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Gauss Principle of Least Compulsion for Relative Motion Dynamics and Differential Equations of Motion

□ **ZHANG Yi, XIA Junling**

College of Civil Engineering, Suzhou University of Science and Technology, Suzhou 215011, Jiangsu, China

Abstract: This paper focuses on Gauss principle of least compulsion for relative motion dynamics and derives differential equations of motion from it. Firstly, starting from the dynamic equation of the relative motion of particles, we give the Gauss principle of relative motion dynamics. By constructing a compulsion function of relative motion, we prove that at any instant, its real motion minimizes the compulsion function under Gaussian variation, compared with the possible motions with the same configuration and velocity but different accelerations. Secondly, the formula of acceleration energy and the formula of compulsion function for relative motion are derived because the carried body is rigid and moving in a plane. Thirdly, the Gauss principle we obtained is expressed as Appell, Lagrange, and Nielsen forms in generalized coordinates. Utilizing Gauss principle, the dynamical equations of relative motion are established. Finally, two relative motion examples also verify the results' correctness.

Key words: relative motion dynamics; Gauss principle of least compulsion; acceleration energy; compulsion function

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0 Introduction

The Gauss principle is a differential variational principle proposed by Gauss in 1829, a general analytical mechanics principle[1]. Chen pointed out[2] that taking the Gauss principle as the fundamental principle in terms of mechanics concepts seems most appropriate. According to Mei[3], the Gauss principle can be used as a basis for analytical dynamics. Udwadia and Kalaba[4] took the Gauss principle as a starting point to derive the basic equations of analytical mechanics by using matrix algebra operations and recommended its application to holonomic and non-holonomic mechanics, which reveals the broad applicability of Gauss principle in describing the motion of constrained mechanical systems. Of all differential variational principles, only the Gauss principle is a stationary principle, which shows that the variation of compulsion function in the sense of Gauss is equal to zero[1]. For a system of particles, applying Gauss minimum compulsion principle, its motion equation can be obtained directly by calculating the extremum of the compulsion function[5,6]. Because of this, the Gauss principle of least compulsion is widely used in dynamics modeling and in

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Biography: ZHANG Yi, male, Ph.D., Professor, research direction: analytical mechanics. E-mail: zhy@mail.usts.edu.cn

finding approximate solutions. For example, robot dynamics[5], multi-body system dynamics[6-14], elastic rod dynamics[15-18], and hybrid dynamics[19], etc. So far, there have been many achievements in the Gauss and least compulsion principles for constrained mechanical systems and their applications[20-28].

Using the analytical mechanics method to study the relative motion dynamics of complex systems can unify the expression forms and show the superiority of analytical mechanics in solving the dynamics problems of complex systems. These complex systems comprise the carrier body and carried bodies moving relative to the former[29]. There are many such systems in practical engineering, such as the relative motion and control of spacecraft, the relative motion of satellites[30-33] and so on. Whittaker analyzed a holonomic system subject to uniform rotation constraints and derived its Lagrange equations[34]. Lurie studied holonomic mechanics with relative motion[35]. Mei *et al* extended it to nonholonomic mechanics[36]. Since then, progress has been made in the variational principle, equations of motion, integral theory, and symmetry of relative motion dynamics[37-44]. The Gauss principle of relative motion dynamics is further studied in this paper. Section 1 introduces the establishment of the Gauss principle for relative motion dynamics by analyzing the virtual displacement of acceleration space. In Section 2, the compulsion function of relative motion dynamics is constructed, and it is proved that real motion causes the compulsion function to reach an extreme value under Gaussian variation. Section 3 gives the formulae of acceleration energy and corresponding compulsion function when the carried body is a rigid body whose relative motion is planar motion. In Section 4, we study Gauss principle of relative motion dynamics and give its Appell, Lagrange, and Nielsen forms in generalized coordinates. In Section 5, from Gauss principle we obtained, we deduce dynamical equations with relative motion. In Section 6, two examples are given. Section 7 is the conclusion of the article.

1 Gauss Principle of Relative Motion Dynamics

Study a system of particles that comprises a rigid body (carrier) and N particles (carried bodies). The carried bodies are moving relative to the carrier. The moving frame of reference $Oxyz$ is attached to the carrier. We use n generalized coordinates q_s to describe the configuration of relative motion $s=1,2,\cdots,n$. The acceleration $\boldsymbol{a}_{\mathrm{O}}$ of the point O in a fixed frame $O_1x_1y_1z_1$, and the angular velocity $\boldsymbol{\omega}$ of moving frame are the given functions of time t. For the i-th particle, let m_i be its mass and $\boldsymbol{r}_i'=\boldsymbol{r}_i'(q_s,t)$ its position vector relative to $Oxyz$. The dynamic equation of relative motion is

$$-m_i\tilde{\ddot{\boldsymbol{r}}}_i'+\boldsymbol{F}_i+\boldsymbol{N}_i+\boldsymbol{F}_{\mathrm{e}i}^{\mathrm{I}}+\boldsymbol{F}_{\mathrm{c}i}^{\mathrm{I}}=\boldsymbol{0},\ i=1,2,\cdots,N \tag{1}$$

where $\boldsymbol{F}_i$, $\boldsymbol{N}_i$, $\boldsymbol{F}_{\mathrm{e}i}^{\mathrm{I}}=-m_i\boldsymbol{a}_{\mathrm{O}}-m_i\dot{\boldsymbol{\omega}}\times\boldsymbol{r}_i'-m_i\boldsymbol{\omega}\times(\boldsymbol{\omega}\times\boldsymbol{r}_i')$, $\boldsymbol{F}_{\mathrm{c}i}^{\mathrm{I}}=-2m_i\boldsymbol{\omega}\times\tilde{\dot{\boldsymbol{r}}}_i'$ are the active force, the constraint force, the convective inertial force, and the Coriolis inertial force, respectively. $\tilde{\dot{\boldsymbol{r}}}_i'$ is the relative velocity, $\tilde{\ddot{\boldsymbol{r}}}_i'$ is the relative acceleration.

By dotting the equation (1) with $\delta_{\mathrm{G}}\tilde{\ddot{\boldsymbol{r}}}_i'$ and summing over i, we get

$$\sum_{i=1}^{N}\left(-m_i\tilde{\ddot{\boldsymbol{r}}}_i'+\boldsymbol{F}_i+\boldsymbol{N}_i+\boldsymbol{F}_{\mathrm{e}i}^{\mathrm{I}}+\boldsymbol{F}_{\mathrm{c}i}^{\mathrm{I}}\right)\cdot\delta_{\mathrm{G}}\tilde{\ddot{\boldsymbol{r}}}_i'=0 \tag{2}$$

where $\delta_{\mathrm{G}}(\cdot)$ stands for the Gaussian variation[3]. Within acceleration space, the condition of ideal constraints yields

$$\sum_{i=1}^{N}\boldsymbol{N}_i\cdot\delta_{\mathrm{G}}\tilde{\ddot{\boldsymbol{r}}}_i'=0 \tag{3}$$

Thus, formula (2) becomes

$$\sum_{i=1}^{N}\left(-m_i\tilde{\ddot{\boldsymbol{r}}}_i'+\boldsymbol{F}_i+\boldsymbol{F}_{\mathrm{e}i}^{\mathrm{I}}+\boldsymbol{F}_{\mathrm{c}i}^{\mathrm{I}}\right)\cdot\delta_{\mathrm{G}}\tilde{\ddot{\boldsymbol{r}}}_i'=0 \tag{4}$$

Formula (4) is the Gauss principle of relative motion dynamics.

