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The Eigenvalue Properties of a Kind of Singular Differential Equations in 3-Dimensional Space

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Abstract: In this paper, we consider the eigenvalue problem of the singular differential equation $-\Delta u_i-\frac{h}{|x|^2}u_i+V(x)u_i=\lambda_i(V,h)u_i$ in a bounded open ball with Dirichlet boundary condition in 3-dimensional space, where, $V\in\mathcal{V}=\{a\in L^\infty(\Omega)|0\leqslant a\leqslant M$ a.e., M is a given constant$\}$. And we have made a detailed characterization of the weak solution space. Furthermore, the existence of the minimum eigenvalue and the fundamental gap are provided.

Key words: singular differential equation; eigenvalue; fundamental gap
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0 Introduction

Consider the following equation:

$$\begin{cases}-\Delta u_i-\dfrac{h}{|x|^2}u_i+V(x)u_i=\lambda_i(V,h)u_i & \text{in } \Omega,\\ u_i=0 & \text{on } \partial\Omega,\end{cases}$$

where $\Omega\subseteq\mathbb{R}^N(N=3)$ is an open and bounded ball with $|\Omega|=1$, $\mathbf{0}=(0,0,0)\in\partial\Omega\in C^1$, and $h\in(0,h_*=\frac{(N-2)^2}{4})$ is a constant, V is the potential function, and $V\in\mathcal{V}=\{a\in L^\infty(\Omega)|0\leqslant a\leqslant M$ a.e., M is a given constant$\}$.

In the Sobolev space $H_0^1(\Omega)$, its norm is:

$$\|u\|_{H_0^1(\Omega)}=\left(\sum_{|\alpha|\leqslant 1}\int_\Omega|D^\alpha u|^2\,\mathrm{d}x\right)^{\frac{1}{2}}$$

or the equivalent norm: $\|u\|_{H_0^1(\Omega)}=\|Du\|_{L^2(\Omega)}$ (by using the Poincaré inequality).

Denote

$$\mathcal{H}=\left\{u\in H_0^1(\Omega)\,\middle|\,\frac{1}{|x|}u\in L^2(\Omega),\lim_{|x|\to 0}|x|^{-\frac{1}{2}}u(x)=0\right\}.$$

In quantum mechanics, actually, the eigenvalue problem of operators in the theory of Hilbert space is mentioned by many researchers. In 2013, Tyagi[1] considered a singular eigenvalue problem involving Hardy's potential with Dirichlet boundary condition and gave that the problem possesses a continuous family of eigenvalues. Gesztesy and Zinchenko[2] showed the case of self-adjoint half-line Schrödinger operators on (a,∞) with a potential strongly singular at the endpoint a. Nursul-

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tanov[3] found that asymptotic formulas for the eigenvalues of the Sturm-Liouville operator on the finite interval, with potentials having a strong negative singular at one endpoint. Haese-Hill[4] studied the spectral properties of its complex regularizations of the form:

$$L=-\frac{\mathrm{d}^2}{\mathrm{d}x^2}+m(m+1)w^2 f(wx+z_0), z_0\in\mathbb{C},$$

where w is one of the half-periods of $f(z)$ and $f(z)$ is the classical Weierstrass elliptic function. Li and Zhang[5] studied the numerical approximation of eigenvalue problems of the Schrödinger operator $-\Delta u+\frac{x^2}{|x|^2}u$. Homa and Hryniv[6] proved analogues of the classical Sturm comparison and oscillation theorems for equations $-(pu')'+qu=\lambda ru$ on a finite interval with real-valued distributional potentials.

Due to the presence of singular terms such as $-\frac{h}{|x|^2}$ in the differential equation, the properties of the equation differ from those of traditional differential equations, constituting singular differential equations. The existence of this particular term makes it challenging to directly apply the classical Sobolev space framework and its related properties to the analysis of such equations. To effectively handle this singularity, this paper introduces weighted Sobolev spaces. Within this newly constructed framework, the existence, uniqueness, and regularity of solutions to singular differential equations are successfully established. The establishment of these properties not only deepens our understanding of solutions to singular differential equations but also lays a solid foundation for further research and applications.

Additionally, a comprehensive description of spectral theory for singular differential equations is provided within this new framework. The characteristics of the equation's eigenvalues and eigenfunctions can be more accurately characterized by introducing weighted Sobolev spaces. This theory holds profound mathematical significance and offers new perspectives and methods for solving practical problems.

