



Article ID 1007-1202(2024)03-0284-09 DOI <https://doi.org/10.1051/wujns/2024293284>

Cite this article: WEI Xingmin, LI Dekui. Control and Stabilization of Chaotic System Based on Linear Feedback Control Method[J]. *Wuhan Univ J of Nat Sci*, 2024, 29(3): 284-292.

Control and Stabilization of Chaotic System Based on Linear Feedback Control Method

□ **WEI Xingmin, LI Dekui[†]**

School of Public Health, Gansu University of Chinese Medicine, Lanzhou 730000, Gansu, China

© Wuhan University 2024

Abstract: In this paper, two kinds of chaotic systems are controlled respectively with and without time-delay to eliminate their chaotic behaviors. First of all, according to the first-order approximation method and the stabilization condition of the linear system, one linear feedback controller is structured to control the chaotic system without time-delay, its chaotic behavior is eliminated and stabilized to its equilibrium. After that, based on the first-order approximation method, the Lyapunov stability theorem, and the matrix inequality theory, the other linear feedback controller is structured to control the chaotic system with time-delay and make it stabilized at its equilibrium. Finally, two numerical examples are given to illustrate the correctness and effectiveness of the two linear feedback controllers.

Key words: chaotic system without time-delay; chaotic system with time-delay; stabilization; linear feedback control method

CLC number: O231

0 Introduction

Due to find that chaotic phenomena commonly exist in nonlinear dynamical systems, in recent years, chaotic and hyperchaotic systems have been studied in many scientific realms, such as mathematics, physics, chemistry, biology, electronic, neural network mechanics, and so forth. Chaos has the potential application value in secure communication because of some particular properties, including strange attractor, ergodicity of phase space, and sensitive dependence on its initial condition, etc.; in this situation, chaotic behavior of the nonlinear system is desired, but in other situation chaotic behavior is undesired because of its harmfulness for people's lives, such as chaos in mechanical systems results in increased noise, increased component cracks, friction and wear, and reduced overall system performance, the harmful chaos need to be suppressed as much as possible or eliminated totally. However, people had been thinking it was impossible to control or apply chaos for a long time because of its extreme complexity.

Until 1990, Ott, Grebogi and Yorke have proposed a chaos control method, which is simply called OGY^[1]; they have used OGY to control a chaotic system successfully. It has opened the door of chaos control, but it is a pity that OGY requires a priori information of the chaotic system. Since then, many chaos control methods are proposed and ap-

Received date: 2023-06-28

Foundation item: Supported by the National Natural Science Foundation of China (61863022), the Natural Science Foundation of Gansu Province(20JR10RA329), and Scientific Research and Innovation Fund Project of Gansu University of Chinese Medicine in 2019 (2019KCYB-10)

Biography: WEI Xingmin, male, Associate professor, research direction: differential equation and dynamical system. E-mail: weixingmin@gszy.edu.cn

[†] Corresponding author. E-mail: dkli2009@163.com

This is an Open Access article distributed under the terms of the Creative Commons Attribution License (<https://creativecommons.org/licenses/by/4.0>), which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

plied, such as the targeting method^[2,3], time-delayed feedback control method^[4-6], stable manifold method^[7-11] etc. On the other hand, many chaos control schemes have been proposed, such as adaptive control^[12], sliding mode control^[13], fuzzy logic control^[14], neural network control^[15] and so on.

In the above chaotic control methods, the targeting method is the direct promotion method of OGY, it takes the subtraction value of the target output signal and the present output signal as the coming origin of the feedback signal. The time-delay feedback control takes the subtraction value of the past output signal and the present output signal as the coming origin of the feedback signal. The targeting and the time-delay feedback control methods all belong to the linear method, but the stable manifold method is a nonlinear method.