2 Gauss Principle of Least Compulsion for Relative Motion Dynamics

The compulsion function of relative motion is explained as

$$Z_{\mathrm{r}}=\sum_{i=1}^{N}\frac{1}{2}m_i\left(\tilde{\ddot{\boldsymbol{r}}}_i'-\frac{\boldsymbol{F}_i+\boldsymbol{F}_{\mathrm{e}i}^{\mathrm{I}}+\boldsymbol{F}_{\mathrm{c}i}^{\mathrm{I}}}{m_i}\right)^2 \tag{5}$$

then we have

$$\delta_G Z_r = \sum_{i=1}^{N} m_i \left(\tilde{\ddot{\boldsymbol{r}}}_i' - \frac{\boldsymbol{F}_i + \boldsymbol{F}_{ei}^I + \boldsymbol{F}_{ci}^I}{m_i} \right) \cdot \delta_G \left(\tilde{\ddot{\boldsymbol{r}}}_i' - \frac{\boldsymbol{F}_i + \boldsymbol{F}_{ei}^I + \boldsymbol{F}_{ci}^I}{m_i} \right)$$

$$= \sum_{i=1}^{N} m_i \left(\tilde{\ddot{\boldsymbol{r}}}_i' - \frac{\boldsymbol{F}_i + \boldsymbol{F}_{ei}^I + \boldsymbol{F}_{ci}^I}{m_i} \right) \cdot \delta_G \tilde{\ddot{\boldsymbol{r}}}_i' \tag{6}$$

Therefore, the principle (4) becomes

$$\delta_G Z_r = 0 \tag{7}$$

If $\tilde{\ddot{\boldsymbol{r}}}_i'$ is the relative acceleration in real motion and $\tilde{\ddot{\boldsymbol{r}}}_i' + \delta_G \tilde{\ddot{\boldsymbol{r}}}_i'$ is of possible motion of which the constraints admit, subsequently, the distinction between the compulsion functions is

$$\Delta Z_r = \sum_{i=1}^{N} \frac{1}{2} m_i \left\{ \left(\tilde{\ddot{\boldsymbol{r}}}_i' - \frac{\boldsymbol{F}_i + \boldsymbol{F}_{ei}^I + \boldsymbol{F}_{ci}^I}{m_i} \right)^2 - \left(\tilde{\ddot{\boldsymbol{r}}}_i' + \delta_G \tilde{\ddot{\boldsymbol{r}}}_i' - \frac{\boldsymbol{F}_i + \boldsymbol{F}_{ei}^I + \boldsymbol{F}_{ci}^I}{m_i} \right)^2 \right\}$$

$$= -\sum_{i=1}^{N} \frac{1}{2} m_i \left(\delta_G \tilde{\ddot{\boldsymbol{r}}}_i' \right)^2 - \sum_{i=1}^{N} m_i \left(\tilde{\ddot{\boldsymbol{r}}}_i' - \frac{\boldsymbol{F}_i + \boldsymbol{F}_{ei}^I + \boldsymbol{F}_{ci}^I}{m_i} \right) \cdot \delta_G \tilde{\ddot{\boldsymbol{r}}}_i' = -\sum_{i=1}^{N} \frac{1}{2} m_i \left(\delta_G \tilde{\ddot{\boldsymbol{r}}}_i' \right)^2 < 0 \tag{8}$$

Thus, equation (7) shows that, at any instant, the real motion of a relative motion dynamic system minimizes the compulsion function Z_r under Gaussian variation when compared with possible motions with the same configuration and the same velocity but with different accelerations. Equation (7) can be called the Gauss principle of least compulsion for relative motion dynamics. When, $\boldsymbol{a}_O = \boldsymbol{0}, \boldsymbol{\omega} = \boldsymbol{0}$, principles (4) and (7) degenerate to the classical Gauss principle and the least compulsion principle on the absolute motion[3].

3 Calculation of Acceleration Energy and Compulsion Function

Expanding formula (5), we have

$$Z_r = \sum_{i=1}^{N} \frac{1}{2} m_i \tilde{\ddot{\boldsymbol{r}}}_i' \cdot \tilde{\ddot{\boldsymbol{r}}}_i' - \sum_{i=1}^{N} \left(\boldsymbol{F}_i + \boldsymbol{F}_{ei}^I + \boldsymbol{F}_{ci}^I \right) \cdot \tilde{\ddot{\boldsymbol{r}}}_i' + \cdots \tag{9}$$

where the ellipsis "$\cdots$" symbolizes the terms that are independent of relative acceleration.

Let S_r denote the acceleration energy of relative motion, i.e.,

$$S_r = \sum_{i=1}^{N} \frac{1}{2} m_i \tilde{\ddot{\boldsymbol{r}}}_i' \cdot \tilde{\ddot{\boldsymbol{r}}}_i' \tag{10}$$

The compulsion function gives

$$Z_r = S_r - \sum_{i=1}^{N} \left(\boldsymbol{F}_i + \boldsymbol{F}_{ei}^I + \boldsymbol{F}_{ci}^I \right) \cdot \tilde{\ddot{\boldsymbol{r}}}_i' + \cdots \tag{11}$$

Next, we study the calculation of acceleration energy of relative motion if the carried body is rigid. First, if the relative motion is translation, denote the center of mass of the carried body as C and its relative acceleration $\boldsymbol{a}_{Cr}$, then

$$S_r = \sum_{i=1}^{N} \frac{1}{2} m_i \tilde{\ddot{\boldsymbol{r}}}_i' \cdot \tilde{\ddot{\boldsymbol{r}}}_i' = \sum_{i=1}^{N} \frac{1}{2} m_i a_{ir}^2 = \frac{1}{2} m a_{Cr}^2 \tag{12}$$

where $\boldsymbol{a}_{ir}$ is the relative acceleration of the i-th particle, and $m = \sum_{i=1}^{N} m_i$ is the total mass. Second, in the case of fixed-axis rotation for relative motion, denote the relative angular velocity of the carried body $\boldsymbol{\omega}_r$, the relative angular acceleration $\boldsymbol{\varepsilon}_r$, and the moment of inertia about the rotation axis $A\xi$ as J_ξ, then

$$S_r = \sum_{i=1}^{N} \frac{1}{2} m_i \tilde{\ddot{\boldsymbol{r}}}_i' \cdot \tilde{\ddot{\boldsymbol{r}}}_i' = \sum_{i=1}^{N} \frac{1}{2} m_i \boldsymbol{a}_{ir} \cdot \boldsymbol{a}_{ir} = \sum_{i=1}^{N} \frac{1}{2} m_i \left(\rho_i \varepsilon_r \boldsymbol{\tau}_i + \rho_i \omega_r^2 \boldsymbol{n}_i \right) \cdot \left(\rho_i \varepsilon_r \boldsymbol{\tau}_i + \rho_i \omega_r^2 \boldsymbol{n}_i \right) \tag{13}$$

where ρ_i is the distance between the i-th particle and $A\xi$ axis, and unit vectors $\boldsymbol{\tau}_i$ and $\boldsymbol{n}_i$ are along tangential and principal normal directions, respectively. Expanding equation (13), we get

$$S_r = \sum_{i=1}^{N} \frac{1}{2} m_i \rho_i^2 \varepsilon_r^2 + \sum_{i=1}^{N} \frac{1}{2} m_i \rho_i^2 \omega_r^4 = \frac{1}{2} J_\xi \varepsilon_r^2 + \cdots \tag{14}$$

Third, in the event that the relative motion is planar motion, denote the relative angular velocity of the carried body with planar motion as $\boldsymbol{\omega}_r$, the relative angular acceleration as $\boldsymbol{\varepsilon}_r$, the relative acceleration as $\boldsymbol{a}_{Cr}$, then

$$S_{\rm r}=\sum_{i=1}^{N}\frac{1}{2}m_i\tilde{\ddot{\boldsymbol r}}_i'\cdot\tilde{\ddot{\boldsymbol r}}_i'=\sum_{i=1}^{N}\frac{1}{2}m_i\boldsymbol a_{i\rm r}\cdot\boldsymbol a_{i\rm r}=\sum_{i=1}^{N}\frac{1}{2}m_i\left[\boldsymbol a_{\rm Cr}+\rho_i\left(\varepsilon_{\rm r}\boldsymbol\tau_i+\omega_{\rm r}^2\boldsymbol n_i\right)\right]\cdot\left[\boldsymbol a_{\rm Cr}+\rho_i\left(\varepsilon_{\rm r}\boldsymbol\tau_i+\omega_{\rm r}^2\boldsymbol n_i\right)\right]$$
$$=\sum_{i=1}^{N}\frac{1}{2}m_i\left[a_{\rm Cr}^2+2\boldsymbol a_{\rm Cr}\cdot\rho_i\left(\varepsilon_{\rm r}\boldsymbol\tau_i+\omega_{\rm r}^2\boldsymbol n_i\right)+\rho_i^2\left(\varepsilon_{\rm r}^2+\omega_{\rm r}^4\right)\right] \tag{15}$$

where ρ_i is the distance between the i-th particle and C, and $\boldsymbol\tau_i$ and $\boldsymbol n_i$ are tangential and normal unit vectors relative to C, respectively. Obviously, from the centroid coordinate formula, we have

$$\sum_{i=1}^{N}m_i\rho_i\boldsymbol a_{\rm Cr}\cdot\boldsymbol\tau_i=0,\ \sum_{i=1}^{N}m_i\rho_i\boldsymbol a_{\rm Cr}\cdot\boldsymbol n_i=0 \tag{16}$$