1 Preliminary

Lemma 1 The space $(\mathcal{H}, \|\cdot\|_{H^1})$ is a Hilbert space.

Proof Let $\{u_n\}_{n\in\mathbb{N}}\subset\mathcal{H}$ is a Cauchy sequence. Note that $\mathcal{H}\subset H_0^1(\Omega)$, then there exists $u^*\in H_0^1(\Omega)$ such that $u_n\to u^*$ strongly in H_0^1.

By using the Sobolev Embedding Theorem, we have $u_n\to u^*$ strongly in $L^q(\Omega)$ for $N=3$.

By using the Hardy-Sobolev inequality[7]:

$$\int_\Omega |Du|^p\,\mathrm{d}x \geqslant \left(\frac{n-p}{p}\right)^p\int_\Omega \frac{|u(x)|^p}{|x|^p}\,\mathrm{d}x \text{ holds for } u\in W_0^{1,p}(\Omega),$$

where $W_0^{1,p}(\Omega)$ is the completion of $C_0^\infty(\Omega)$ in the norm:

$$\|u\|_{1,p,\Omega}:=\left(\int_\Omega |u(x)|^p\,\mathrm{d}x+\int_\Omega |Du|^p\,\mathrm{d}x\right)^{\frac{1}{p}}.$$

We have:

$$\int_\Omega |Du|^2\,\mathrm{d}x\geqslant\left(\frac{3-2}{2}\right)^2\int_\Omega\frac{u(x)^2}{|x|^2}\,\mathrm{d}x=\frac{1}{4}\int_\Omega\frac{u(x)^2}{|x|^2}\,\mathrm{d}x,$$

$$\int_\Omega\frac{u(x)^2}{|x|^2}\,\mathrm{d}x\leqslant 4\int_\Omega |Du|^2\,\mathrm{d}x,$$

which means that

$$\left\|\frac{u}{|x|}\right\|_{L^2}\leqslant\|Du\|_{L^2}. \tag{1}$$

So, we have $\{u_n\}_{n\in\mathbb{N}}\subset\mathcal{H}$, and

$$\left\|\frac{1}{|x|}u_n-\frac{1}{|x|}u_m\right\|_{L^2}\leqslant C\|u_n-u_m\|_{H_0^1}\to 0,\quad n,m\to\infty.$$

Then, there exists $v^*\in L^2(\Omega)$ such that

$$\frac{1}{|x|}u_n\to v^* \text{ strongly in } L^2(\Omega)$$

and

$$\int_\Omega |u_n-v^*|x||^2\,\mathrm{d}x=\int_\Omega |x|^2\left|\frac{1}{|x|}u_n-v^*\right|^2\mathrm{d}x$$

$$\leqslant\int_\Omega\left|\frac{1}{|x|}u_n-v^*\right|^2\mathrm{d}x\to 0,\ \text{as } n\to\infty,$$

which shows that $u_n\to v^*|x|$ strongly in $L^2(\Omega)$. From the assumptions $v^*|x|=u^*$ in $L^2(\Omega)$, then $\frac{1}{|x|}u_n=v^*\in L^2(\Omega)$.

Denote

$$L:=-\Delta-\frac{1}{|x|}+V(x). \tag{2}$$

In the sense of distribution, for $g\in L^2(\Omega)$, $u\in\mathcal{H}$ is a solution of the following equation

$$\begin{cases} Lu=g, & \text{in } \Omega;\\ u=0, & \text{on } \partial\Omega,\end{cases} \tag{3}$$

which is equivalent to the following equality:

$$\int_\Omega g\varphi(x)\,\mathrm{d}x=\int_\Omega Lu(x)\varphi(x)\,\mathrm{d}x,\quad\forall\varphi(x)\in C_c^\infty(\Omega).$$

Definition 1 (i) The bilinear form $B(\cdot,\cdot)$ associated with the elliptic operator L defined by (2) is

$$B(u,v)=\int_\Omega\left(Du\cdot Dv-\frac{h}{|x|^2}u\cdot v+Vu\cdot v\right)\mathrm{d}x,\quad\forall u,v\in\mathcal{H}.$$

(ii) We say that $u\in\mathcal{H}$ is a weak solution of the boundary-value problem (3) if

$$B(u,v)=(g,v)_{L^2(\Omega)}$$

For all $v\in\mathcal{H}$, where $(\cdot,\cdot)$ denotes the inner product in $L^2(\Omega)$.

