(r)}{K(r)}$.

Furthermore, the inverse of (11) could be characterized by

$$\varphi = \left[\frac{u(\ln r)v(\ln r)^k}{c_{n,k} r^{2k} K(r)} \right]^{1/(q-k)}. \tag{13}$$

In particular, when $K(r) = r^{\lambda-2}/(1+r^2)^{\lambda/2}$, we have $p(t) = n - 2 + \lambda - \dfrac{\lambda \mathrm{e}^{2t}}{1+\mathrm{e}^{2t}}$.

Define $p := n - 2 + \lambda$. Thus $\lim_{t\to-\infty} p(t) = p$. Then the limiting system of system (12) as $t \to -\infty$ could be written as

$$\begin{cases} \dot{u} = u\left[\, p - u - qv \,\right], \\ \dot{v} = v\left[-\dfrac{n-2k}{k} + \dfrac{u}{k} + v \right]. \end{cases} \tag{14}$$

2 Singular Solutions

In this section, we mainly study singular solutions φ in $(0,R)$ and their corresponding solutions ϕ in $(-\infty,T)$, where $0 < R < +\infty$ and $T = \ln R$. To establish more precisely asymptotic expansions of singular solutions (i.e., the M-solutions) near the origin, we need to divide $p > [(n-2k)q]/k$ into the following three subcases:

(a) $[(n-2k)q]/k < p < [(n-2k)(q+1)]/k$;

(b) $p=[(n-2k)(q+1)]/k$;

(c) $p>[(n-2k)(q+1)]/k$.

For the case $p\leqslant[(n-2k)q]/k$ one can apply a dynamical system approach to obtain asymptotic expansions of singular solutions, see Refs. [10,12,16]. Firstly, we give a prior estimates of singular solution for $p>[(n-2k)q]/k$.

Lemma 2 Let $p>[(n-2k)q]/k$. Then the following statements are equivalent:

(i) φ is an M-solution.

(ii) There exists a constant $c>0$ such that $\varphi=\dfrac{c}{r^{(n-2k)/k}}[1+o(1)],\ \ r\to 0$.

(iii) There exists a constant $c>0$ such that $u(t)=\dfrac{k^k c^{q-k}}{(n-2k)^k c_{n,k}}\mathrm{e}^{\{p-[(n-2k)q]/k\}t}[1+o(1)],\ v(t)=\dfrac{n-2k}{k}[1+o(1)],\ \ t\to-\infty$.

Moreover, φ satisfies

$$r^{n-k}(-\varphi')^k=\bar{c}+\int_0^r c_{n,k}^{-1}s^{n-1}K(s)\varphi^q\,\mathrm{d}s \tag{15}$$

where $\bar{c}=\left(\dfrac{(n-2k)c}{k}\right)^k$ and c are uniquely determined.

Proof This proof can be established by similar argument as in Refs. [4,10], and is omitted here.

We proceed to prove that the M-solution φ has a splitting form $\varphi=S+\Theta$, where S is the singular term in the form of $S=cP/r^{(n-2k)/k}$ with $P=1+o(1)$ as $r\to 0$, and Θ is the regular term which satisfies

$$\begin{cases}\Theta'=-\dfrac{1}{r^{(n-k)/k}}\left[\bar{c}+\displaystyle\int_0^r c_{n,k}^{-1}s^{n-1}K(s)(\Theta+S)^q\,\mathrm{d}s\right]^{1/k}-S',0<r<R,\\ \Theta(0):=\lim\limits_{r\to 0^+}\Theta=\beta\in\mathbb{R}.\end{cases} \tag{16}$$

It follows that

$$\begin{aligned}\Theta'&=-\frac{1}{r^{(n-k)/k}}\left[\bar{c}+\int_0^r c_{n,k}^{-1}s^{n-1}K(s)\left(\Theta+\frac{cP}{s^{(n-2k)/k}}\right)^q\mathrm{d}s\right]^{1/k}-\left(\frac{cP}{r^{(n-2k)/k}}\right)'\\
&=-\frac{(n-2k)c}{kr^{(n-k)/k}}\left[1+\frac{c^q}{\bar{c}c_{n,k}}\int_0^r s^{\omega-\lambda+1}K(s)P^q\left(1+\frac{s^{(n-2k)/k}}{cP}\Theta\right)^q\mathrm{d}s\right]^{1/k}+\frac{(n-2k)c}{kr^{(n-k)/k}}P-\frac{c}{r^{(n-2k)/k}}P'\\
&=-\frac{(n-2k)c}{kr^{(n-k)/k}}\left[1+\frac{c^q}{\bar{c}c_{n,k}}\int_0^r s^{\omega-\lambda+1}K(s)P^q\left(1+O\left(\frac{s^{(n-2k)/k}}{P}\Theta\right)\right)\mathrm{d}s\right]^{1/k}+\frac{(n-2k)c}{kr^{(n-k)/k}}P-\frac{c}{r^{(n-2k)/k}}P'\\
&=-\frac{(n-2k)c}{kr^{(n-k)/k}}\left[1+\frac{c^q}{\bar{c}c_{n,k}}\int_0^r s^{\omega-\lambda+1}K(s)P^q\mathrm{d}s+O(r^{\omega+(n-2k)/k})\right]^{1/k}+\frac{(n-2k)c}{kr^{(n-k)/k}}P-\frac{c}{r^{(n-2k)/k}}P'\\
&=:f(r,\Theta)\end{aligned} \tag{17}$$

It is clear that if $f(r,\Theta)$ satisfies the assumptions in Lemma 1, then the problem (16) admits a unique solution. In the following, we will discuss three different subcases $p=[(n-2k)(q+1)]/k, p>[(n-2k)(q+1)]/k$ and $[(n-2k)q]/k<p<[(n-2k)(q+1)]/k$, and establish the expansions of S and Θ near the origin, respectively. In order to do so, we need the following Taylor expansions

$$(1+r)^\alpha=1+\alpha r+\frac{\alpha(\alpha-1)}{2!}r^2+\cdots+\frac{\alpha(\alpha-1)\cdots(\alpha-h+1)}{h!}r^h+o(r^h),$$

$$\frac{1}{(1+r^2)^{\lambda/2}}=1-\frac{\lambda}{2}r^2+\cdots+(-1)^{k_0}\frac{\lambda(\lambda+2)\cdots(\lambda+2k_0-2)}{(2k_0)!!}r^{2k_0}+O(r^{2k_0+2}),$$

which are crucial in the following process of proof.

Theorem 1 Let $[(n-2k)q]/k<p<[(n-2k)(q+1)]/k$. Define $\omega:=p-[(n-2k)q]/k$ and choose $n_0,k_0\in\mathbb{N}$ such that $2(n_0+1)k_0+2n_0<(n-2k)/k\leqslant(n_0+1)\omega$ and $2k_0+2n_0/(n_0+1)<\omega\leqslant 2(k_0+1)$. Then there exist numbers $\hat{a}_{ij}\ (i=0,1,\cdots,n_0+3;j=0,1,\cdots,\eta_i$, where $\eta_i=(n_0-i+1)k_0+n_0-i;\hat{a}_{i0}:=\hat{a}_i)$ depending on $c,\lambda,q,k,k_0,n,$ and n_0 such that

(i) Every M-solution φ has the form $\varphi = S + \Theta$, where

$$S=\begin{cases}\dfrac{c}{r^{(n-2k)/k}}\left(1+\sum_{i=1}^{n_0}\sum_{j=0}^{\eta_i}\hat{a}_{ij}r^{i\omega+2j}\right), & (n-2k)/k<(n_0+1)\omega,\\ \dfrac{c}{r^{(n_0+1)\omega}}\left(1+\sum_{i=1}^{n_0}\sum_{j=0}^{\eta_i}\hat{a}_{ij}r^{i\omega+2j}+\hat{a}_{n_0+1}r^{(n_0+1)\omega}\ln r\right), & (n-2k)/k=(n_0+1)\omega,\end{cases}\tag{18}$$

and Θ is the solution of (16) with the expansion

$$\Theta=\begin{cases}\beta+\hat{a}_{n_0+1}r^{(n_0+1)\omega-(n-2k)/k}+o(r^{(n_0+1)\omega-(n-2k)/k}), & (n-2k)/k<(n_0+1)\omega,\\ \beta+\sum_{i=1}^{n_0}\sum_{j=\eta_i+1}^{\eta_{i-1}}\hat{a}_{ij}r^{(i-n_0-1)\omega+2j}+\sum_{j=1}^{k_0}\hat{a}_{n_0+1,j}r^{2j}+\hat{a}_{n_0+2}r^{\omega}\ln r+\hat{a}_{n_0+3}r^{\omega}+o(r^{\omega}), & (n-2k)/k=(n_0+1)\omega,\end{cases}$$

for uniquely determined constants $c>0, \beta\in\mathbb{R}$.