Hence, we obtain

$$S_{\rm r}=\frac{1}{2}ma_{\rm Cr}^2+\frac{1}{2}J_{\rm C}\varepsilon_{\rm r}^2 \tag{17}$$

where $J_{\rm C}=\sum_{i=1}^{N}m_i\rho_i^2$ represents the moment of inertia. Equation (17) shows that the acceleration energy of the relative motion of the carried body with planar motion equals the sum of the acceleration energy of relative translation with and relative rotation around the center of mass. Let

$$\boldsymbol F=\sum_{i=1}^{N}\boldsymbol F_i,\ \boldsymbol F_{\rm e}^{\rm I}=\sum_{i=1}^{N}\boldsymbol F_{{\rm e}i}^{\rm I},\ \boldsymbol F_{\rm c}^{\rm I}=\sum_{i=1}^{N}\boldsymbol F_{{\rm c}i}^{\rm I} \tag{18}$$

represent the principal vectors of the active forces, the convective inertial forces, and the Coriolis inertial forces, respectively, and

$$M_{\rm C}=\sum_{i=1}^{N}M_{\rm C}\left(\boldsymbol F_i\right),M_{\rm Ce}^{\rm I}=\sum_{i=1}^{N}M_{\rm C}\left(\boldsymbol F_{{\rm e}i}^{\rm I}\right),M_{\rm Cc}^{\rm I}=\sum_{i=1}^{N}M_{\rm C}\left(\boldsymbol F_{{\rm c}i}^{\rm I}\right) \tag{19}$$

represent the principal moment about point C, then

$$\sum_{i=1}^{N}\left(\boldsymbol F_i+\boldsymbol F_{{\rm e}i}^{\rm I}+\boldsymbol F_{{\rm c}i}^{\rm I}\right)\cdot\tilde{\ddot{\boldsymbol r}}_i'=\sum_{i=1}^{N}\left(\boldsymbol F_i+\boldsymbol F_{{\rm e}i}^{\rm I}+\boldsymbol F_{{\rm c}i}^{\rm I}\right)\cdot\left[\boldsymbol a_{\rm Cr}+\rho_i\left(\varepsilon_{\rm r}\boldsymbol\tau_i+\omega_{\rm r}^2\boldsymbol n_i\right)\right]$$
$$=\boldsymbol a_{\rm Cr}\cdot\sum_{i=1}^{N}\left(\boldsymbol F_i+\boldsymbol F_{{\rm e}i}^{\rm I}+\boldsymbol F_{{\rm c}i}^{\rm I}\right)+\varepsilon_{\rm r}\sum_{i=1}^{N}\rho_i\boldsymbol\tau_i\cdot\left(\boldsymbol F_i+\boldsymbol F_{{\rm e}i}^{\rm I}+\boldsymbol F_{{\rm c}i}^{\rm I}\right)+\omega_{\rm r}^2\sum_{i=1}^{N}\rho_i\boldsymbol n_i\cdot\left(\boldsymbol F_i+\boldsymbol F_{{\rm e}i}^{\rm I}+\boldsymbol F_{{\rm c}i}^{\rm I}\right)$$
$$=\boldsymbol a_{\rm Cr}\cdot\left(\boldsymbol F+\boldsymbol F_{\rm e}^{\rm I}+\boldsymbol F_{\rm c}^{\rm I}\right)+\varepsilon_{\rm r}\left(M_{\rm C}+M_{\rm Ce}^{\rm I}+M_{\rm Cc}^{\rm I}\right)+\cdots \tag{20}$$

Substituting formulas (17) and (20) into (11), we get

$$Z_{\rm r}=\frac{1}{2}ma_{\rm Cr}^2+\frac{1}{2}J_{\rm C}\varepsilon_{\rm r}^2-\boldsymbol a_{\rm Cr}\cdot\left(\boldsymbol F+\boldsymbol F_{\rm e}^{\rm I}+\boldsymbol F_{\rm c}^{\rm I}\right)-\varepsilon_{\rm r}\left(M_{\rm C}+M_{\rm Ce}^{\rm I}+M_{\rm Cc}^{\rm I}\right)+\cdots \tag{21}$$

This formula calculates the compulsion function of relative motion for the carried body in planar motion.

If the relative motion is translation, then the compulsion function (21) gives

$$Z_{\rm r}=\frac{1}{2}ma_{\rm Cr}^2-\boldsymbol a_{\rm Cr}\cdot\left(\boldsymbol F+\boldsymbol F_{\rm e}^{\rm I}+\boldsymbol F_{\rm c}^{\rm I}\right)+\cdots \tag{22}$$

If the relative motion is fixed axis rotation, then the compulsion function (21) provides

$$Z_{\rm r}=\frac{1}{2}J_\xi\varepsilon_{\rm r}^2-\varepsilon_{\rm r}\left(M_\xi+M_{\xi\rm e}^{\rm I}+M_{\xi\rm c}^{\rm I}\right)+\cdots \tag{23}$$

4 Gauss Principle of Relative Motion Dynamics in Generalized Coordinates

Taking the relative derivative of $\boldsymbol r_i'=\boldsymbol r_i'\left(q_s,t\right)$, we get

$$\tilde{\dot{\boldsymbol r}}_i'=\sum_{s=1}^{n}\frac{\partial\boldsymbol r_i'}{\partial q_s}\dot q_s+\frac{\partial\boldsymbol r_i'}{\partial t} \tag{24}$$

$$\tilde{\ddot{\boldsymbol r}}_i'=\sum_{s=1}^{n}\frac{\partial\boldsymbol r_i'}{\partial q_s}\ddot q_s+\sum_{s=1}^{n}\sum_{k=1}^{n}\frac{\partial^2\boldsymbol r_i'}{\partial q_s\partial q_k}\dot q_s\dot q_k+2\sum_{s=1}^{n}\frac{\partial^2\boldsymbol r_i'}{\partial t\partial q_s}\dot q_s+\frac{\partial^2\boldsymbol r_i'}{\partial t^2} \tag{25}$$

Hence, we have

$$\delta_G \tilde{\ddot{\boldsymbol{r}}}_i' = \sum_{s=1}^{n} \frac{\partial \boldsymbol{r}_i'}{\partial q_s} \delta_G \ddot{q}_s \tag{26}$$

Calculating the Gaussian variation of equation (11), we get

$$\delta_G Z_r = \delta_G S_r - \sum_{i=1}^{N} \left(\boldsymbol{F}_i + \boldsymbol{F}_{ei}^{I} + \boldsymbol{F}_{ci}^{I} \right) \cdot \delta_G \tilde{\ddot{\boldsymbol{r}}}_i' \tag{27}$$

Notice that

$$\delta_G S_r = \sum_{s=1}^{n} \frac{\partial S_r}{\partial \ddot{q}_s} \delta_G \ddot{q}_s \tag{28}$$

$$\sum_{i=1}^{N} \boldsymbol{F}_i \cdot \delta_G \tilde{\ddot{\boldsymbol{r}}}_i' = \sum_{s=1}^{n} \sum_{i=1}^{N} \boldsymbol{F}_i \cdot \frac{\partial \boldsymbol{r}_i'}{\partial q_s} \delta_G \ddot{q}_s = \sum_{s=1}^{n} Q_s \delta_G \ddot{q}_s \tag{29}$$

$$\begin{aligned}
\sum_{i=1}^{N} \left(\boldsymbol{F}_{ei}^{I} + \boldsymbol{F}_{ci}^{I} \right) \cdot \delta_G \tilde{\ddot{\boldsymbol{r}}}_i' &= \sum_{i=1}^{N} \left[-m_i \boldsymbol{a}_o - m_i \dot{\boldsymbol{\omega}} \times \boldsymbol{r}_i' - m_i \boldsymbol{\omega} \times \left(\boldsymbol{\omega} \times \boldsymbol{r}_i' \right) - 2 m_i \boldsymbol{\omega} \times \tilde{\dot{\boldsymbol{r}}}_i' \right] \cdot \sum_{s=1}^{n} \frac{\partial \boldsymbol{r}_i'}{\partial q_s} \delta_G \ddot{q}_s \\
&= \sum_{s=1}^{n} \sum_{i=1}^{N} \left\{ -m_i \boldsymbol{a}_o \cdot \frac{\partial \boldsymbol{r}_i'}{\partial q_s} - m_i \left(\dot{\boldsymbol{\omega}} \times \boldsymbol{r}_i' \right) \cdot \frac{\partial \boldsymbol{r}_i'}{\partial q_s} - m_i \left[\boldsymbol{\omega} \times \left(\boldsymbol{\omega} \times \boldsymbol{r}_i' \right) \right] \cdot \frac{\partial \boldsymbol{r}_i'}{\partial q_s} - 2 m_i \left(\boldsymbol{\omega} \times \tilde{\dot{\boldsymbol{r}}}_i' \right) \cdot \frac{\partial \boldsymbol{r}_i'}{\partial q_s} \right\} \delta_G \ddot{q}_s \\
&= \sum_{s=1}^{n} \left[-\frac{\partial}{\partial q_s} \left(\Pi^{o} + \Pi^{\omega} \right) + Q_s^{\dot{\omega}} + \Gamma_s \right] \delta_G \ddot{q}_s
\end{aligned} \tag{30}$$