Theorem 1 Let $B:\mathcal{H}\times\mathcal{H}\to\mathbb{R}$ be defined as above. Then, there exist constants $C>0$ such that

$$|B(u,v)|\leqslant C\|u\|_{H^1}\|v\|_{H^1}$$

and

$$C\|u\|_{H^1}^2\leqslant B(u,u) \tag{4}$$

for all $u,v\in H_0^1(\Omega)$.

Proof We readily check that:

$$\begin{aligned}|B(u,v)|&=\left|\int_\Omega\left(Du\cdot Dv-\frac{h}{|x|^2}u\cdot v+V(x)u\cdot v\right)\mathrm{d}x\right|\\&\leqslant\int_\Omega|Du\cdot Dv|\mathrm{d}x+h\int_\Omega\left|\frac{u}{|x|}\right|\left|\frac{v}{|x|}\right|\mathrm{d}x+\int_\Omega|V(x)u\cdot v|\mathrm{d}x\\&\leqslant\|Du\|_{L^2}\|Dv\|_{L^2}+h\left\|\frac{1}{|x|}u\right\|_{L^2}\left\|\frac{1}{|x|}v\right\|_{L^2}+M\|u\|_{L^2}\|v\|_{L^2}\end{aligned}$$

for all $u,v\in H_0^1(\Omega)$ by Cauchy-Schwarz inequality.

Then we have

$$\begin{aligned}\left|B(u,v)\right|&\leqslant\|Du\|_{L^2}\|Dv\|_{L^2}+C\|Du\|_{L^2}\|Dv\|_{L^2}\\&\quad+C\|Du\|_{L^2}\|Dv\|_{L^2}\\&\leqslant C\|Du\|_{L^2}\|Dv\|_{L^2}\end{aligned}$$

by Lemma 1 and Poincaré inequality.

What's more, since $V(x)\geqslant0$ a.e., we can obtain that

$$\begin{aligned}B(u,u)&=\int_\Omega\left(|Du|^2-\frac{h}{|x|^2}u^2+Vu^2\right)\mathrm{d}x\\&\geqslant\int_\Omega|Du|^2\mathrm{d}x-h\int_\Omega\frac{u^2}{|x|^2}\mathrm{d}x\\&\geqslant\int_\Omega|Du|^2\mathrm{d}x-4h\|Du\|_{L^2}^2\\&=(1-4h)\|Du\|_{L^2}^2\\&=C\|Du\|_{L^2}^2=C\|u\|_{H^1}^2.\end{aligned}$$

Corollary 1 Let $g\in L^2(\Omega)$ be a given function, then the elliptic equation $Lu=g$, $u\in\mathcal{H}$ has a unique weak solution.

Proof By Theorem 1 and Lax-Milgram theorem, we can conclude the corollary.

Corollary 2 The operator $L:\mathcal{H}\to\mathcal{H}'$ is a continuous, coercive, surjective, and linear operator, where H' is the dual space of $\mathcal{H}$.

Proof It is evident that L is a linear operator.

$$\begin{aligned}\|Lu\|_{\mathcal{H}'}&=\sup_{v\in\mathcal{H},\|v\|_{H_0^1}=1}<Lu,v>_{\mathcal{H}',\mathcal{H}}\geqslant<Lu,\frac{u}{\|u\|}>_{\mathcal{H}',\mathcal{H}}\\&\geqslant\frac{C<u,u>_{\mathcal{H}}}{\|u\|_{H^1}}=C\|u\|_{H^1}\end{aligned}$$

$$\begin{aligned}\|Lu\|_{\mathcal{H}'}&=\sup_{v\in\mathcal{H},\|v\|_{H_0^1}=1}<Lu,v>_{\mathcal{H}',\mathcal{H}}\\&=\sup_{v\in\mathcal{H},\|v\|_{H_0^1}=1}\int_\Omega\left(Du\cdot Dv-\frac{h}{|x|^2}u\cdot v+V(x)u\cdot v\right)\mathrm{d}x\\&=\sup_{v\in\mathcal{H},\|v\|_{H_0^1}=1}B(u,v)\leqslant C\|u\|_{H_0^1}.\end{aligned}$$

Let $g\in\mathcal{H}'$, by using the Lax-Milgram theorem[8], we can see that there exists a unique element $u\in\mathcal{H}$ such that

$$<Lu,v>_{\mathcal{H}',\mathcal{H}}=B(u,v)=<g,v>_{\mathcal{H}',\mathcal{H}},\quad\forall v\in\mathcal{H}.$$

Note that $C_c^\infty(\Omega)\subset\mathcal{H}$, thus

$$Lu=g$$

in the sense of distribution.