(ii) Conversely, given any $c>0$, $\beta\in\mathbb{R}$, there exists a unique solution Θ of (16) such that $\varphi = S+\Theta$ is an M-solution, where S is given by (18), and Θ satisfies (17) with

$$f(r,\Theta)=\begin{cases}O(r^{(n_0+1)\omega-(n-k)/k}), & (n-2k)/k<(n_0+1)\omega,\\ O(r^{n_0(2k_0+2-\omega)-1}), & (n-2k)/k=(n_0+1)\omega.\end{cases}$$

Moreover, the solution (u, v) possesses the following expansion:

$$\begin{cases}u(t)=\dfrac{c^q}{\bar{c}c_{n,k}}\mathrm{e}^{\omega t}\left[1-\dfrac{\lambda}{2}\mathrm{e}^{2t}+\cdots+(-1)^{k_0}\dfrac{\lambda(\lambda+2)\cdots(\lambda+2k_0-2)}{(2k_0)!!}\mathrm{e}^{2k_0t}-\dfrac{[\omega k^2+(n-2k)(q-k)]c^q}{\omega[\omega-(n-2k)/k]k^2\bar{c}c_{n,k}}\mathrm{e}^{\omega t}+o(\mathrm{e}^{\omega t})\right],\\ v(t)=\dfrac{n-2k}{k}\left[1+\dfrac{[\omega k+(n-2k)(q-1)]c^q}{\omega[\omega-(n-2k)/k]k^2\bar{c}c_{n,k}}\mathrm{e}^{\omega t}+o(\mathrm{e}^{\omega t})\right],\quad t\to-\infty.\end{cases}$$

Proof (i) Let c, $\bar{c}>0$ be determined by Lemma 2. We compute

$$s^{n-1}K(s)\varphi^q=s^{p-1}[1+O(s^2)]\left(\frac{c}{s^{(n-2k)/k}}[1+o(1)]\right)^q=c^qs^{\omega-1}[1+O(s^2)][1+o(1)]=c^qs^{\omega-1}[1+o(1)].$$

Using (15), we have

$$(-\varphi')^k=\frac{\bar{c}}{r^{n-k}}\left[1+\frac{1}{\bar{c}c_{n,k}}\int_0^r s^{n-1}K(s)\varphi^q\mathrm{d}s\right]=\frac{\bar{c}}{r^{n-k}}\left[1+\frac{c^q}{\omega\bar{c}c_{n,k}}r^{\omega}+o(r^{\omega})\right],$$

$$\varphi'=-\frac{(n-2k)c}{kr^{(n-k)/k}}\left[1+\frac{c^q}{\omega\bar{c}c_{n,k}}r^{\omega}+o(r^{\omega})\right]^{1/k}=-\frac{(n-2k)c}{kr^{(n-k)/k}}\left[1+\frac{c^q}{\omega k\bar{c}c_{n,k}}r^{\omega}+o(r^{\omega})\right].$$

Thus

$$\varphi=\frac{c}{r^{(n-2k)/k}}-\frac{(n-2k)c^{q+1}}{\omega[\omega-(n-2k)/k]k^2\bar{c}c_{n,k}}r^{\omega-(n-2k)/k}+o(r^{\omega-(n-2k)/k})+\hat{C}=\frac{c}{r^{(n-2k)/k}}\left[1-\frac{(n-2k)c^q}{\omega[\omega-(n-2k)/k]k^2\bar{c}c_{n,k}}r^{\omega}+o(r^{\omega})\right],$$

where $\hat{C}$ is a constant.

Repeating the above iterative process, we deduce

$$\begin{aligned}s^{n-1}K(s)\varphi^q&=c^qs^{\omega-1}(1+s^2)^{-\lambda/2}\left[1-\frac{(n-2k)c^q}{\omega[\omega-(n-2k)/k]k^2\bar{c}c_{n,k}}s^{\omega}+o(s^{\omega})\right]^q\\&=c^qs^{\omega-1}\left[1-\frac{\lambda}{2}s^2+\cdots+(-1)^{k_0}\frac{\lambda(\lambda+2)\cdots(\lambda+2k_0-2)}{(2k_0)!!}s^{2k_0}+O(s^{2k_0+2})\right]\cdot\left[1-\frac{q(n-2k)c^q}{\omega[\omega-(n-2k)/k]k^2\bar{c}c_{n,k}}s^{\omega}+o(s^{\omega})\right]\\&=c^qs^{\omega-1}\left[1-\frac{\lambda}{2}s^2+\cdots+(-1)^{k_0}\frac{\lambda(\lambda+2)\cdots(\lambda+2k_0-2)}{(2k_0)!!}s^{2k_0}-\frac{q(n-2k)c^q}{\omega[\omega-(n-2k)/k]k^2\bar{c}c_{n,k}}s^{\omega}+o(s^{\omega})\right].\end{aligned}$$

Hence,

$$(-\varphi')^k=\frac{\bar{c}}{r^{n-k}}\left[1+\frac{1}{\bar{c}c_{n,k}}\int_0^r s^{n-1}K(s)\varphi^q\mathrm{d}s\right]$$

$$= \frac{\bar{c}}{r^{n-k}}\left\{1+\frac{c^q}{\bar{c}c_{n,k}}\int_0^r s^{\omega-1}\left[1+\sum_{j=1}^{k_0}(-1)^j\frac{\lambda(\lambda+2)\cdots(\lambda+2j-2)}{(2j)!!}s^{2j}-\frac{q(n-2k)c^q}{\omega[\omega-(n-2k)/k]k^2\bar{c}c_{n,k}}s^{\omega}+o(s^{\omega})\right]\mathrm{d}s\right\}$$

$$= \frac{\bar{c}}{r^{n-k}}\left[1+\frac{c^q}{\omega\bar{c}c_{n,k}}r^{\omega}+\sum_{j=1}^{k_0}(-1)^j\frac{\lambda(\lambda+2)\cdots(\lambda+2j-2)c^q}{(2j)!!(\omega+2j)\bar{c}c_{n,k}}r^{\omega+2j}-\frac{q(n-2k)c^{2q}}{2\omega^2[\omega-(n-2k)/k]k^2(\bar{c}c_{n,k})^2}r^{2\omega}+o(r^{2\omega})\right].$$

Then

$$\varphi'=-\frac{(n-2k)c}{kr^{(n-k)/k}}\left[1+\frac{c^q}{\omega k\bar{c}c_{n,k}}r^{\omega}+\sum_{j=1}^{k_0}(-1)^j\frac{\lambda(\lambda+2)\cdots(\lambda+2j-2)c^q}{(2j)!!(\omega+2j)k\bar{c}c_{n,k}}r^{\omega+2j}-\frac{[q(n-2k)+(k-1)(\omega k-n+2k)]c^{2q}}{2\omega^2[\omega-(n-2k)/k]k^3(\bar{c}c_{n,k})^2}r^{2\omega}+o(r^{2\omega})\right].$$

If $n_0=1$, i.e., $4k_0+2<(n-2k)/k\leqslant 2\omega$, then there exists $\beta\in\mathbb{R}$ such that when $(n-2k)/k=2\omega$,

$$\varphi'=-\frac{2\omega c}{r^{2\omega+1}}\left[1+\frac{c^q}{\omega k\bar{c}c_{n,k}}r^{\omega}+\sum_{j=1}^{k_0}(-1)^j\frac{\lambda(\lambda+2)\cdots(\lambda+2j-2)c^q}{(2j)!!(\omega+2j)k\bar{c}c_{n,k}}r^{\omega+2j}+\frac{(2q-k+1)c^{2q}}{2\omega^2k^2(\bar{c}c_{n,k})^2}r^{2\omega}+o(r^{2\omega})\right],$$

$$\varphi=\frac{c}{r^{2\omega}}\left[1+\frac{2c^q}{\omega k\bar{c}c_{n,k}}r^{\omega}+\sum_{j=1}^{k_0}(-1)^j\frac{2\omega\lambda(\lambda+2)\cdots(\lambda+2j-2)c^q}{(2j)!!(\omega-2j)(\omega+2j)k\bar{c}c_{n,k}}r^{\omega+2j}-\frac{(2q-k+1)c^{2q}}{\omega k^2(\bar{c}c_{n,k})^2}r^{2\omega}\ln r+o(r^{2\omega}\ln r)\right].$$