where

$$\Pi^{o} = \sum_{i=1}^{N} m_i \boldsymbol{a}_o \cdot \boldsymbol{r}_i' = m \boldsymbol{a}_o \cdot \boldsymbol{r}_c' \tag{31}$$

is the potential energy of a uniform force field[29], and

$$\Pi^{\omega} = \frac{1}{2} \sum_{i=1}^{N} m_i \left[\boldsymbol{\omega} \times \left(\boldsymbol{\omega} \times \boldsymbol{r}_i' \right) \right] \cdot \boldsymbol{r}_i' = -\frac{1}{2} \sum_{i=1}^{N} m_i \left(\boldsymbol{\omega} \times \boldsymbol{r}_i' \right) \cdot \left(\boldsymbol{\omega} \times \boldsymbol{r}_i' \right) = -\frac{1}{2} \boldsymbol{\omega} \cdot \boldsymbol{\theta}^{o} \cdot \boldsymbol{\omega} \tag{32}$$

is the potential energy of centrifugal forces, $\boldsymbol{\theta}^{o} = \sum_{i=1}^{N} m_i \left[\left(r_i' \right)^2 \boldsymbol{E} - \boldsymbol{r}_i' \boldsymbol{r}_i' \right]$ is the inertia tensor. And

$$Q_s^{\dot{\omega}} = -\sum_{i=1}^{N} m_i \left(\dot{\boldsymbol{\omega}} \times \boldsymbol{r}_i' \right) \cdot \frac{\partial \boldsymbol{r}_i'}{\partial q_s} \tag{33}$$

represents the generalized rotational inertia force and

$$\Gamma_s = -\sum_{i=1}^{N} 2 m_i \left(\boldsymbol{\omega} \times \tilde{\dot{\boldsymbol{r}}}_i' \right) \cdot \frac{\partial \boldsymbol{r}_i'}{\partial q_s} \tag{34}$$

is the generalized gyroscopic force. By substituting formulas (28), (29) and (30) into formula (27), we get

$$\delta_G Z_r = \sum_{s=1}^{n} \left[\frac{\partial S_r}{\partial \ddot{q}_s} - Q_s + \frac{\partial}{\partial q_s} \left(\Pi^{o} + \Pi^{\omega} \right) - Q_s^{\dot{\omega}} - \Gamma_s \right] \delta_G \ddot{q}_s \tag{35}$$

Therefore, principle (7) gives

$$\sum_{s=1}^{n} \left[\frac{\partial S_r}{\partial \ddot{q}_s} - Q_s + \frac{\partial}{\partial q_s} \left(\Pi^{o} + \Pi^{\omega} \right) - Q_s^{\dot{\omega}} - \Gamma_s \right] \delta_G \ddot{q}_s = 0 \tag{36}$$

Equation (36) is the Appell form of the Gauss principle of relative motion dynamics in generalized coordinates.

Now, two alternative forms of the principle are derived: the Lagrange form and the Nielsen form. First of all, we give some formulas for the subsequent derivation. From equations (24) and (25), we can easily obtain

$$\frac{\partial \tilde{\ddot{\boldsymbol{r}}}_i'}{\partial \ddot{q}_s} = \frac{\partial \tilde{\dot{\boldsymbol{r}}}_i'}{\partial \dot{q}_s} = \frac{\partial \boldsymbol{r}_i'}{\partial q_s} \tag{37}$$

$$\frac{\partial \tilde{\ddot{\boldsymbol{r}}}_i'}{\partial \dot{q}_s} = 2 \sum_{k=1}^{n} \frac{\partial^2 \boldsymbol{r}_i'}{\partial q_s \partial q_k} \dot{q}_k + 2 \frac{\partial^2 \boldsymbol{r}_i'}{\partial t \partial q_s} = \frac{\partial \tilde{\dot{\boldsymbol{r}}}_i'}{\partial q_s} \tag{38}$$

$$\frac{\tilde{\mathrm{d}}}{\mathrm{d}t} \frac{\partial \boldsymbol{r}_i'}{\partial q_s} = \frac{\partial \tilde{\dot{\boldsymbol{r}}}_i'}{\partial q_s} \tag{39}$$

By the relation between the absolute derivative and the relative derivative of a vector, for any vector $\boldsymbol{A}$, there is

$$\frac{\mathrm{d}}{\mathrm{d}t} \boldsymbol{A} = \frac{\tilde{\mathrm{d}}}{\mathrm{d}t} \boldsymbol{A} + \boldsymbol{\omega} \times \boldsymbol{A} \tag{40}$$

Thus, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\tilde{\dot{\boldsymbol{r}}}_i'=\frac{\tilde{\mathrm{d}}}{\mathrm{d}t}\tilde{\dot{\boldsymbol{r}}}_i'+\boldsymbol{\omega}\times\tilde{\dot{\boldsymbol{r}}}_i'=\tilde{\ddot{\boldsymbol{r}}}_i'+\boldsymbol{\omega}\times\tilde{\dot{\boldsymbol{r}}}_i' \tag{41}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial\boldsymbol{r}_i'}{\partial q_s}=\frac{\tilde{\mathrm{d}}}{\mathrm{d}t}\frac{\partial\boldsymbol{r}_i'}{\partial q_s}+\boldsymbol{\omega}\times\frac{\partial\boldsymbol{r}_i'}{\partial q_s}=\frac{\partial\tilde{\dot{\boldsymbol{r}}}_i'}{\partial q_s}+\boldsymbol{\omega}\times\frac{\partial\boldsymbol{r}_i'}{\partial q_s} \tag{42}$$

Secondly, denote T_{r} as the kinetic energy of relative motion, i.e.,

$$T_{\mathrm{r}}=\frac{1}{2}\sum_{i=1}^{N}m_i\tilde{\dot{\boldsymbol{r}}}_i'\cdot\tilde{\dot{\boldsymbol{r}}}_i' \tag{43}$$

then we have

$$\frac{\partial T_{\mathrm{r}}}{\partial\dot{q}_s}=\sum_{i=1}^{N}m_i\tilde{\dot{\boldsymbol{r}}}_i'\cdot\frac{\partial\tilde{\dot{\boldsymbol{r}}}_i'}{\partial\dot{q}_s}=\sum_{i=1}^{N}m_i\tilde{\dot{\boldsymbol{r}}}_i'\cdot\frac{\partial\boldsymbol{r}_i'}{\partial q_s} \tag{44}$$

By using equations (41) and (42), we can obtain

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial T_{\mathrm{r}}}{\partial\dot{q}_s}&=\sum_{i=1}^{N}m_i\frac{\mathrm{d}}{\mathrm{d}t}\tilde{\dot{\boldsymbol{r}}}_i'\cdot\frac{\partial\boldsymbol{r}_i'}{\partial q_s}+\sum_{i=1}^{N}m_i\tilde{\dot{\boldsymbol{r}}}_i'\cdot\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial\boldsymbol{r}_i'}{\partial q_s}=\sum_{i=1}^{N}m_i\left(\tilde{\ddot{\boldsymbol{r}}}_i'+\boldsymbol{\omega}\times\tilde{\dot{\boldsymbol{r}}}_i'\right)\cdot\frac{\partial\boldsymbol{r}_i'}{\partial q_s}+\sum_{i=1}^{N}m_i\tilde{\dot{\boldsymbol{r}}}_i'\cdot\left(\frac{\partial\tilde{\dot{\boldsymbol{r}}}_i'}{\partial q_s}+\boldsymbol{\omega}\times\frac{\partial\boldsymbol{r}_i'}{\partial q_s}\right)\\&=\sum_{i=1}^{N}m_i\tilde{\ddot{\boldsymbol{r}}}_i'\cdot\frac{\partial\boldsymbol{r}_i'}{\partial q_s}+\sum_{i=1}^{N}m_i\tilde{\dot{\boldsymbol{r}}}_i'\cdot\frac{\partial\tilde{\dot{\boldsymbol{r}}}_i'}{\partial q_s}\end{aligned} \tag{45}$$

and

$$\frac{\partial T_{\mathrm{r}}}{\partial q_s}=\sum_{i=1}^{N}m_i\tilde{\dot{\boldsymbol{r}}}_i'\cdot\frac{\partial\tilde{\dot{\boldsymbol{r}}}_i'}{\partial q_s} \tag{46}$$