Control and synchronization of chaotic system has been largely studied in recent years. The authors have studied the stabilization and synchronization of a class of chaotic systems with both discrete and distributed time-varying delays by periodically intermittent control^[16]. The stabilization of nonlinear systems with time-delay is realized under flexible delayed impulsive control^[17]. The size of the delay can be smaller or larger than the impulsive intervals, and there is no magnitude relationship between the delay in continuous flow and impulsive delay. Based on the Lyapunov stability theorem, the nonlinear controllers are obtained to realize the dislocated function projective partial synchronization and the general dislocated function projective partial synchronization of the two dynamical systems, respectively^[18].

In most situations, we find that the stabilization or synchronization of chaotic systems were achieved by imposing some very complex nonlinear controllers on chaotic systems. However, the complex nonlinear controllers make the chaotic circuit design very difficult, leading to its practical application being inconvenient^[16-21]. The linear feedback controller has an advantage of structure and form, and circuit realization is simple, so people should use the linear feedback controller to control and stabilize chaotic system as much as possible. In this paper, we use the linear feedback control method to study the stabilization of the two chaotic systems with and without time delay.

The rest of this paper is arranged as follows: the stabilization of the chaotic system without time-delay is realized by the linear feedback controller in Section 1, and in Section 2, we studied the stabilization of the chaotic system with time-delay. In Section 3, numerical examples are presented to illustrate the effectiveness and correctness of the two linear feedback controllers constructed in Section 1 and 2. Finally, conclusion will be presented in Section 4.

1 Stabilization of the Chaotic System without Time-Delay

In fact, the state equations of many chaotic or hyperchaotic systems can be divided into two linear and nonlinear parts: such as the Lorenz system, the Chen system, the Chua circuit system, and the Lorenz hyperchaotic system etc. So the state equations of the chaotic or hyperchaotic systems can be written as follows,

$$\dot{\mathbf{x}} = \mathbf{D}\mathbf{x} + \mathbf{f}(\mathbf{x}) \quad (1)$$

where $\mathbf{x} = (x_1, x_2, x_3, \dots, x_n)^T$ is state vector, $\mathbf{D}\mathbf{x}$ and $\mathbf{f}(\mathbf{x})$ are linear and nonlinear parts of the system, respectively. If let Ω is the chaotic region of the system, then its equilibrium point $\bar{\mathbf{x}}$ must satisfy $\bar{\mathbf{x}} \in \Omega$. Our goal is to stabilize chaotic trajectories of the system (1) to the point $\bar{\mathbf{x}}$ by the following linear feedback controller,

$$\mathbf{u} = \mathbf{R}(\mathbf{x} - \bar{\mathbf{x}}) \quad (2)$$

where $\mathbf{u}$ is the controller added on the system (1). Because $\bar{\mathbf{x}}$ is an equilibrium point of the chaotic system (1), $\bar{\mathbf{x}}$ is a solution of the equation (1), and we have

$$\dot{\bar{\mathbf{x}}} = \mathbf{D}\bar{\mathbf{x}} + \mathbf{f}(\bar{\mathbf{x}}) \quad (3)$$

In order to realize the goal, we define the error of the system and its equilibrium point as follows,

$$\mathbf{e} = \mathbf{x} - \bar{\mathbf{x}} + \mathbf{u} \quad (4)$$

According to (1), (2), (3) and (4), we can get the error system as follows,

$$\dot{\mathbf{e}} = (\mathbf{D} + \mathbf{R})\mathbf{e} + \mathbf{f}(\mathbf{x}) - \mathbf{f}(\bar{\mathbf{x}}) = (\mathbf{D} + \mathbf{R})\mathbf{e} + \mathbf{f}(\mathbf{e} + \bar{\mathbf{x}}) - \mathbf{f}(\bar{\mathbf{x}}) = (\mathbf{D} + \mathbf{R})\mathbf{e} + \mathbf{F}(\mathbf{e}) \quad (5)$$

where $\mathbf{F}(\mathbf{e}) = \mathbf{f}(\mathbf{e} + \bar{\mathbf{x}}) - \mathbf{f}(\bar{\mathbf{x}})$.