Similarly, we repeat above process again and get

$$s^{n-1}K(s)\varphi^q=c^qs^{\omega-1}\Bigg[1-\frac{\lambda}{2}s^2+\cdots+(-1)^{2k_0+1}\frac{\lambda(\lambda+2)\cdots(\lambda+4k_0)}{(4k_0+2)!!}s^{4k_0+2}+\frac{2qc^q}{\omega k\bar{c}c_{n,k}}s^{\omega}$$
$$+\sum_{j=1}^{k_0}(-1)^j\frac{2q\omega\lambda(\lambda+2)\cdots(\lambda+2j-2)c^q}{(2j)!!(\omega-2j)(\omega+2j)k\bar{c}c_{n,k}}s^{\omega+2j}-\frac{q(2q-k+1)c^{2q}}{\omega k^2(\bar{c}c_{n,k})^2}s^{2\omega}\ln s+o(s^{2\omega}\ln s)\Bigg]$$

and

$$(-\varphi')^k=\frac{\bar{c}}{r^{n-k}}\Bigg[1+\frac{c^q}{\omega\bar{c}c_{n,k}}r^{\omega}+\sum_{j=1}^{2k_0+1}(-1)^j\frac{\lambda(\lambda+2)\cdots(\lambda+2j-2)c^q}{(2j)!!(\omega+2j)\bar{c}c_{n,k}}r^{\omega+2j}+\frac{qc^{2q}}{\omega^2k(\bar{c}c_{n,k})^2}r^{2\omega}$$
$$+\sum_{j=1}^{k_0}(-1)^j\frac{2q\omega\lambda(\lambda+2)\cdots(\lambda+2j-2)c^{2q}}{(2j)!!(\omega-2j)(\omega+2j)(2\omega+2j)k(\bar{c}c_{n,k})^2}r^{2\omega+2j}-\frac{q(2q-k+1)c^{3q}}{3\omega^2k^2(\bar{c}c_{n,k})^3}r^{3\omega}\ln r+\frac{q(2q-k+1)c^{3q}}{9\omega^3k^2(\bar{c}c_{n,k})^3}r^{3\omega}+o(r^{3\omega})\Bigg].$$

Therefore,

$$\varphi'=-\frac{2\omega c}{r^{2\omega+1}}\left[1+\sum_{j=0}^{2k_0+1}\tilde{a}_{0,j}r^{\omega+2j}+\sum_{j=0}^{k_0}\tilde{a}_{1,j}r^{2\omega+2j}+\tilde{a}_2r^{3\omega}\ln r+\tilde{a}_3r^{3\omega}+o(r^{3\omega})\right]$$

with some adequate constants $\tilde{a}_{ij}\,(i=1,2,3;j=0,1,\cdots,2k_0+1)$ depending on $c,\lambda,q,k,k_0,$ and n. Integration yields

$$\varphi=\frac{c}{r^{2\omega}}\left[1+\sum_{j=0}^{k_0}\frac{2\omega a_{0,j}}{\omega-2j}r^{\omega+2j}+2\omega a_1r^{2\omega}\ln r\right]+\beta+\sum_{j=k_0+1}^{2k_0+1}\frac{2\omega ca_{0,j}}{\omega-2j}r^{-\omega+2j}-\sum_{j=0}^{k_0}\frac{\omega ca_{1,j}}{j}r^{2j}-2ca_2r^{\omega}\ln r+2c\left(\frac{a_2}{\omega}-a_3\right)r^{\omega}+o(r^{\omega}).$$

The singular term S and regular term Θ can be acquired. By induction, for the case $(n-2k)/k=(n_0+1)\omega$, we may suppose that $\varphi=\frac{c}{r^{(n_0+1)\omega}}\left[1+\sum_{i=1}^{n_0}\sum_{j=0}^{\eta_i}\hat{a}_{ij}r^{i\omega+2j}+\hat{a}_{n_0+1}r^{(n_0+1)\omega}\ln r+o(r^{(n_0+1)\omega}\ln r)\right]$, where $\hat{a}_{ij}\,(i=0,1,\cdots,n_0+1;j=0,1,\cdots,\eta_i)$ are some constants depending upon $c,\lambda,q,k,k_0,n,$ and n_0. In a similar manner, we compute

$$s^{n-1}K(s)\varphi^q=c^qs^{\omega-1}(1+s^2)^{-\lambda/2}\left[1+\sum_{i=1}^{n_0}\sum_{j=0}^{\eta_i}\hat{a}_{ij}s^{i\omega+2j}+\hat{a}_{n_0+1}s^{(n_0+1)\omega}\ln s+o(s^{(n_0+1)\omega}\ln s)\right]^q$$

$$=c^qs^{\omega-1}\left[1-\frac{\lambda}{2}s^2+\cdots+(-1)^{k_0}\frac{\lambda(\lambda+2)\cdots(\lambda+2k_0-2)}{(2k_0)!!}s^{2k_0}+O(s^{2k_0+2})\right]\cdot\left[1+\sum_{i=1}^{n_0}\sum_{j=0}^{\eta_i}\breve{a}_{ij}s^{i\omega+2j}+q\hat{a}_{n_0+1}s^{(n_0+1)\omega}\ln s+o(s^{(n_0+1)\omega}\ln s)\right]$$

$$=c^qs^{\omega-1}\Bigg[1-\frac{\lambda}{2}s^2+\cdots+(-1)^{(n_0+1)k_0+n_0}\frac{\lambda(\lambda+2)\cdots(\lambda+2(n_0+1)k_0-2(n_0-1))}{[2(n_0+1)k_0+2n_0]!!}\cdot s^{2(n_0+1)k_0+2n_0}$$
$$+\sum_{i=1}^{n_0}\sum_{j=0}^{\eta_i}\bar{a}_{ij}s^{i\omega+2j}+q\hat{a}_{n_0+1}s^{(n_0+1)\omega}\ln s+o(s^{(n_0+1)\omega}\ln s)\Bigg].$$

where $\breve{a}_{ij}$ and $\bar{a}_{ij}$ are some appropriate constants which depend on $c,\lambda,q,k,k_0,n,$ and n_0. Hence,

$$(-\varphi')^k = \frac{\bar{c}}{r^{n-k}}\left\{1+\frac{c^q}{\bar{c}c_{n,k}}\int_0^r s^{\omega-1}\left[1-\frac{\lambda}{2}s^2+\cdots+(-1)^{(n_0+1)k_0+n_0}\frac{\lambda(\lambda+2)\cdots(\lambda+2(n_0+1)k_0-2(n_0-1))}{[2(n_0+1)k_0+2n_0]!!}s^{2(n_0+1)k_0+2n_0}\right.\right.$$

$$\left.\left.+\sum_{i=1}^{n_0}\sum_{j=0}^{\eta_i}\bar{a}_{ij}s^{i\omega+2j}+q\hat{a}_{n_0+1}s^{(n_0+1)\omega}\ln s+o(s^{(n_0+1)\omega}\ln s)\right]\mathrm{d}s\right\}$$

$$=\frac{\bar{c}}{r^{n-k}}\left[1+\frac{c^q}{\omega\bar{c}c_{n,k}}r^{\omega}-\frac{\lambda c^q}{2(\omega+2)\bar{c}c_{n,k}}r^{\omega+2}+\cdots+(-1)^{(n_0+1)k_0+n_0}\cdot\frac{\lambda(\lambda+2)\cdots(\lambda+2(n_0+1)k_0-2(n_0-1))c^q}{[2(n_0+1)k_0+2n_0]!!(\omega+2(n_0+1)k_0+2n_0)\bar{c}c_{n,k}}r^{\omega+2(n_0+1)k_0+2n_0}\right.$$

$$\left.+\sum_{i=1}^{n_0}\sum_{j=0}^{\eta_i}\frac{\bar{a}_{ij}c^q}{[(i+1)\omega+2j]\bar{c}c_{n,k}}r^{(i+1)\omega+2j}+\frac{q\hat{a}_{n_0+1}c^q}{(n_0+2)\omega\bar{c}c_{n,k}}r^{(n_0+2)\omega}\ln r-\frac{q\hat{a}_{n_0+1}c^q}{(n_0+2)^2\omega^2\bar{c}c_{n,k}}r^{(n_0+2)\omega}+o(r^{(n_0+2)\omega})\right].$$