Consequently, $L:\mathcal{H}\to\mathcal{H}'$ is a continuous, coercive, surjective, and linear operator.

Lemma 2 Assume $g\in L^2(\Omega)$, let u be a weak solution to the equation

$$Lu=g,\ u\in\mathcal{H}. \tag{5}$$

Then $u\in H_{\mathrm{loc}}^2(\Omega)$. Moreover,

$$\|u\|_{H^2(\Omega'')}\leqslant C\left(\|g\|_{L^2}+\sigma^{-1}\|u\|_{H'}\right),\quad\forall\sigma\in(0,1), \tag{6}$$

where $\Omega''=\Omega\backslash B(0,\sigma)$.

Proof (1) For all $x\in\Omega$, let $s_0>0$ be small such that $\mathrm{dist}(x,\partial\Omega)>2s_0$. We take

$$\Omega'=\{x|x\in\Omega,\ \mathrm{dist}(x,\partial\Omega)>2s_0\}.$$

Select a smooth function ζ satisfying

$$\zeta\equiv1\text{ on }\Omega'\text{ and }\zeta\equiv0\text{ on }\Omega\backslash\Omega',\quad0\leqslant\zeta\leqslant1.$$

Since u is a weak solution of (5), we have $B(u,v)=(g,v)$ for all $v\in\mathcal{H}$. Then

$$\int_\Omega Du\cdot Dv\mathrm{d}x=\int_\Omega\left(gv+\frac{h}{|x|^2}uv-Vuv\right)\mathrm{d}x=\int_\Omega\tilde{g}v\mathrm{d}x,$$

where $\tilde{g}=g+\frac{h}{|x|^2}u-Vu$.

Set

$$D^hu(x)=\frac{u(x+s)-u(x)}{s},\quad|s|\leqslant s_0,s\neq0.$$

$$v=D^{-s}\zeta^2D^su.$$

Then, on the one hand, by the Cauchy equality with weight and difference quotient estimate, we have

$$\begin{aligned}\int_\Omega Du\cdot Dv\mathrm{d}x&=-\int_\Omega Du\cdot D(D^{-s}\zeta^2D^su)\mathrm{d}x\\&=\int_\Omega(D^s(Du))(D(\zeta^2D^su))\mathrm{d}x\\&=\int_\Omega(D^s(Du))(2\zeta D\zeta D^su)\mathrm{d}x+\int_\Omega(D^s(Du))\zeta^2(D^s(Du))\mathrm{d}x\end{aligned}$$

$$\geqslant \frac{1}{2}\int_{\Omega}\zeta^2|D^s(Du)|^2\mathrm{d}x - C\int_{\Omega}|D^s u|^2\mathrm{d}x$$
$$\geqslant \frac{1}{2}\int_{\Omega}\zeta^2|D^s(Du)|^2\mathrm{d}x - C\int_{\Omega}|Du|^2\mathrm{d}x.$$

On the other hand, by difference quotient estimate, we have

$$\int_{\Omega}|v|^2\mathrm{d}x = \int_{\Omega}\left|D^{-s}(\zeta^2 D^s u)\right|^2\mathrm{d}x$$
$$\leqslant C\int_{\Omega}\left|D(\zeta^2 D^s u)\right|^2\mathrm{d}x \leqslant C\int_{\Omega}(|Du|^2 + \zeta^2\left|D^s(Du)\right|^2)\mathrm{d}x.$$

Thus, the Cauchy inequality with weight implies

$$\left|\int_{\Omega}\tilde{g}v\mathrm{d}x\right| \leqslant \frac{1}{4}\int_{\Omega}\zeta^2\left|D^s(Du)\right|^2\mathrm{d}x + C\int_{\Omega}g^2 + u^2 + |Du|^2\mathrm{d}x,$$

where C is depend on $\frac{1}{s_0^2}$.

Consequently,

$$\int_{\Omega'}\left|D^s(Du)\right|^2\mathrm{d}x \leqslant C\int_{\Omega}\left(g^2 + u^2 + |Du|^2\right)\mathrm{d}x, \quad \forall|s| \leqslant |s_0|.$$

From the difference quotient, we deduce $Du \in H^1_{\mathrm{loc}}(\Omega)$ and thus $u \in H^2_{\mathrm{loc}}(\Omega)$.