Therefore, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial T_{\mathrm{r}}}{\partial\dot{q}_s}-\frac{\partial T_{\mathrm{r}}}{\partial q_s}=\sum_{i=1}^{N}m_i\tilde{\ddot{\boldsymbol{r}}}_i'\cdot\frac{\partial\boldsymbol{r}_i'}{\partial q_s} \tag{47}$$

From equations (10) and (37), we get

$$\frac{\partial S_{\mathrm{r}}}{\partial\ddot{q}_s}=\sum_{i=1}^{N}m_i\tilde{\ddot{\boldsymbol{r}}}_i'\cdot\frac{\partial\tilde{\ddot{\boldsymbol{r}}}_i'}{\partial\ddot{q}_s}=\sum_{i=1}^{N}m_i\tilde{\ddot{\boldsymbol{r}}}_i'\cdot\frac{\partial\boldsymbol{r}_i'}{\partial q_s} \tag{48}$$

By comparing formula (47) and formula (48), principle (36) can be expressed as

$$\sum_{s=1}^{n}\left[\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial T_{\mathrm{r}}}{\partial\dot{q}_s}-\frac{\partial T_{\mathrm{r}}}{\partial q_s}-Q_s+\frac{\partial}{\partial q_s}\left(\Pi^{\mathrm{o}}+\Pi^{\omega}\right)-Q_s^{\dot{\omega}}-\Gamma_s\right]\delta_{\mathrm{G}}\ddot{q}_s=0 \tag{49}$$

Equation (49) is the Lagrange form of the Gauss principle of relative motion dynamics in generalized coordinates.

Calculating the time derivative of T_{r}, we get

$$\frac{\mathrm{d}}{\mathrm{d}t}T_{\mathrm{r}}=\sum_{i=1}^{N}m_i\tilde{\dot{\boldsymbol{r}}}_i'\cdot\frac{\mathrm{d}}{\mathrm{d}t}\tilde{\dot{\boldsymbol{r}}}_i'=\sum_{i=1}^{N}m_i\tilde{\dot{\boldsymbol{r}}}_i'\cdot\left(\tilde{\ddot{\boldsymbol{r}}}_i'+\boldsymbol{\omega}\times\tilde{\dot{\boldsymbol{r}}}_i'\right)=\sum_{i=1}^{N}m_i\tilde{\dot{\boldsymbol{r}}}_i'\cdot\tilde{\ddot{\boldsymbol{r}}}_i' \tag{50}$$

Here, the following relationship is applied, i.e.,

$$\tilde{\dot{\boldsymbol{r}}}_i'\cdot\left(\boldsymbol{\omega}\times\tilde{\dot{\boldsymbol{r}}}_i'\right)=\boldsymbol{\omega}\cdot\left(\tilde{\dot{\boldsymbol{r}}}_i'\times\tilde{\dot{\boldsymbol{r}}}_i'\right)=0 \tag{51}$$

Taking the partial derivative of $\frac{\mathrm{d}}{\mathrm{d}t}T_{\mathrm{r}}$ with respect to $\dot{q}_s$, we get

$$\frac{\partial}{\partial\dot{q}_s}\frac{\mathrm{d}}{\mathrm{d}t}T_{\mathrm{r}}=\sum_{i=1}^{N}m_i\tilde{\ddot{\boldsymbol{r}}}_i'\cdot\frac{\partial\tilde{\dot{\boldsymbol{r}}}_i'}{\partial\dot{q}_s}+\sum_{i=1}^{N}m_i\frac{\partial\tilde{\ddot{\boldsymbol{r}}}_i'}{\partial\dot{q}_s}\cdot\tilde{\dot{\boldsymbol{r}}}_i'=\sum_{i=1}^{N}m_i\tilde{\ddot{\boldsymbol{r}}}_i'\cdot\frac{\partial\boldsymbol{r}_i'}{\partial q_s}+2\sum_{i=1}^{N}m_i\tilde{\dot{\boldsymbol{r}}}_i'\cdot\frac{\partial\tilde{\dot{\boldsymbol{r}}}_i'}{\partial q_s} \tag{52}$$

From formula (52) and formula (46), we obtain

$$\frac{\partial}{\partial\dot{q}_s}\frac{\mathrm{d}}{\mathrm{d}t}T_{\mathrm{r}}-2\frac{\partial T_{\mathrm{r}}}{\partial q_s}=\sum_{i=1}^{N}m_i\tilde{\ddot{\boldsymbol{r}}}_i'\cdot\frac{\partial\boldsymbol{r}_i'}{\partial q_s} \tag{53}$$

By comparing formula (53) and formula (48), principle (36) can also be expressed as

$$\sum_{s=1}^{n}\left[\frac{\partial}{\partial\dot{q}_s}\frac{\mathrm{d}}{\mathrm{d}t}T_{\mathrm{r}}-2\frac{\partial T_{\mathrm{r}}}{\partial q_s}-Q_s+\frac{\partial}{\partial q_s}\left(\Pi^{\mathrm{o}}+\Pi^{\omega}\right)-Q_s^{\dot{\omega}}-\Gamma_s\right]\delta_{\mathrm{G}}\ddot{q}_s=0 \tag{54}$$

Equation (54) is the Nielsen form of the Gauss principle of relative motion dynamics in generalized coordinates.

5 Dynamical Equations of Relative Motion

For a holonomic system, $\delta_{\mathrm{G}}\ddot{q}_s$ is independent and arbitrary, so from principle (36), we get

$$\frac{\partial S_{\mathrm{r}}}{\partial \ddot{q}_s}=Q_s-\frac{\partial}{\partial q_s}\left(\Pi^{\mathrm{o}}+\Pi^{\omega}\right)+Q_s^{\dot{\omega}}+\Gamma_s \tag{55}$$

This is the Appell equation of relative motion dynamics, and $s=1,2,\cdots,n$. From principle (49), we get

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial T_{\mathrm{r}}}{\partial \dot{q}_s}-\frac{\partial T_{\mathrm{r}}}{\partial q_s}=Q_s-\frac{\partial}{\partial q_s}\left(\Pi^{\mathrm{o}}+\Pi^{\omega}\right)+Q_s^{\dot{\omega}}+\Gamma_s \tag{56}$$

This is the Lagrange equation of relative motion dynamics. From principle (54), we get

$$\frac{\partial}{\partial \dot{q}_s}\frac{\mathrm{d}}{\mathrm{d}t}T_{\mathrm{r}}-2\frac{\partial T_{\mathrm{r}}}{\partial q_s}=Q_s-\frac{\partial}{\partial q_s}\left(\Pi^{\mathrm{o}}+\Pi^{\omega}\right)+Q_s^{\dot{\omega}}+\Gamma_s \tag{57}$$

This is the Nielsen equation of relative motion dynamics. For a nonholonomic system, let g ideal two-sided nonholonomic constraints be

$$\phi_\alpha\left(t,q_s,\dot{q}_s\right)=0,\ \alpha=1,2,\cdots,g \tag{58}$$

By differentiating equation (58), we obtain

$$\frac{\partial \phi_\alpha}{\partial \dot{q}_s}\ddot{q}_s+\frac{\partial \phi_\alpha}{\partial q_s}\dot{q}_s+\frac{\partial \phi_\alpha}{\partial t}=0 \tag{59}$$

Then we have

$$\frac{\partial \phi_\alpha}{\partial \dot{q}_s}\delta_{\mathrm{G}}\ddot{q}_s=0 \tag{60}$$

From the Gauss principle (36) and formula (60) in Appell form, using the Lagrange multiplier method, we get

$$\frac{\partial S_{\mathrm{r}}}{\partial \ddot{q}_s}=Q_s-\frac{\partial}{\partial q_s}\left(\Pi^{\mathrm{o}}+\Pi^{\omega}\right)+Q_s^{\dot{\omega}}+\Gamma_s+\mu_\alpha\frac{\partial \phi_\alpha}{\partial \dot{q}_s} \tag{61}$$

where μ_α is the Lagrange multiplier, $s=1,2,\cdots,n$. Formula (61) is the Appell equation with multipliers in generalized coordinates for nonholonomic systems in relative motion. From the Lagrange form of Gauss principle (49) and formula (60), we get