Then

$$\varphi'=-\frac{(n_0+1)\omega c}{r^{(n_0+1)\omega+1}}\left[1+\frac{c^q}{\omega k\bar{c}c_{n,k}}r^{\omega}+\sum_{j=1}^{\eta_0}\tilde{a}_{0,j}r^{\omega+2j}+\sum_{i=1}^{n_0}\sum_{j=0}^{\eta_i}\tilde{a}_{ij}r^{(i+1)\omega+2j}+\tilde{a}_{n_0+1}r^{(n_0+2)\omega}\ln r-\tilde{a}_{n_0+2}r^{(n_0+2)\omega}+o(r^{(n_0+2)\omega})\right]$$

with some adequate constants $\tilde{a}_{ij}\,(i=0,1,\cdots,n_0+2;j=0,1,\cdots,\eta_i)$ depending upon $c,\lambda,q,k,$ and n. By integrating, we derive

$$\varphi=\frac{c}{r^{(n_0+1)\omega}}\left[1+\frac{(n_0+1)c^q}{n_0\omega k\bar{c}c_{n,k}}r^{\omega}+\sum_{j=1}^{\eta_0-k_0-1}\frac{\tilde{a}_{0,j}(n_0+1)\omega}{n_0\omega-2j}r^{\omega+2j}+\sum_{i=1}^{n_0-1}\sum_{j=0}^{\eta_i-k_0-1}\frac{\tilde{a}_{ij}(n_0+1)\omega}{(n_0+1)\omega-(i+1)\omega-2j}r^{(i+1)\omega+2j}-\tilde{a}_{n_0}(n_0+1)\omega r^{(n_0+1)\omega}\ln r\right]$$

$$+\beta+\frac{c}{r^{(n_0+1)\omega}}\left[\sum_{j=\eta_0-k_0}^{\eta_0}\frac{\tilde{a}_{0,j}(n_0+1)\omega}{n_0\omega-2j}r^{\omega+2j}+\sum_{i=1}^{n_0-1}\sum_{j=\eta_i-k_0}^{\eta_i}\frac{\tilde{a}_{ij}(n_0+1)\omega}{(n_0+1)\omega-(i+1)\omega-2j}r^{(i+1)\omega+2j}\right]-\sum_{j=1}^{k_0}\frac{\tilde{a}_{n_0,j}(n_0+1)\omega c}{2j}r^{2j}$$

$$-\tilde{a}_{n_0+1}(n_0+1)cr^{\omega}\ln r+(n_0+1)\left(\frac{\tilde{a}_{n_0+1}}{\omega}-\tilde{a}_{n_0+2}\right)cr^{\omega}+o(r^{\omega})$$

$$=\frac{c}{r^{(n_0+1)\omega}}\left[1+\sum_{i=1}^{n_0}\sum_{j=0}^{\eta_i}\hat{a}_{ij}r^{i\omega+2j}+\hat{a}_{n_0+1}r^{(n_0+1)\omega}\ln r\right]+\beta+\sum_{i=1}^{n_0}\sum_{j=\eta_i+1}^{\eta_{i-1}}\hat{a}_{ij}r^{(i-n_0-1)\omega+2j}+\sum_{j=1}^{k_0}\hat{a}_{n_0+1,j}r^{2j}+\hat{a}_{n_0+2}r^{\omega}\ln r+\hat{a}_{n_0+3}r^{\omega}+o(r^{\omega}),$$

where

$$\begin{cases}\hat{a}_1=\dfrac{(n_0+1)c^q}{n_0\omega k\bar{c}c_{n,k}},\\[2mm] \hat{a}_{i+1,j}=\dfrac{\tilde{a}_{ij}(n_0+1)\omega c}{(n_0+1)\omega-(i+1)\omega-2j}\,(i=0,1,\cdots,n_0-1;j=0,1,\cdots,\eta_i),\\[2mm] \hat{a}_{n_0+1}=-\tilde{a}_{n_0}(n_0+1)\omega,\quad \hat{a}_{n_0+1,j}=\dfrac{\tilde{a}_{n_0,j}(n_0+1)\omega c}{2j}\,(j=0,1,\cdots,\eta_i),\\[2mm] \hat{a}_{n_0+2}=-\tilde{a}_{n_0+1}(n_0+1)c,\quad \hat{a}_{n_0+3}=(n_0+1)\left(\dfrac{\tilde{a}_{n_0+1}}{\omega}-\tilde{a}_{n_0+2}\right)c.\end{cases}\tag{19}$$

Thus the singular term S and regular term Θ could be obtained precisely. For the case $2(n_0+1)k_0+2n_0<(n-2k)/k<(n_0+1)\omega$, we can apply similar arguments to obtain its corresponding conclusion.

(ii) To prove that (16) admits a unique solution Θ, it suffices to verify that $f(r,\Theta)$ fulfills the assumptions in Lemma 1. For the case $(n-2k)/k=(n_0+1)\omega$, we find

$$P=1+\sum_{i=1}^{n_0}\sum_{j=0}^{\eta_i}\hat{a}_{ij}r^{i\omega+2j}+\hat{a}_{n_0+1}r^{(n_0+1)\omega}\ln r,\ P'=\sum_{i=1}^{n_0}\sum_{j=0}^{\eta_i}\hat{a}_{ij}(i\omega+2j)r^{i\omega+2j-1}+\hat{a}_{n_0+1}(n_0+1)\omega r^{(n_0+1)\omega-1}\ln r+\hat{a}_{n_0+1}r^{(n_0+1)\omega-1},$$

and

$$f(r,\Theta)=-\frac{(n_0+1)\omega c}{r^{(n_0+1)\omega+1}}\left[1+\frac{c^q}{\bar{c}c_{n,k}}\int_0^r s^{\omega-1}(1+s^2)^{-\lambda/2}\left(1+\sum_{i=1}^{n_0}\sum_{j=0}^{\eta_i}\hat{a}_{ij}s^{i\omega+2j}+\hat{a}_{n_0+1}s^{(n_0+1)\omega}\ln s\right)^q\mathrm{d}s+O(r^{(n_0+2)\omega})\right]^{1/k}$$

$$+\frac{(n_0+1)\omega c}{r^{(n_0+1)\omega+1}}\left(1+\sum_{i=1}^{n_0}\sum_{j=0}^{\eta_i}\hat{a}_{ij}r^{i\omega+2j}+\hat{a}_{n_0+1}r^{(n_0+1)\omega}\ln r\right)-$$

$$\frac{c}{r^{(n_0+1)\omega}}\left(\sum_{i=1}^{n_0}\sum_{j=0}^{\eta_i}\hat{a}_{ij}(i\omega+2j)r^{i\omega+2j-1}+\hat{a}_{n_0+1}(n_0+1)\omega r^{(n_0+1)\omega-1}\ln r+\hat{a}_{n_0+1}r^{(n_0+1)\omega-1}\right)$$

$$
=-\frac{(n_0+1)\omega c}{r^{(n_0+1)\omega+1}}\left[1+\frac{c^q}{\omega\bar{c}c_{n,k}}r^{\omega}-\frac{\lambda c^q}{2(\omega+2)\bar{c}c_{n,k}}r^{\omega+2}+\cdots+(-1)^{(n_0+1)k_0+n_0}\cdot\right.
$$

$$
\frac{\lambda(\lambda+2)\cdots(\lambda+2(n_0+1)k_0+2(n_0-1))c^q}{[2(n_0+1)k_0+2n_0]!!(\omega+2(n_0+1)k_0+2n_0)\bar{c}c_{n,k}}r^{\omega+2(n_0+1)k_0+2n_0}
$$