(2) For any $\sigma \in (0, 1)$, according to (5)

$$|\Delta u| \leqslant \left|\frac{h}{|x^2|}u - Vu + gu\right| \leqslant h\left|\frac{u}{|x|^2}\right| + |Vu| + |gu|, \forall x \in \Omega'',$$

From which, Cauchy-Schwarz inequality and (1), we deduce that:

$$\int_{\Omega''}|\Delta u|^2\mathrm{d}x \leqslant C\left(\int_{\Omega''}\left|\frac{h}{|x|^2}u\right|^2\mathrm{d}x + \int_{\Omega''}|Vu|^2\mathrm{d}x + \int_{\Omega''}|g|^2\mathrm{d}x\right)$$
$$\leqslant C\left(\frac{1}{\sigma^2}\int_{\Omega}|Du|^2\mathrm{d}x + \int_{\Omega}|u|^2\mathrm{d}x + \|g\|_{L^2}\right)$$
$$\leqslant C\left(\frac{1}{\sigma^2}\|Du\|_{L^2} + \|u\|_{L^2} + \|g\|_{L^2}\right)$$
$$\leqslant C\left(\frac{1}{\sigma}\|u\|_{H^1} + \|g\|_{L^2}\right),$$

where we used Poincaré inequality in the equality, consequently, (6) holds.

Remark 1 Under the assumptions of the Lemma 2, (i) It seems that $u \notin H^2(\Omega)$ since L is singular at $x = 0$; (ii) From the Sobolev Embedding Theorem, it is clear that $u \in C^{1,\frac{1}{2}}(\Omega'')$.

Proposition 1 Let $\sigma(L)$ denotes the spectrum of L. Then $\sigma(L) \subset \mathbb{R}$ and $\sigma(L)$ is at most countable. Let $\sigma(L) = \{\lambda_k\}_{k=1}^{\infty}$, where the eigenvalue is counted according to its multiplicity, then

$$0 < \lambda_1 \leqslant \lambda_2 \leqslant \cdots, \text{ and } \lambda_k \to \infty \text{ as } k \to \infty.$$

Finally, there exists an orthonormal basis $\{w_k\}_{k=1}^{\infty}$ of $L^2(\Omega)$, where $w_k \in \mathcal{H}$ is an eigenfunction corresponding to λ_k, that is

$$Lw_k = \lambda_k w_k, \quad w_k \in \mathcal{H}.$$

Proof Since L is a coercive, surjective, linear operator, set $T = L^{-1}$. From the Sobolev Compact Embedding theorem, we deduce that $T: L^2(\Omega) \to L^2(\Omega)$ is a bounded, linear, compact operator. Hence $\sigma(T)$, the spectrum of T, is at most countable and has 0 as the unique limit point.

For any $g, f \in L^2(\Omega)$, denote $u = Tg$, $v = Tf$, then

$$Lu = g, u \in \mathcal{H};\ Lv = f, v \in \mathcal{H}.$$

Thus,

$$(Tg, f)_{L^2} = (u, Lv)_{L^2}$$
$$= \int_{\Omega}\left(Du \cdot Dv - \frac{h}{|x|^2}u \cdot v + V(x)u \cdot v\right)\mathrm{d}x$$
$$= (Lu, v)_{L^2} = (g, Tf)_{L^2},$$

which shows that T is a symmetric operator.

So, we say that T is a self-adjoint linear operator. And we thereby obtain the assertion of the Proposition by the Standard Functional Analysis 8[8].

Lemma 3 Let $u_i \in \mathcal{H}$ be the i-th eigenfunction with respect to the eigenfunction λ_i of L with the potential $V \in \mathcal{V}$. Then,

$$\lambda_1(V) = \min_{u \in \mathcal{H}, u \neq 0}\frac{B(u, u)}{\|u\|_{L^2}^2}, \tag{7}$$

$$\lambda_2(V) = \min_{u \in E_1^{\perp}, u \neq 0}\frac{B(u, u)}{\|u\|_{L^2}^2}, \tag{8}$$

where,

$$E_1 = \{u \in \mathcal{H} | (u, u_1)_{L^2} = 0\}.$$

Proof (i) Let $\sigma(L) = \{\lambda_m\}_{m=1}^{\infty}$ and $\{w_m\}_{m=1}^{\infty}$ be defined as in Proposition 1. Then

$$B(w_m, w_m) = \lambda_m(w_m, w_m) = \lambda_m$$

and

$$B(w_m, w_n) = \lambda_m(w_m, w_n) = 0, \quad \forall m, n \in \mathbb{N}, m \neq n.$$