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial T_{\mathrm{r}}}{\partial \dot{q}_s}-\frac{\partial T_{\mathrm{r}}}{\partial q_s}=Q_s-\frac{\partial}{\partial q_s}\left(\Pi^{\mathrm{o}}+\Pi^{\omega}\right)+Q_s^{\dot{\omega}}+\Gamma_s+\mu_\alpha\frac{\partial \phi_\alpha}{\partial \dot{q}_s} \tag{62}$$

This is the Lagrange equation with multipliers in generalized coordinates for nonholonomic systems in relative motion, also known as Routh equation. From the Nielsen form of the Gauss principle (54) and formula (60), we get

$$\frac{\partial}{\partial \dot{q}_s}\frac{\mathrm{d}}{\mathrm{d}t}T_{\mathrm{r}}-2\frac{\partial T_{\mathrm{r}}}{\partial q_s}=Q_s-\frac{\partial}{\partial q_s}\left(\Pi^{\mathrm{o}}+\Pi^{\omega}\right)+Q_s^{\dot{\omega}}+\Gamma_s+\mu_\alpha\frac{\partial \phi_\alpha}{\partial \dot{q}_s} \tag{63}$$

This is the Nielsen equation with multipliers in generalized coordinates for nonholonomic systems in relative motion.

6 Examples

Example 1 A physical pendulum with mass m is suspended at point O on a given block AB, as shown in Fig. 1. Let block AB do circumferential translation with radius l_0, whose motion is determined by the angle θ and known as $\theta=\theta(t)$. The angle describes the position of the pendulum relative to AB φ, and the distance from its center of mass C to O is R, and its radius of gyration to C is ρ. Try to establish the dynamic equation of relative motion using the Gauss principle.

In this example, the carrier is the block AB, and the carried body is the physical pendulum. The carrier's motion is circumferential translation, and the relative motion of the carried body is fixed axis rotation around the axis $O\xi$. The acceleration energy of the relative motion of the pendulum is

$$S_{\mathrm{r}}=\frac{1}{2}J_\xi\varepsilon_{\mathrm{r}}^2+\cdots=\frac{1}{2}\left(J_{\mathrm{C}}+mR^2\right)\varepsilon_{\mathrm{r}}^2+\cdots=\frac{1}{2}m\left(\rho^2+R^2\right)\ddot{\varphi}^2+\cdots \tag{64}$$

The active force is only gravity mg, and the moment to the axis $O\xi$ is

$$M_{\xi}=-mgR\sin\varphi \tag{65}$$

Since the convected motion is translation, there is no Coriolis inertia force, and the convected inertia force is

$$F_{\mathrm{e}}^{\mathrm{I\tau}}=ml_0\ddot{\theta},F_{\mathrm{e}}^{\mathrm{In}}=ml_0\dot{\theta}^2 \tag{66}$$

The moment of convected inertia force about $O\xi$ is

$$M_{\xi\mathrm{e}}=-F_{\mathrm{e}}^{\mathrm{In}}R\sin\left(\varphi-\theta\right)-F_{\mathrm{e}}^{\mathrm{I\tau}}R\cos\left(\varphi-\theta\right) \tag{67}$$

Therefore, from formula (22), the compulsion function of relative motion is

$$\begin{aligned}Z_{\mathrm{r}}&=\frac{1}{2}J_{\xi}\varepsilon_{\mathrm{r}}^2-\varepsilon_{\mathrm{r}}\left(M_{\xi}+M_{\xi\mathrm{e}}^{\mathrm{I}}+M_{\xi\mathrm{c}}^{\mathrm{I}}\right)+\cdots\\&=\frac{1}{2}m\left(\rho^2+R^2\right)\ddot{\varphi}^2+\left[mgR\sin\varphi+ml_0R\dot{\theta}^2\sin\left(\varphi-\theta\right)+ml_0R\ddot{\theta}\cos\left(\varphi-\theta\right)\right]\ddot{\varphi}+\cdots\end{aligned} \tag{68}$$

To calculate the Gaussian variation $\delta_{\mathrm{G}}Z_{\mathrm{r}}$ and set it to zero, we get

$$\delta_{\mathrm{G}}Z_{\mathrm{r}}=m\left(\rho^2+R^2\right)\ddot{\varphi}\delta_{\mathrm{G}}\ddot{\varphi}+ml_0R\dot{\theta}^2\sin\left(\varphi-\theta\right)\delta_{\mathrm{G}}\ddot{\varphi}+ml_0R\ddot{\theta}\cos\left(\varphi-\theta\right)\delta_{\mathrm{G}}\ddot{\varphi}+mgR\sin\varphi\delta_{\mathrm{G}}\ddot{\varphi}=0 \tag{69}$$

Due to the arbitrariness of $\delta_{\mathrm{G}}\ddot{\varphi}$, we get

$$m\left(\rho^2+R^2\right)\ddot{\varphi}+ml_0R\dot{\theta}^2\sin\left(\varphi-\theta\right)+ml_0R\ddot{\theta}\cos\left(\varphi-\theta\right)+mgR\sin\varphi=0 \tag{70}$$

This represents the differential equation governing the relative motion of a physical pendulum. Equation (70) is consistent with the results obtained using the Lagrange equation in Ref. [29].

Example 2 As shown in Fig. 2, a uniform rod *AB* with mass *m* and length *l* has one end, *A*, sliding along the vertical fixed axis *Oz* and the other end, *B*, sliding along the horizontal axis *Ox*. In contrast, *Ox* rotates around *Oz* at a uniform angular velocity $\boldsymbol{\omega}$. The point *B* is connected to the spring *BD*, and *D* is fixed on the *Ox* axis. Let θ indicate the angle between *AB* and the plumb line, when $\theta=0$, the spring has its original length. Suppose that the spring stiffness is *k*, friction is ignored, and $0\leqslant\theta\leqslant\frac{\pi}{2}$, find the dynamic equation of relative motion.

In this scenario, the carrier rotates at a uniform angular speed around *Oz*. The relative motion of the carried body *AB* is planar. With θ the generalized coordinate, the acceleration energy of the relative motion of rod *AB* is

$$S_{\mathrm{r}}=\frac{1}{2}ma_{\mathrm{Cr}}^2+\frac{1}{2}J_{\mathrm{C}}\varepsilon_{\mathrm{r}}^2 \tag{71}$$

where $J_{\mathrm{C}}=\frac{1}{12}ml^2$. Since

$$x_{\mathrm{C}}=\frac{1}{2}l\sin\theta,\ z_{\mathrm{C}}=\frac{1}{2}l\cos\theta \tag{72}$$

Taking the time derivative of (72) twice, we have

$$\ddot{x}_{\mathrm{C}}=\frac{1}{2}l\ddot{\theta}\cos\theta-\frac{1}{2}l\dot{\theta}^2\sin\theta,\ \ddot{z}_{\mathrm{C}}=-\frac{1}{2}l\ddot{\theta}\sin\theta-\frac{1}{2}l\dot{\theta}^2\cos\theta \tag{73}$$

Hence, we have

$$a_{\mathrm{Cr}}^2=\ddot{x}_{\mathrm{C}}^2+\ddot{z}_{\mathrm{C}}^2=\frac{1}{4}l^2\ddot{\theta}^2+\cdots \tag{74}$$

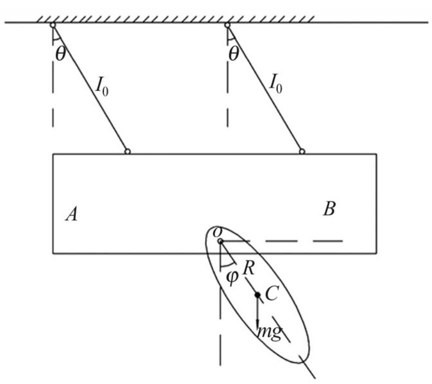

Fig. 1 A physical pendulum in relative motion

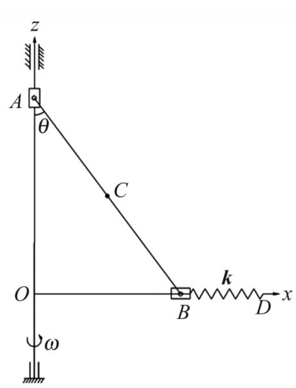

Fig. 2 A uniform rod *AB* in relative motion

Substituting equation (74) into equation (71) and noting that $\varepsilon_\mathrm{r}=\ddot{\theta}$, we get

$$S_\mathrm{r}=\frac{1}{2}m\frac{1}{4}l^2\ddot{\theta}^2+\frac{1}{2}\frac{1}{12}ml^2\ddot{\theta}^2+\cdots=\frac{1}{6}ml^2\ddot{\theta}^2+\cdots \tag{75}$$