$$
\left.+\sum_{i=1}^{n_0}\sum_{j=0}^{\eta_i}\frac{\bar{a}_{ij}c^q}{[(i+1)\omega+2j]\bar{c}c_{n,k}}r^{(i+1)\omega+2j}+\frac{q\hat{a}_{n_0+1}c^q}{(n_0+2)\omega\bar{c}c_{n,k}}r^{(n_0+2)\omega}\ln r+o(r^{(n_0+2)\omega}\ln r)\right]^{1/k}
$$

$$
+\frac{(n_0+1)\omega c}{r^{(n_0+1)\omega+1}}+\sum_{i=1}^{n_0}\sum_{j=1}^{\eta_i}\hat{a}_{ij}[(n_0-i+1)\omega-2j]cr^{(i-n_0-1)\omega+2j-1}-\hat{a}_{n_0+1}cr^{-1}
$$

$$
=-\frac{(n_0+1)\omega c}{r^{(n_0+1)\omega+1}}\left[1+\frac{c^q}{\omega k\bar{c}c_{n,k}}r^{\omega}+\sum_{j=1}^{\eta_0}\tilde{a}_{0,j}r^{\omega+2j}+\sum_{i=1}^{n_0}\sum_{j=0}^{\eta_i}\tilde{a}_{ij}r^{(i+1)\omega+2j}+\tilde{a}_{n_0+1}r^{(n_0+2)\omega}\ln r+o(r^{(n_0+2)\omega}\ln r)\right]+\frac{(n_0+1)\omega c}{r^{(n_0+1)\omega+1}}
$$

$$
+\sum_{i=1}^{n_0}\sum_{j=0}^{\eta_i}\hat{a}_{ij}[(n_0-i+1)\omega-2j]cr^{(i-n_0-1)\omega+2j-1}-\hat{a}_{n_0+1}cr^{-1}
$$

$$
=\hat{a}_{1,n_0(k_0+1)}(2k_0+2-\omega)n_0cr^{n_0(2k_0+2-\omega)-1}+o(r^{n_0(2k_0+2-\omega)-1})
$$

due to the relations (19) between $\hat{a}_{ij}$ and $\tilde{a}_{ij}$. The other case can be similarly handled.

For the case $2(n_0+1)k_0+2n_0<(n-2k)/k<(n_0+1)\omega$, we obtain

$$
u(t)=r\frac{K(r)\varphi^q}{c_{n,k}(-\varphi')^k}=\frac{r^{\lambda+k-2}}{c_{n,k}(1+r^2)^{\lambda/2}}\cdot\frac{\varphi^q}{(-\varphi')^k}
$$

$$
=\frac{c^q r^{\omega}}{\bar{c}c_{n,k}(1+r^2)^{\lambda/2}}\cdot\left(1+\sum_{i=1}^{n_0}\sum_{j=0}^{\eta_i}\hat{a}_{ij}r^{i\omega+2j}+o(r^{n_0\omega+2\eta_i})\right)^q\cdot\left[1+\sum_{i=1}^{n_0}\sum_{j=0}^{\eta_i}\frac{\hat{a}_{ij}k[(n-2k)/k-(i\omega+2j)]}{n-2k}r^{i\omega+2j}+o(r^{n_0\omega+2\eta_i})\right]^{-k}
$$

$$
=\frac{c^q}{\bar{c}c_{n,k}}r^{\omega}\left[1-\frac{\lambda}{2}r^2+\cdots+(-1)^{k_0}\frac{\lambda(\lambda+2)\cdots(\lambda+2k_0-2)}{(2k_0)!!}r^{2k_0}+O(r^{2k_0+2})\right]
$$

$$
\cdot\left[1-\frac{q(n-2k)c^q}{\omega[\omega-(n-2k)/k]k^2\bar{c}c_{n,k}}r^{\omega}+o(r^{\omega})\right]\cdot\left[1-\frac{c^q}{\omega\bar{c}c_{n,k}}r^{\omega}+o(r^{\omega})\right]
$$

$$
=\frac{c^q}{\bar{c}c_{n,k}}r^{\omega}\left[1-\frac{\lambda}{2}r^2+\cdots+(-1)^{k_0}\frac{\lambda(\lambda+2)\cdots(\lambda+2k_0-2)}{(2k_0)!!}r^{2k_0}-\frac{[\omega k^2+(n-2k)(q-k)]c^q}{\omega[\omega-(n-2k)/k]k^2\bar{c}c_{n,k}}r^{\omega}+o(r^{\omega})\right]
$$

and

$$
v(t)=r\frac{-\varphi'}{\varphi}=r\left[\frac{(n-2k)c}{kr^{(n-k)/k}}+\sum_{i=1}^{n_0}\sum_{j=0}^{\eta_i}\hat{a}_{ij}[(n-2k)/k-(i\omega+2j)]cr^{i\omega+2j-(n-k)/k}+o(r^{n_0\omega+2\eta_i-(n-k)/k})\right]
$$

$$
\cdot\left[\frac{c}{r^{(n-2k)/k}}\left(1+\sum_{i=1}^{n_0}\sum_{j=0}^{\eta_i}\hat{a}_{ij}r^{i\omega+2j}+o(r^{n_0\omega+2\eta_i})\right)\right]^{-1}
$$

$$
=\frac{n-2k}{k}\left[1+\frac{c^q}{\omega k\bar{c}c_{n,k}}r^{\omega}+o(r^{\omega})\right]\cdot\left[1-\frac{q(n-2k)c^q}{\omega[\omega-(n-2k)/k]k^2\bar{c}c_{n,k}}r^{\omega}+o(r^{\omega})\right]
$$

$$
=\frac{n-2k}{k}\left[1+\frac{[\omega k+(n-2k)(q-1)]c^q}{\omega[\omega-(n-2k)/k]k^2\bar{c}c_{n,k}}r^{\omega}+o(r^{\omega})\right].
$$

Remark 3 The ranges of the parameters (n,k,q,λ,n_0,k_0) are not empty under the assumptions in Theorem 1. Since $n>2k$, $q>k$, $\lambda+2k-2>0$, and $0<\omega:=p-[(n-2k)q]/k<(n-2k)/k$, we can choose $n=50, k=2, q=3$, and $\lambda=29$ so that $(n-2k)/k=23$ and $\omega=8$. By some computations, it is not difficult to find $n_0=2$ and $k_0=3$ such that $2(n_0+1)k_0+2n_0<(n-2k)/k\leqslant(n_0+1)\omega$ and $2k_0+2n_0/(n_0+1)<\omega\leqslant 2(k_0+1)$.

Theorem 2 Let $p=[(n-2k)(q+1)]/k$ and select $k_0\in\mathbb{N}$ such that $2k_0<(n-2k)/k\leqslant 2(k_0+1)$. Then the following results are valid.

(i) Every M-solution φ has the form $\varphi=S+\Theta$, where

$$S=\frac{c}{r^{(n-2k)/k}}-\frac{c^{q+1}}{k\bar{c}c_{n,k}}\ln r \tag{20}$$

and Θ solves the initial value problem (16) with the following expansion

$$\Theta=\beta-\sum_{i=1}^{k_0}(-1)^i\frac{\lambda(\lambda+2)\cdots(\lambda+2i-2)(n-2k)c^{q+1}}{2i[2i+(n-2k)/k](2i)!!k^2\bar{c}c_{n,k}}r^{2i}+\frac{qc^{2q+1}}{2k(n-2k)(\bar{c}c_{n,k})^2}r^{(n-2k)/k}\ln r-\frac{qc^{2q+1}}{[2(n-2k)\bar{c}c_{n,k}]^2}r^{(n-2k)/k}+o(r^{(n-2k)/k})$$

as $r\to 0$ for some uniquely determined constants $c>0,\ \beta\in\mathbb{R}$.