We observe that $\{\lambda_m^{-\frac{1}{2}}w_m\}_{m=1}^{\infty}$ is an orthonormal subset of $\mathcal{H}$ under the new inner product $B(\cdot, \cdot)$. In fact, according to Theorem 1, Lemma 1, and the Poincaré inequality, we deduce that

$$C\|u\|_{H^1}^2 \leqslant B(u, u) = \int_{\Omega}(|Du|^2 - \frac{h}{|x|}^2 u^2 + Vu^2)\mathrm{d}x$$
$$\leqslant C\|Du\|_{L^2}^2 = C\|u\|_{H^1}^2,$$

i. e., these two norms $\|\cdot\|_{H^1}$ and $\sqrt{B(\cdot, \cdot)}$ are equivalent on $\mathcal{H}$.

For arbitrary $u \in \mathcal{H}$ with $B(w_m, u) = 0$, $\forall m \in \mathbb{N}$, it follows that $u = \sum_{m=1}^{\infty}d_m w_m$ since $\{w_m\}_{m=1}^{\infty}$ is an orthonormal basis of $L^2(\Omega)$. Therefore

$$0=B(w_m,u)=d_m\lambda_m,\quad \forall m\in\mathbb{N}.$$

We obtain $d_m=0$, $\forall m\in\mathbb{N}$. Hence $u=0$.

Finally, we have $\left\{\lambda_m^{-\frac{1}{2}}w_m\right\}_{m=1}^{\infty}$ is an orthonormal subset of $\mathcal{H}$ under the new inner product $B(\cdot,\cdot)$.

(ii) Let $u\in\mathcal{H}$ with $\|u\|_{L^2}=1$. Assume that $\sum_{m=1}^{\infty}d_m^2=1$. We conclude that

$$B(u,u)=\sum_{m=1}^{\infty}d_m^2B(w_m,w_m)=\sum_{m=1}^{\infty}d_m^2\lambda_m\geqslant\lambda_1$$

according to (i), this proves (7).

(iii) Suppose we have obtained u_1, for any $u\perp u_1$, with $\|u\|_{L^2(\Omega)}=1$, then $u=\sum_{i=2}^{\infty}(u,w_i)w_i=\sum_{i=2}^{\infty}d_iw_i$ and $\sum_{i=2}^{\infty}d_i^2=1$. Moreover,

$$B(u,u)=\sum_{i=2}^{\infty}B(w_i,w_i)=\sum_{i=2}^{\infty}d_i^2\lambda_i\geqslant\lambda_2,$$

which implies that $\min\limits_{u\in E_1^{\perp},u\neq0}\dfrac{B(u,u)}{\|u\|_{L^2}^2}\geqslant\lambda_2$.

By (i), we have $B(w_2,w_2)=\lambda_2$, then the result (8) is obtained.

2 Main Results

Theorem 2 There exists V^*, V_1^*, $V_2^*\in\mathcal{V}$, such that

$$\lambda_1(V_1^*)=\inf_{v\in\mathcal{V}}\lambda_1(V),\quad \lambda_2(V_2^*)=\inf_{v\in\mathcal{V}}\lambda_2(V)$$

and

$$\Gamma(V^*)=\inf_{V\in\mathcal{V}}\Gamma(V),$$

where $\Gamma(V)=\lambda_2(V)-\lambda_1(V)$ is called the fundamental gap with potential $V\in\mathcal{V}$.

Proof we will show the proof by the following steps.

Step 1: $\inf\limits_{V\in\mathcal{V}}\lambda_1(V)$, $\inf\limits_{V\in\mathcal{V}}\lambda_2(V)$ exists.

For arbitrary $V_0\in\mathcal{V}$, we have $\inf\limits_{V\in\mathcal{V}}\lambda_1(V)\leqslant\lambda_1(V_0)$. Let $\{V_n'\}_{n\in\mathbb{N}}\subset\mathcal{V}$ be such that

$$\lambda_1(V_n^1)\downarrow\inf_{V\in\mathcal{V}}\lambda_1(V)=\lambda_1^*,\quad \lambda_1(V_n^1)\leqslant\lambda_1(V_0),\quad \forall n\in\mathbb{N}.$$

Then there exists a subsequence of $\{V_n^1\}_{n\in\mathbb{N}}$, still denoted by itself, and $V_1^*\in\mathcal{V}$, such that $V_n^1\to V_1^*$ weakly star in $L^{\infty}(\Omega)$.