Suppose we take a small segment $\mathrm{d}l_i$ on AB at a distance l_i from end A; then its mass is $\mathrm{d}m_i=\frac{m}{l}\mathrm{d}l_i$. The coordinates in the moving coordinate system $Oxyz$ are

$$x_i=l_i\sin\theta,\ y_i=0,\ z_i=(l-l_i)\cos\theta \tag{76}$$

Then we have

$$\dot{x}_i=l_i\dot{\theta}\cos\theta,\ \dot{y}_i=0,\ \dot{z}_i=-(l-l_i)\dot{\theta}\sin\theta \tag{77}$$

$$\ddot{x}_i=l_i\left(\ddot{\theta}\cos\theta-\dot{\theta}^2\sin\theta\right),\ \ddot{y}_i=0, \ddot{z}_i=-(l-l_i)\left(\ddot{\theta}\sin\theta+\dot{\theta}^2\cos\theta\right) \tag{78}$$

Now let us calculate the relevant terms in the compulsion function formula (11), and we get

$$\boldsymbol{\omega}=\omega\boldsymbol{k} \tag{79}$$

$$\tilde{\dot{\boldsymbol{r}}}_i'=l_i\dot{\theta}\cos\theta\boldsymbol{i}-(l-l_i)\dot{\theta}\sin\theta\boldsymbol{k} \tag{80}$$

$$\tilde{\ddot{\boldsymbol{r}}}_i'=l_i\left(\ddot{\theta}\cos\theta-\dot{\theta}^2\sin\theta\right)\boldsymbol{i}-(l-l_i)\left(\ddot{\theta}\sin\theta+\dot{\theta}^2\cos\theta\right)\boldsymbol{k} \tag{81}$$

$$\boldsymbol{F}_i=-\mathrm{d}m_i g\boldsymbol{k} \tag{82}$$

$$\boldsymbol{F}_{\mathrm{e}i}^1=-\mathrm{d}m_i\boldsymbol{a}_{\mathrm{e}i}=\mathrm{d}m_i x_i\omega^2\boldsymbol{i}=\mathrm{d}m_i l_i\omega^2\sin\theta\boldsymbol{i} \tag{83}$$

$$\boldsymbol{F}_{\mathrm{c}i}^1=-2\mathrm{d}m_i\boldsymbol{\omega}\times\tilde{\dot{\boldsymbol{r}}}_i'=-2\mathrm{d}m_i l_i\omega\dot{\theta}\cos\theta\boldsymbol{j} \tag{84}$$

In addition, the elastic force $\boldsymbol{F}_{BD}$ of the spring and the relative acceleration of its action point B are

$$\boldsymbol{F}_{BD}=-x_B\boldsymbol{i}=-kl\sin\theta\boldsymbol{i} \tag{85}$$

$$\tilde{\ddot{\boldsymbol{r}}}_B'=l\left(\ddot{\theta}\cos\theta-\dot{\theta}^2\sin\theta\right)\boldsymbol{i} \tag{86}$$

Thus, we have

$$\begin{aligned}
&\sum_{i=1}^{N}\left(\boldsymbol{F}_i+\boldsymbol{F}_{\mathrm{e}i}^1+\boldsymbol{F}_{\mathrm{c}i}^1\right)\cdot\tilde{\ddot{\boldsymbol{r}}}_i'\\
&=\sum_{i=1}^{N}\left\{\left(-\mathrm{d}m_i g\boldsymbol{k}_i+\mathrm{d}m_i l_i\omega^2\sin\theta\boldsymbol{i}-2\mathrm{d}m_i l_i\omega\dot{\theta}\cos\theta\boldsymbol{j}\right)\cdot\left[l_i\left(\ddot{\theta}\cos\theta-\dot{\theta}^2\sin\theta\right)\boldsymbol{i}-(l-l_i)\left(\ddot{\theta}\sin\theta+\dot{\theta}^2\cos\theta\right)\boldsymbol{k}\right]\right\}\\
&\quad+(-kl\sin\theta\boldsymbol{i})\cdot\left(l\ddot{\theta}\cos\theta-l\dot{\theta}^2\sin\theta\right)\boldsymbol{i}\\
&=\int_0^l\left[\frac{m}{l}\omega^2 l_i^2\sin\theta\left(\ddot{\theta}\cos\theta-\dot{\theta}^2\sin\theta\right)+\frac{m}{l}g(l-l_i)\left(\ddot{\theta}\sin\theta+\dot{\theta}^2\cos\theta\right)\right]\mathrm{d}l_i-kl^2\sin\theta\left(\ddot{\theta}\cos\theta-\dot{\theta}^2\sin\theta\right)\\
&=\left(\frac{1}{3}ml^2\omega^2-kl^2\right)\left(\ddot{\theta}\cos\theta-\dot{\theta}^2\sin\theta\right)\sin\theta+\frac{1}{2}mgl\left(\ddot{\theta}\sin\theta+\dot{\theta}^2\cos\theta\right)
\end{aligned} \tag{87}$$

By substituting equations (75) and (87) into equation (11), we get

$$Z_\mathrm{r}=\frac{1}{6}ml^2\ddot{\theta}^2-\left(\frac{1}{3}ml^2\omega^2-kl^2\right)\left(\ddot{\theta}\cos\theta-\dot{\theta}^2\sin\theta\right)\sin\theta-\frac{1}{2}mgl\left(\ddot{\theta}\sin\theta+\dot{\theta}^2\cos\theta\right)+\cdots \tag{88}$$

To calculate the Gaussian variation $\delta_\mathrm{G}Z_\mathrm{r}$ and set it to zero, we get

$$\delta_\mathrm{G}Z_\mathrm{r}=\left[\frac{1}{3}ml^2\ddot{\theta}-\left(\frac{1}{3}ml^2\omega^2-kl^2\right)\sin\theta\cos\theta-\frac{1}{2}mgl\sin\theta\right]\delta_\mathrm{G}\ddot{\theta}=0 \tag{89}$$

Due to the arbitrariness of $\delta_\mathrm{G}\ddot{\theta}$, we get

$$\frac{1}{3}ml^2\ddot{\theta}-\left(\frac{1}{3}ml^2\omega^2-kl^2\right)\sin\theta\cos\theta-\frac{1}{2}mgl\sin\theta=0 \tag{90}$$

i.e.,

$$\ddot{\theta}-\left(\omega^2-\frac{3k}{m}\right)\sin\theta\cos\theta-\frac{3g}{2l}\sin\theta=0 \tag{91}$$

This is the dynamic equation of the relative motion of rod AB. It is consistent with the results obtained using the Lagrange equation in Ref. [29].

7 Conclusion

Complex mechanical systems, including the carrier and the carried bodies, are ubiquitous, so their study is significant. Using the theory of analytical mechanics to study the relative motion dynamics of complex systems not only has the unity of expression form but also shows the superiority of analytical mechanics in solving the problems of complex system dynamics. Unlike other differential variational principles, such as d'Alembert-Lagrange's or Jourdain's principle, Gauss principle is an extreme value principle from which the motion of a system can be directly obtained. The work conducted in this article includes the following aspects:

① The Gauss principle of relative motion dynamics and its least compulsion principle were established. Based on the dynamic equation of relative motion and the concept of virtual displacement in acceleration space, the Gauss principle for relative motion dynamics was presented. The compulsion function of relative motion was constructed, and it was proved that real motion makes the compulsion function yield its extreme value under Gaussian variation.

② The formulation of acceleration energy and compulsion function of relative motion was presented. The acceleration energy and compulsion function of relative motion were obtained when the carried rigid body was in planar motion.

③ The Appell, Lagrange, and Nielsen forms in generalized coordinates of the Gauss principle of relative motion were derived. According to the above documents of the Gauss principle, using the Lagrange multiplier method, we established the Appell equation, the Lagrange equation, and the Nielsen equation with multipliers for relative motion dynamics.

References

[1] Mei F X, Wu H B, Li Y M. *A Brief History of Analytical Mechanics*[M]. Beijing: Science Press, 2019(Ch).