(ii) Conversely, for any given $c>0$ and $\beta\in\mathbb{R}$, there exists a unique solution Θ of (16) such that $\varphi=S+\Theta$ is an M-solution, where S is given by (20), and Θ satisfies (17) with $f(r,\Theta)=O(r)$. In addition, the solution (u,v) has the following expansion:

$$\begin{cases} u(t)=\dfrac{c^q}{\bar{c}c_{n,k}}\mathrm{e}^{(n-2k)t/k}\left[1+\displaystyle\sum_{i=1}^{k_0}(-1)^i\frac{\lambda(\lambda+2)\cdots(\lambda+2i-2)}{(2i)!!}\mathrm{e}^{2it}-\frac{qc^q}{k\bar{c}c_{n,k}}t\mathrm{e}^{(n-2k)t/k}+\left(\frac{q\beta}{c}-\frac{kc^q}{(n-2k)\bar{c}c_{n,k}}\right)\mathrm{e}^{(n-2k)t/k}+o(\mathrm{e}^{(n-2k)t/k})\right], \\ v(t)=\dfrac{n-2k}{k}\left[1+\dfrac{c^q}{k\bar{c}c_{n,k}}t\mathrm{e}^{(n-2k)t/k}+\left(\dfrac{c^q}{(n-2k)\bar{c}c_{n,k}}-\dfrac{\beta}{c}\right)\mathrm{e}^{(n-2k)t/k}+o(\mathrm{e}^{(n-2k)t/k})\right],\quad t\to-\infty. \end{cases}$$

Proof (i) Let $c,\bar{c}>0$ be determined in Lemma 2. By some calculations, we have

$$s^{n-1}K(s)\varphi^q=\frac{s^{p-1}}{(1+s^2)^{\lambda/2}}\cdot\frac{c^q}{s^{[(n-2k)q]/k}}\left(\frac{s^{(n-2k)/k}}{c}\varphi(s)\right)^q=c^q s^{(n-2k)/k-1}[1+O(s^2)][1+o(1)]^q=c^q s^{(n-2k)/k-1}[1+o(1)].$$

It follows from (15) that

$$\varphi'=-\frac{1}{r^{(n-k)/k}}\left[\bar{c}+\int_0^r c_{n,k}^{-1}s^{n-1}K(s)\varphi^q\mathrm{d}s\right]^{1/k}=-\frac{(n-2k)c}{kr^{(n-k)/k}}\left[1+\frac{1}{\bar{c}c_{n,k}}\int_0^r s^{n-1}K(s)\varphi^q\mathrm{d}s\right]^{1/k}$$
$$=-\frac{(n-2k)c}{kr^{(n-k)/k}}\left[1+\frac{1}{k\bar{c}c_{n,k}}\int_0^r s^{n-1}K(s)\varphi^q\mathrm{d}s+o\left(\int_0^r s^{n-1}K(s)\varphi^q\mathrm{d}s\right)\right]=-\frac{(n-2k)c}{kr^{(n-k)/k}}-\frac{c^{q+1}}{k\bar{c}c_{n,k}r}+o\left(\frac{1}{r}\right).$$

Hence,

$$\varphi=\frac{c}{r^{(n-2k)/k}}-\frac{c^{q+1}}{k\bar{c}c_{n,k}}\ln r+o(\ln r). \tag{21}$$

It is clear that each term on the expansion of φ is still singular. To obtain the regular term of φ, we need to repeat the above arguments. Using (21), we find

$$s^{n-1}K(s)\varphi^q=\frac{c^q s^{(n-2k)/k-1}}{(1+s^2)^{\lambda/2}}\left[1-\frac{c^q}{k\bar{c}c_{n,k}}s^{(n-2k)/k}\ln s+o(s^{(n-2k)/k}\ln s)\right]^q$$
$$=c^q s^{(n-2k)/k-1}\left[1-\frac{\lambda}{2}s^2+\cdots+(-1)^{k_0}\frac{\lambda(\lambda+2)\cdots(\lambda+2k_0-2)}{(2k_0)!!}s^{2k_0}+O(s^{2k_0+2})\right]\cdot\left[1-\frac{qc^q}{k\bar{c}c_{n,k}}s^{(n-2k)/k}\ln s+o(s^{(n-2k)/k}\ln s)\right]$$
$$=c^q s^{(n-2k)/k-1}\left[1+\sum_{i=1}^{k_0}(-1)^i\frac{\lambda(\lambda+2)\cdots(\lambda+2i-2)}{(2i)!!}s^{2i}-\frac{qc^q}{k\bar{c}c_{n,k}}s^{(n-2k)/k}\ln s+o(s^{(n-2k)/k}\ln s)\right].$$

Therefore,

$$\varphi'=-\frac{(n-2k)c}{kr^{(n-k)/k}}\left[1+\frac{1}{\bar{c}c_{n,k}}\int_0^r s^{n-1}K(s)\varphi^q\mathrm{d}s\right]^{1/k}$$
$$=-\frac{(n-2k)c}{kr^{(n-k)/k}}\left[1+\frac{kc^q r^{(n-2k)/k}}{(n-2k)\bar{c}c_{n,k}}+\sum_{i=1}^{k_0}(-1)^i\frac{\lambda(\lambda+2)\cdots(\lambda+2i-2)c^q r^{(n-2k)/k+2i}}{(2i+(n-2k)/k)(2i)!!\bar{c}c_{n,k}}\right]$$
$$=-\frac{(n-2k)c}{kr^{(n-k)/k}}\left[1+\frac{c^q r^{(n-2k)/k}}{(n-2k)\bar{c}c_{n,k}}+\sum_{i=1}^{k_0}(-1)^i\frac{\lambda(\lambda+2)\cdots(\lambda+2i-2)c^q r^{(n-2k)/k+2i}}{(2i+(n-2k)/k)(2i)!!k\bar{c}c_{n,k}}\right.$$
$$\left.-\frac{qc^{2q}r^{2(n-2k)/k}\ln r}{2k(n-2k)(\bar{c}c_{n,k})^2}+\frac{[q-2(k-1)]c^{2q}r^{2(n-2k)/k}}{4(n-2k)^2(\bar{c}c_{n,k})^2}+o(r^{2(n-2k)/k})\right].$$

Let $S'=-\dfrac{(n-2k)c}{kr^{(n-k)/k}}-\dfrac{c^{q+1}}{k\bar{c}c_{n,k}r}$, $\Theta'=\varphi'-S'$. Note that Θ' is integrable and $\Theta(0):=\beta$ exists. Therefore,

$$\varphi=\frac{c}{r^{(n-2k)/k}}-\frac{c^{q+1}}{k\bar{c}c_{n,k}}\ln r+\beta-\sum_{i=1}^{k_0}(-1)^i\frac{\lambda(\lambda+2)\cdots(\lambda+2i-2)(n-2k)c^{q+1}}{2ik(2ik+n-2k)(2i)!!\bar{c}c_{n,k}}r^{2i}$$
$$+\frac{qc^{2q+1}}{2k(n-2k)(\bar{c}c_{n,k})^2}r^{(n-2k)/k}\ln r-\frac{[3q-2(k-1)]c^{2q}}{4(n-2k)^2(\bar{c}c_{n,k})^2}r^{(n-2k)/k}+o(r^{(n-2k)/k}).$$

(ii) For any given c and β, we denote $S=\frac{c}{r^{(n-2k)/k}}-\frac{c^{q+1}}{k\bar{c}c_{n,k}}\ln r$. Then

$$P=1-\frac{c^q}{k\bar{c}c_{n,k}}r^{(n-2k)/k}\ln r,\ P'=-\frac{(n-2k)c^q}{k^2\bar{c}c_{n,k}}r^{(n-2k)/k-1}\ln r-\frac{c^q}{k\bar{c}c_{n,k}}r^{(n-2k)/k-1},$$