Denote u_n^1 be the normalized first eigenfunction of V_n^1, which means that $\|u_n^1\|_{L^2}=1$. From (4), we see

$$C\|u_n^1\|_{H^1}\leqslant B(u_n^1,u_n^1)=\lambda_1(V_n^1)\leqslant C,$$

where $\lambda_1(V_n^1)$ is the first eigenvalue of L with the potential V_n^1. Thus, there exists a subsequence of $\{u_n^1\}_{n\in\mathbb{N}}$, still denoted by itself, and $u_1^*\in\mathcal{H}$ such that

$$u_n^1\to u_1^*\text{ weakly in }\mathcal{H}.\tag{9}$$

By $\mathcal{H}\in H_0^1(\Omega)$ and Sobolev Embedding theorem, we deduce that

$$u_n^1\to u_1^*\text{ strongly in }L^2(\Omega),\tag{10}$$

which implies that;

$$\|u_1^*\|_{L^2}=1.\tag{11}$$

In view of Lemma 1, we have

$$\left\|\frac{u_n^1}{|x|}\right\|_{L^2}\leqslant C\|u_n{}'\|_{H^1}\leqslant C,$$

then there exists a subsequence of $\left\{\dfrac{u_n^1}{|x|}\right\}_{n\in\mathbb{N}}$, still denoted by itself, and $v^*\in L^2(\Omega)$ such that

$$\frac{1}{|x|}u_n^1\to v^*\text{ weakly in }L^2(\Omega).$$

Hence,

$$\int_{\Omega}(u_n{}'-v^*|x|)\varphi\mathrm{d}x$$
$$=\int_{\Omega}\left(\frac{1}{|x|}u_n{}'-v^*\right)|x|\varphi\mathrm{d}x\to0,\forall\varphi\in L^2(\Omega),$$

which shows that $u_n^1\to v^*|x|$ weakly in $L^2(\Omega)$.

Thus (10) implies that

$$u_1^*=v^*|x|\text{ in }L^2(\Omega),$$

or we can say $\dfrac{u_1^*}{|x|}=v^*$ in $L^2(\Omega)$, i.e.

$$\frac{1}{|x|}u_n^1\to\frac{1}{|x|}u_1^*\text{ weakly in }L^2(\Omega).\tag{12}$$

For all $v\in\mathcal{H}$,

$$B(u_n^1,v)=\int_{\Omega}\left((u_n^1)'v'-\frac{h}{|x|}u_n^1\left(\frac{v}{|x|}\right)+V_n^1u_n^1v\right)\mathrm{d}x.$$

Since $\dfrac{1}{|x|}v\in L^2(\Omega)$ and (12), we have

$$\begin{aligned}B(u_n^1,v)&=\int_{\Omega}\left((u_n^1)'v'-\frac{h}{|x|}u_n^1\left(\frac{v}{|x|}\right)+V_n^1u_n^1v\right)\mathrm{d}x\\&\to\int_{\Omega}\left((u_1^*)'v'-\frac{h}{|x|}u_1^*\left(\frac{v}{|x|}\right)+V_1^*u_1^*v\right)\mathrm{d}x\\&=B(u_1^*,v),\qquad\forall v\in\mathcal{H}.\end{aligned}$$

Combining (9) (10) (11), we can obtain that

$$\lambda_1(V_n^1)\int_{\Omega}u_n^1v\mathrm{d}x\to\lambda_1^*\int_{\Omega}u_1^*v\mathrm{d}x,\quad\forall v\in\mathcal{H}.$$

These imply that

$$B(u_1^*,v)=\lambda_1^*(u_1^*,v)_{L^2},\quad v\in\mathcal{H}.$$

Note that $C_c^{\infty}(\Omega)\subset\mathcal{H}$, so in the sense of distribution, we have

$$-\Delta(u_1^*)-\frac{h}{|x|^2}u_1^*+V_1^*u_1^*=\lambda_1^*u_1^*,\quad u_1^*\in\mathcal{H}.$$

This proves λ_1^* is an eigenvalue of L with potential V_1^*. Therefore, $\lambda_1^*=\lambda_1(V_1^*)$ by the definition of λ_1^*.