[2] Chen B. *Analytical Dynamics*[M]. 2nd Ed. Beijing: Peking University Press, 2012(Ch).

[3] Mei F X. *Analytical Mechanics* II[M]. Beijing: Beijing Institute of Technology Press, 2013(Ch).

[4] Udwadia F E, Kalaba R E. *Analytical Dynamics — A New Approach*[M]. New York: Cambridge University Press, 2008.

[5] Попов Е П. *Operating Robot Dynamics and Algorithm*[M]. Yu L J, Chen X J trans. Beijing: Mechanical Industry Press, 1983(Ch).

[6] Liu Y Z, Pan Z K, Ge X S. *Dynamics of Multibody Systems*[M]. 2nd Ed. Beijing: Higher Education Press, 2014(Ch).

[7] Liu Y Z. Dynamic modeling of multi-body system based on Gauss's principle[J]. *Chinese Journal of Theoretical and Applied Mechanics*, 2014, **46**(6): 940-945(Ch).

[8] Liu Y Z. Dynamical modeling of a net system of rods based on Gauss's principle[J]. *Journal of Dynamics and Control*, 2018, **16**(4): 289-294(Ch).

[9] Yao W L, Liu Y P, Yang L S. Dynamic modeling of nonideal system based on Gauss's principle[J]. *Chinese Journal of Theoretical and Applied Mechanics*, 2020, **52**(4): 945-953(Ch).

[10] Yao W L, Yang L S, Song K W, *et al*. Optimization method for dynamics of non-holonomic system based on Gauss's principle[J]. *Acta Mechanica Sinica*, 2020, **36**(5): 1133-1141.

[11] Yao W L, Yang L S, Guo M M. Gauss optimization method for the dynamics of unilateral contact of rigid multibody systems[J]. *Acta Mechanica Sinica*, 2021, **37**(3): 494-506.

[12] Orsino R M M. Extended constraint enforcement formulations for finite-DOF systems based on Gauss's principle of least constraint [J]. *Nonlinear Dynamics*, 2020, **101**(4): 2577-2597.

[13] Yao W L, Dai B Q. Gauss principle of least constraint in generalized coordinates and its generalization[J]. *Mechanics in Engineering*, 2014, **36**(6): 779-782, 785(Ch).

[14] Yang L S, Yao W L, Xue S F. Application of particle swarm optimization on the multi-body system dynamics with singular positions [J]. *Acta Scientiarum Naturalium Universitatis Pekinensis*, 2021, **57**(5): 795-803(Ch).

[15] Xue Y, Liu Y Z, Chen L Q. On analytical mechanics for a super-thin elastic rod[J]. *Chinese Journal of Theoretical and Applied Mechanics*, 2005, **37**(4): 485-493(Ch).

[16] Xue Y, Weng D W. Gauss principle for a super-thin elastic rod dynamics[J]. *Acta Physica Sinica*, 2009, **58**(1): 34-39(Ch).

[17] Liu Y Z, Xue Y. Dynamical model of Cosserat elastic rod based on Gauss principle[J]. *Acta Physica Sinica*, 2015, **64**(4): 044601(Ch).

[18] Xue Y, Qu J L, Chen L Q. Gauss principle of least constraint for Cosserat growing elastic rod dynamics[J]. *Applied Mathematics and*

Mechanics, 2015, **36**(7): 700-709(Ch).

[19] Зегжда СА, Солтаханов ШХ, Юшков МП. *Equations of Motion for Nonholonomic Systems and Variational Principles of Mechanics*: *A New Class of Control Problems*[M]. Mei F X trans. Beijing: Beijing Institute of Technology Press, 2007(Ch).

[20] Li B L, Song F P. On Gauss's principle of least constraint for impulsive motion[J]. *Journal of Hunan University*, 1995, **22**(4): 23-28 (Ch).

[21] Ivanov A P. On the variational formulation of the dynamics of systems with friction[J]. *Regular and Chaotic Dynamics*, 2014, **19**(1): 100-115.

[22] Wang L S, Pao Y H. Jourdain's variational equation and Appell's equation of motion for nonholonomic dynamical systems[J]. *American Journal of Physics*, 2003, **71**(1): 72-82.

[23] Yan C C. Hamilton's principle and Schrodinger's equation derived from Gauss's principle of least squares[J]. *Foundations of Physics Letters*, 2000, **13**(1): 79-87.

[24] Zegzhda S A, Soltakhanov S K. Application of the generalized Gaussian principle to the problem of damping vibrations of mechanical systems[J]. *Journal of Computer and Systems Sciences International*, 2010, **49**(2): 186-191.

[25] Lewis A D. The geometry of the Gibbs-Appell equations and Gauss's principle of least constraint[J]. *Reports on Mathematical Physics*, 1996, **38**(1): 11-28.

[26] Kalaba R E, Natsuyama H H, Udwadia F E. An extension Gauss's principle of least constraint[J]. *International Journal of General Systems*, 2004, **33**(1): 63-69.

[27] Zhang Y, Chen X Y. The generalized Gauss principle for mechanical system with variable mass and its generalization to higher order nonholonomic systems[J]. *Chinese Journal of Theoretical and Applied Mechanics*, 2022, **54**(10): 2883-2891(Ch).

[28] Zhang Y, Song C J, Zhai X H. Generalized Gauss principle of least compulsion for variable acceleration dynamical systems[J]. *Chinese Journal of Theoretical and Applied Mechanics*, 2023, **55**(5): 1174-1180(Ch).

[29] Mei F X. *Analytical Mechanical Special Problem*[M]. Beijing: Beijing Institute of Technology Press, 1988(Ch).

[30] Markeyev A P. Approximate equations of rotational motion of a rigid body carrying a movable point mass[J]. *Journal of Applied Mathematics and Mechanics*, 2013, **77**(2): 137-144.

[31] Prioroc C L, Mikkola S. Simple algorithms for relative motion of satellites[J]. *New Astronomy*, 2015, **34**: 41-46.

[32] Vepa R. Feedback tracking control of optimal reference trajectories for spacecraft relative motion[J]. *Advances in Space Research*, 2022, **69**(9): 3478-3489.

[33] Burnett E R, Schaub H. Geometric perspectives on fundamental solutions in the linearized satellite relative motion problem[J]. *Acta Astronautica*, 2022, **190**: 48-61.

[34] Whittaker E T. *A Treatise on the Analytical Dynamics of Particles and Rigid Bodies*[M]. Fourth Ed. Cambridge: Cambridge University Press, 1952.

[35] Lure A I. *Analytical Mechanics*[M]. Moscow: GIFML, 1961 (in Russian).

[36] Mei F X, Wu H B. *Dynamics of Constrained Mechanical Systems*[M]. Beijing: Beijing Institute of Technology Press, 2009.

[37] Liu G L, Qiao Y F, Zhang J F, *et al*. Relative motion dynamics of variable mass non-holonomic mechanical systems[J]. *Acta Mechanica Sinica*, 1989, **21**(6): 742-748(Ch).

[38] Chen X W, Luo S K. Integration method for the dynamics equation of relative motion of variable mass nonlinear nonholonomic system[J]. *Applied Mathematics and Mechanics*, 1998, **19**(5): 479-488.

[39] Luo S K. Integral theory for the dynamics of nonlinear nonholonomic system in noninertial reference frames[J]. *Applied Mathematics and Mechanics*, 1993, **14**(10): 907-918.

[40] Zhang X W. The universal forms of the dynamic equations of holonomic mechanical system in relative motion[J]. *Acta Physica Sinica*, 2006, **55**(6): 2669-2675(Ch).

[41] Xie Y L, Jia L Q, Luo S K. Special Lie symmetry and Hojman conserved quantity of Appell equations in a dynamical system of relative motion[J]. *Chinese Physics B*, 2011, **20**(1): 010203.

[42] Xie Y L, Jia L Q, Yang X F. Lie symmetry and Hojman conserved quantity of Nielsen equation in a dynamical system of the relative motion[J]. *Acta Physica Sinica*, 2011, **60**(3): 030201(Ch).

[43] Mei F X, Wu H B. Lagrange symmetry for a dynamical system of relative motion[J]. *Acta Physica Sinica*, 2009, **58**(9): 5919-5922 (Ch).

[44] Zhang Y, Mei F X. Algebraic structure of the dynamical equations of holonomic mechanical system in relative motion[J]. *Journal of Beijing Intitute of Technology* (*English Edition*), 1998, **7**(1): 12-18.

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