and

$$f(r,\Theta)=-\frac{(n-2k)c}{kr^{(n-k)/k}}\left[1+\frac{c^q}{\bar{c}c_{n,k}}\int_0^r\frac{s^{(n-2k)/k-1}}{(1+s^2)^{\lambda/2}}\left(1-\frac{c^q}{k\bar{c}c_{n,k}}s^{(n-2k)/k}\ln s\right)^q\mathrm{d}s+O(r^{2(n-2k)/k})\right]^{1/k}+\frac{(n-2k)c}{kr^{(n-k)/k}}\left(1-\frac{c^q}{k\bar{c}c_{n,k}}r^{(n-2k)/k}\ln r\right)$$
$$+\frac{c}{r^{(n-2k)/k}}\left(\frac{(n-2k)c^q}{k^2\bar{c}c_{n,k}}r^{(n-2k)/k-1}\ln r+\frac{c^q}{k\bar{c}c_{n,k}}r^{(n-2k)/k-1}\right)$$
$$=-\frac{(n-2k)c}{kr^{(n-k)/k}}\left[1+\frac{kc^qr^{(n-2k)/k}}{(n-2k)\bar{c}c_{n,k}}+\sum_{i=1}^{k_0}(-1)^i\frac{\lambda(\lambda+2)\cdots(\lambda+2i-2)c^qr^{(n-2k)/k+2i}}{(2i+(n-2k)/k)(2i)!!\bar{c}c_{n,k}}-\frac{qc^{2q}r^{2(n-2k)/k}\ln r}{2(n-2k)(\bar{c}c_{n,k})^2}+o(r^{2(n-2k)/k}\ln r)\right]^{1/k}$$
$$+\frac{(n-2k)c}{kr^{(n-k)/k}}+\frac{c^{q+1}}{k\bar{c}c_{n,k}r}$$
$$=-\frac{(n-2k)c}{kr^{(n-k)/k}}\left[1+\frac{c^qr^{(n-2k)/k}}{(n-2k)\bar{c}c_{n,k}}+\sum_{i=1}^{k_0}(-1)^i\frac{\lambda(\lambda+2)\cdots(\lambda+2i-2)c^qr^{(n-2k)/k+2i}}{(2ik+n-2k)(2i)!!\bar{c}c_{n,k}}-\frac{qc^{2q}r^{2(n-2k)/k}\ln r}{2k(n-2k)(\bar{c}c_{n,k})^2}+o(r^{2(n-2k)/k}\ln r)\right]$$
$$+\frac{(n-2k)c}{kr^{(n-k)/k}}+\frac{c^{q+1}}{k\bar{c}c_{n,k}r}$$
$$=\frac{\lambda(n-2k)c^{q+1}r}{2nk\bar{c}c_{n,k}}+o(r),$$

which satisfies the assumptions of Lemma 1, thus (16) possesses a unique solution Θ. Hence $\varphi=S+\Theta$ is an M-solution. We compute

$$u(t)=r\frac{K(r)\varphi^q}{c_{n,k}(-\varphi')^k}=\frac{r^{\lambda+k-2}}{c_{n,k}(1+r^2)^{\lambda/2}}\cdot\frac{\varphi^q}{(-\varphi')^k}=\frac{c^qr^{(n-2k)/k}}{\bar{c}c_{n,k}(1+r^2)^{\lambda/2}}\cdot\frac{\left[1-\frac{c^q}{k\bar{c}c_{n,k}}r^{(n-2k)/k}\ln r+\frac{r^{(n-2k)/k}}{c}\Theta\right]^q}{\left[1+\frac{c^q}{(n-2k)\bar{c}c_{n,k}}r^{(n-2k)/k}-\frac{kr^{(n-k)/k}}{(n-2k)c}\Theta'\right]^k}$$
$$=\frac{c^qr^{(n-2k)/k}}{\bar{c}c_{n,k}}\left[1-\frac{\lambda}{2}r^2+\cdots+(-1)^{k_0}\frac{\lambda(\lambda+2)\cdots(\lambda+2k_0-2)}{(2k_0)!!}r^{2k_0}+O(r^{2k_0+2})\right]$$
$$\cdot\left[1-\frac{qc^q}{k\bar{c}c_{n,k}}r^{(n-2k)/k}\ln r+\frac{q\beta}{c}r^{(n-2k)/k}+o(r^{(n-2k)/k})\right]\cdot\left[1-\frac{kc^q}{(n-2k)\bar{c}c_{n,k}}r^{(n-2k)/k}+o(r^{(n-2k)/k})\right]$$
$$=\frac{c^q}{\bar{c}c_{n,k}}r^{(n-2k)/k}\left[1-\frac{\lambda}{2}r^2+\cdots+(-1)^{k_0}\frac{\lambda(\lambda+2)\cdots(\lambda+2k_0-2)}{(2k_0)!!}r^{2k_0}\right.$$
$$\left.-\frac{qc^q}{k\bar{c}c_{n,k}}r^{(n-2k)/k}\ln r+\frac{q\beta}{c}r^{(n-2k)/k}-\frac{kc^q}{(n-2k)\bar{c}c_{n,k}}r^{(n-2k)/k}+o(r^{(n-2k)/k})\right]$$

and

$$v(t)=r\frac{-\varphi'}{\varphi}=r\frac{\frac{(n-2k)c}{kr^{(n-k)/k}}+\frac{c^{q+1}}{k\bar{c}c_{n,k}r}-\Theta'}{\frac{c}{r^{(n-2k)/k}}-\frac{c^{q+1}}{k\bar{c}c_{n,k}}\ln r+\Theta}$$
$$=\left[\frac{n-2k}{k}+\frac{c^q}{k\bar{c}c_{n,k}}r^{(n-2k)/k}+o(r^{(n-2k)/k})\right]\cdot\left[1-\frac{c^q}{k\bar{c}c_{n,k}}r^{(n-2k)/k}\ln r+\frac{\beta}{c}r^{(n-2k)/k}+o(r^{(n-2k)/k})\right]^{-1}$$

$$=\frac{n-2k}{k}\left[1+\frac{c^q}{k\bar{c}c_{n,k}}r^{(n-2k)/k}\ln r+\left(\frac{c^q}{(n-2k)\bar{c}c_{n,k}}-\frac{\beta}{c}\right)r^{(n-2k)/k}+o(r^{(n-2k)/k})\right].$$

Theorem 3 Let $p>[(n-2k)(q+1)]/k$. Define $\omega:=p-[(n-2k)q]/k$ and choose $k_0\in\mathbb{N}$ such that $2k_0<(n-2k)/k\leqslant 2(k_0+1)$. Then the following statements are true.

(i) Every M-solution φ has the form $\varphi=S+\Theta$, where $S=\frac{c}{r^{(n-2k)/k}}$, and Θ solves the initial value problem (16). Moreover,

$$\Theta=\beta-\frac{(n-2k)c^{q+1}}{k^2\bar{c}c_{n,k}}\left[\frac{1}{\omega[\omega-(n-2k)/k]}r^{\omega-(n-2k)/k}+\sum_{i=1}^{k_0}(-1)^i\frac{\lambda(\lambda+2)\cdots(\lambda+2i-2)}{(\omega+2i)[\omega-(n-2k)/k+2i](2i)!!}r^{\omega-(n-2k)/k+2i}\right.$$
$$\left.+\frac{q\beta}{c\omega[\omega-(n-2k)/k]}r^{\omega}+o(r^{\omega})\right],\quad r\to 0$$

for some uniquely determined constants $c>0,\beta\in\mathbb{R}$.

(ii) Conversely, given any $c>0$ and $\beta\in\mathbb{R}$, there exists a unique solution Θ of (16) such that $\varphi=S+\Theta$ is an M-solution with $S=c/r^{(n-2k)/k}$. In this case, Θ satisfies (17) with $f(r,\Theta)=O(r^{\omega-(n-k)/k})$. In addition, the solution (u,v) has the following expansion:

$$\begin{cases}u(t)=\dfrac{c^q}{\bar{c}c_{n,k}}\mathrm{e}^{\omega t}\left[1+\displaystyle\sum_{i=1}^{k_0}(-1)^i\frac{\lambda(\lambda+2)\cdots(\lambda+2i-2)}{(2i)!!}\mathrm{e}^{2it}+O(\mathrm{e}^{(n-2k)t/k})\right],\\ v(t)=\dfrac{n-2k}{k}\left[1-\dfrac{\beta}{c}\mathrm{e}^{(n-2k)t/k}+\dfrac{c^q}{(\omega k-n+2k)\bar{c}c_{n,k}}\mathrm{e}^{\omega t}+o(\mathrm{e}^{\omega t})\right],\quad t\to-\infty.\end{cases}$$

Proof (i) Let $c,\bar{c}>0$ be determined by Lemma 2. It follows from (15) that φ' is integrable near the origin. We compute

$$\varphi'=-\frac{(n-2k)c}{kr^{(n-k)/k}}\left[1+\frac{1}{\bar{c}}\int_0^r c_{n,k}^{-1}s^{n-1}K(s)\varphi^q\mathrm{d}s\right]^{1/k}=-\frac{(n-2k)c}{kr^{(n-k)/k}}-\frac{(n-2k)c}{k^2\bar{c}c_{n,k}r^{(n-k)/k}}\int_0^r s^{n-1}K(s)\varphi^q\mathrm{d}s+o\left(\frac{1}{r^{(n-k)/k}}\int_0^r s^{n-1}K(s)\varphi^q\mathrm{d}s\right).$$