Step 2: Choose $V_0\in\mathcal{V}$ and $\{V_n^2\}_{n\in\mathbb{N}}\subset L^\infty(\Omega)$ such that

$$\lambda_2(V_n^2)\downarrow\inf_{V\in\mathcal{V}}\lambda_2(V)=\lambda_2^*,\text{ and }\lambda_2(V_n^2)\leqslant\lambda_2(V_0),\quad\forall n\in\mathbb{N}.$$

For each $n\in\mathbb{N}$, there exist u_n^1, $u_n^2\in\mathcal{H}$, such that u_n^1, u_n^2 are the first two normalized eigenfunctions with respect to the eigenvalues $\lambda_1(V_n^2)$ and $\lambda_2(V_n^2)$ with regard to V_n^2. By the same argument as in Step 1, by abstracting subsequence, there exist $\lambda_1^*\in R$, $V_2^*\in L^\infty(\Omega)$ and $u_1^*,u_2^*\in\mathcal{H}$ such that

$$\lambda_1(V_n^2)\to\lambda_1^*;\ V_n^2\to V_2^*\text{ weakly in }L^2(\Omega);$$
$$u_n^1\to u_1^*\text{ and }u_n^2\to u_2^*\text{ weakly in }\mathcal{H}.$$

Moreover,

$$u_n^1\to u_1^*\text{ strongly in }L^2(\Omega),\quad u_n^2\to u_2^*\text{ strongly in }L^2(\Omega),$$

and

$$\frac{1}{|x|}u_n^1\to\frac{1}{|x|}u_1^*\text{ and }\frac{1}{|x|}u_n^2\to\frac{1}{|x|}u_2^*\text{ weakly in }L^2(\Omega)$$
$$(\|u_1^*\|_{L^2}=\|u_2^*\|_{L^2}=1).$$

Consequently, on one side, owing to $(u_n^1,u_n^2)_{L^2}=0$, we obtain $(u_1^*,u_2^*)_{L^2}=0$. On the other side, by the same argument as Step 1, we deduce that:

$$-\Delta u_1^*-\frac{h}{|x|^2}u_1^*+V_2^*u_1^*=\lambda_1^*u_1^*,\ -\Delta u_2^*-\frac{h}{|x|^2}u_2^*+V_2^*u_2^*=\lambda_2^*u_2^*.$$

This proves that λ_1^* and λ_2^* are eigenvalues of L with potential V_2^*, in particular $\lambda_1^*\leqslant\lambda_2^*$.

Obviously, the definition of λ_2^* implies $\lambda_2(V_2^*)\geqslant\lambda_2^*$. Suppose $\lambda_2^*<\lambda_2(V_2^*)$, then $\lambda_1^*\leqslant\lambda_2^*=\lambda_1(V_2^*)$, i.e., $u_1^*=u_2^*$ is the only eigenfunction of $\lambda_1(V_2^*)$ with potential according to Lemma 1. But the case $(u_1^*,u_2^*)_{L^2}=0$ leads us to $\|u_1^*\|_{L^2}=0$, a contradiction. This shows that $\lambda_2^*=\lambda_2(V_2^*)$.

Step 3: Take arbitrary $V_0\in\mathcal{V}$, then

$$\inf_{V\in\mathcal{V}}(\lambda_2(V)-\lambda_1(V))\leqslant(\lambda_2-\lambda_1)(V_0).$$

Then there exists a subsequence of V_n, still denoted by itself, and $V^*\in L^\infty(\Omega)$ such that

$$V_n\to V^*\text{ weakly star in }L^\infty(\Omega).$$

Denote u_n^1, u_n^2 $(n\in\mathbb{N})$ be the first and second normalized eigenfunctions with respect to $\lambda_1(V_n)$, $\lambda_2(V_n)$, respectively. By the same argument as in Step 1, by abstracting subsequence, there exists u_1^*, $u_2^*\in\mathcal{H}$ such that

$$u_n^1\to u_1^*\text{ and }u_n^2\to u_2^*\text{ weakly in }\mathcal{H}.$$

By the same argument as in Step 2, we can deduce that

$$\lambda_1^*=\lambda_1(V^*),\ \lambda_2^*=\lambda_2(V^*),\text{ and }\Gamma(V^*)=\Gamma^*.$$

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