Thus, we obtain for certain $r_0\in(0,R)$,

$$\phi=\frac{c}{r^{(n-2k)/k}}-\frac{c}{r_0^{(n-2k)/k}}+\phi(r_0)-\int_{r_0}^r\frac{(n-2k)c}{k^2\bar{c}c_{n,k}t^{(n-k)/k}}\int_0^t s^{n-1}K(s)\phi^q\mathrm{d}s\mathrm{d}t+o(r^{\omega-(n-2k)/k})=:\frac{c}{r^{(n-2k)/k}}+\Theta=:S+\Theta,\quad 0<r<R.$$

Clearly, Θ satisfies (16) and (17). Suppose that $c_1/r^{(n-2k)/k}+\Theta_1=c_2/r^{(n-2k)/k}+\Theta_2$. It is obvious that $c_1=c_2$ and $\Theta_1=\Theta_2$, which implies that c,β and Θ are unique. It follows from $\Theta=\beta+o(1)$ that

$$\Theta'=-\frac{(n-2k)c}{k^2\bar{c}c_{n,k}r^{(n-k)/k}}\int_0^r s^{n-1}K(s)\varphi^q\mathrm{d}s+o\left(\frac{1}{r^{(n-k)/k}}\int_0^r s^{n-1}K(s)\varphi^q\mathrm{d}s\right)$$
$$=-\frac{(n-2k)c^{q+1}}{k^2\bar{c}c_{n,k}r^{(n-k)/k}}\int_0^r s^{\omega-1}(1+s^2)^{-\lambda/2}\left[1+\frac{s^{(n-2k)/k}}{c}\Theta(s)\right]^q\mathrm{d}s+o(r^{\omega-1})$$
$$=-\frac{(n-2k)c^{q+1}}{k^2\bar{c}c_{n,k}r^{(n-k)/k}}\int_0^r s^{\omega-1}\left[1+\sum_{i=1}^{k_0}(-1)^i\frac{\lambda(\lambda+2)\cdots(\lambda+2i-2)}{(2i)!!}s^{2i}+O(s^{2k_0+2})\right]\cdot\left[1+\frac{q\beta}{c}s^{(n-2k)/k}+o(s^{(n-2k)/k})\right]\mathrm{d}s+o(r^{\omega-1})$$
$$=-\frac{(n-2k)c^{q+1}}{k^2\bar{c}c_{n,k}r^{(n-k)/k}}\int_0^r s^{\omega-1}\left[1+\sum_{i=1}^{k_0}(-1)^i\frac{\lambda(\lambda+2)\cdots(\lambda+2i-2)}{(2i)!!}s^{2i}+\frac{q\beta}{c}s^{(n-2k)/k}+o(s^{(n-2k)/k})\right]\mathrm{d}s+o(r^{\omega-1})$$
$$=-\frac{(n-2k)c^{q+1}r^{\omega-(n-k)/k}}{\omega k^2\bar{c}c_{n,k}}\left[1+\sum_{i=1}^{k_0}(-1)^i\frac{\omega\lambda(\lambda+2)\cdots(\lambda+2i-2)}{(\omega+2i)(2i)!!}r^{2i}+\frac{\omega q\beta}{c[\omega-(n-2k)/k]}r^{(n-2k)/k}+o(r^{(n-2k)/k})\right].$$

The expansion for Θ could be obtained directly by integrating.

(ii) For any given c and β, we denote $S=\frac{c}{r^{(n-2k)/k}}$. Then $P=1$ and

$$f(r,\Theta)=-\frac{(n-2k)c}{kr^{(n-k)/k}}\left[1+\frac{c^q}{\bar{c}c_{n,k}}\int_0^r\frac{s^{\omega-1}}{(1+s^2)^{\lambda/2}}\mathrm{d}s+O(r^{\omega+(n-2k)/k})\right]^{1/k}+\frac{(n-2k)c}{kr^{(n-k)/k}}$$

$$=-\frac{(n-2k)c}{kr^{(n-k)/k}}\left[1+\frac{c^q}{\omega\bar{c}c_{n,k}}r^\omega+\sum_{i=1}^{k_0}(-1)^i\frac{\lambda(\lambda+2)\cdots(\lambda+2i-2)c^q}{(\omega+2i)(2i)!!\bar{c}c_{n,k}}r^{\omega+2i}+O(r^{\omega+(n-2k)/k})\right]^{1/k}+\frac{(n-2k)c}{kr^{(n-k)/k}}$$

$$=-\frac{(n-2k)c}{kr^{(n-k)/k}}\left[1+\frac{c^q}{\omega k\bar{c}c_{n,k}}r^\omega+o(r^\omega)\right]+\frac{(n-2k)c}{kr^{(n-k)/k}}=-\frac{(n-2k)c^{q+1}}{\omega k^2\bar{c}c_{n,k}}r^{\omega-(n-k)/k}+o(r^{\omega-(n-k)/k}).$$

Clearly, $f(r,\Theta)$ satisfies the assumptions in Lemma 1. Therefore, the problem (16) has a unique solution Θ. Thus $\varphi=S+\Theta$ is an M-solution. We compute

$$u(t)=r\frac{K(r)\varphi^q}{c_{n,k}(-\varphi')^k}=\frac{r^{\lambda+k-2}}{c_{n,k}(1+r^2)^{\lambda/2}}\cdot\frac{\varphi^q}{(-\varphi')^k}=\frac{c^q r^\omega}{\bar{c}c_{n,k}(1+r^2)^{\lambda/2}}\cdot\frac{\left[1+\frac{r^{(n-2k)/k}}{c}\Theta\right]^q}{\left[1-\frac{k}{(n-2k)c}r^{(n-k)/k}\Theta'\right]^k}$$

$$=\frac{c^q r^\omega}{\bar{c}c_{n,k}}\left[1+\sum_{i=1}^{k_0}(-1)^i\frac{\lambda(\lambda+2)\cdots(\lambda+2i-2)}{(2i)!!}r^{2i}+O(r^{2k_0+2})\right]$$

$$\cdot\left[1+\frac{q\beta}{c}r^{(n-2k)/k}-\frac{(n-2k)qc^q}{\omega(\omega k-n+2k)k\bar{c}c_{n,k}}r^\omega+o(r^\omega)\right]\cdot\left[1-\frac{c^q}{\omega\bar{c}c_{n,k}}r^\omega+o(r^\omega)\right]$$

$$=\frac{c^q}{\bar{c}c_{n,k}}r^\omega\left[1+\sum_{i=1}^{k_0}(-1)^i\frac{\lambda(\lambda+2)\cdots(\lambda+2i-2)}{(2i)!!}r^{2i}+\frac{q\beta}{c}r^{(n-2k)/k}+o(r^{(n-2k)/k})\right]$$

and

$$v(t)=r\frac{-\varphi'}{\varphi}=\frac{\frac{n-2k}{k}-\frac{r^{(n-k)/k}}{c}\Theta'}{1+\frac{r^{(n-2k)/k}}{c}\Theta}$$

$$=\left[\frac{n-2k}{k}+\frac{(n-2k)c^q}{\omega k^2\bar{c}c_{n,k}}r^\omega+o(r^\omega)\right]\cdot\left[1-\frac{\beta}{c}r^{(n-2k)/k}+\frac{(n-2k)c^q}{\omega(\omega k-n+2k)k\bar{c}c_{n,k}}r^\omega+o(r^\omega)\right]$$

$$=\frac{n-2k}{k}\left[1-\frac{\beta}{c}r^{(n-2k)/k}+\frac{c^q}{(\omega k-n+2k)\bar{c}c_{n,k}}r^\omega+o(r^{(n-2k)/k})\right].$$

Remark 4 We discuss the expansions of singular solutions in different intervals with p as the index in this section. For each specific subinterval of p, the singular solution is unique, and its corresponding singular and regular terms are also unique. However, for the entire interval, the singular solution is not unique, and its singular and regular terms are not unique either.

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