Because the zero point of the error system (5) is its fixed point, $\mathbf{F}(0) = 0$. Taylor series of the nonlinear function $\mathbf{F}(\mathbf{e})$ at the zero point is given as follows,

$$F(e) = F'(0)e + [\text{H.O.T.}] \quad (6)$$

where [H.O.T.] is the high order term of Taylor series with respect to e . Let $a = F'(0)$, we can obtain the linearized system of the error system (5) as follows,

$$\dot{e} = (D + R + aI)e \quad (7)$$

where I is an identity matrix.

According to the stable theory of the linear system, we can give the following conclusion for stabilizing the system (1).

Theorem 1 If there exists a real matrix R which makes all the eigenvalues of the matrix $D + R + aI$ have negative real parts, then the chaotic system (1) can be stabilized to the equilibrium point $\bar{x}$ as the trajectories are very close to $\bar{x}$.

Remark 1 Theorem 1 shows that the chaotic system (1) can be stabilized to the equilibrium point $\bar{x}$ as the chaotic trajectories are very close to $\bar{x}$. Can the trajectories be very close to the point $\bar{x}$ under the arbitrary initial condition? The answer is definite. Because of the ergodicity of the chaotic trajectories in phase space, the trajectories can close to any point of the chaotic region.

In this section, we give the linear feedback control strategy to stabilize the chaotic system without time-delay. However, the time-delay phenomenon is universal existing in complex system, so we will necessarily study the stabilization of the chaotic system with time-delay in Section 2.

2 Stabilization of the Chaotic System with Time-Delay

We consider a chaotic system with time-delay, and its dynamical equation can be written as the following vector form,

$$\dot{y} = Ay + By(t - \tau) + f(y) + g(y(t - \tau)) \quad (8)$$

where Ay and $By(t - \tau)$ are linear parts without and with time-delay, respectively. $f(y)$ and $g(y(t - \tau))$ are nonlinear parts without and with time-delay, respectively, $y = (y_1, y_2, y_3, \dots, y_n)^T$ is the state vector. Let Ω is chaotic region, the equilibrium point $\bar{y}$ of the system satisfies $\bar{y} \in \Omega$. Our goal is to stabilize the chaotic trajectories of the system (8) to $\bar{y}$ by the following linear feedback controller,

$$u = R(y - \bar{y}) \quad (9)$$

where R is an undetermined real matrix. Because $\bar{y}$ is an equilibrium point of the chaotic system (8), $\bar{y}$ satisfies the equation (8), and we can get

$$\dot{\bar{y}} = A\bar{y} + B\bar{y}(t - \tau) + f(\bar{y}) + g(\bar{y}(t - \tau)) \quad (10)$$

We impose the controller u on the system (8), let the error $e = y - \bar{y}$ of the systems (8) and (10), the error system is described as follows,

$$\begin{aligned} \dot{e} &= (A + R)e + Be(t - \tau) + f(y) - f(\bar{y}) + g(y(t - \tau)) - g(\bar{y}(t - \tau)) \\ &= (A + R)e + Be(t - \tau) + F(e) + G(e(t - \tau)) \end{aligned} \quad (11)$$

where,

$$\begin{cases} F(e) = f(y) - f(\bar{y}) = f(e + \bar{y}) - f(\bar{y}) \\ G(e(t - \tau)) = g(y(t - \tau)) - g(\bar{y}(t - \tau)) = g(e(t - \tau) + \bar{y}(t - \tau)) - g(\bar{y}(t - \tau)) \end{cases}$$

Thus, stabilizing chaotic system (8) to its equilibrium point $\bar{y}$ will be changed into $\|e\| \rightarrow 0$ as $t \rightarrow +\infty$. It is obvious that the zero point is the fixed point of the nonlinear function $F(e) + G(e(t - \tau))$. According to Taylor's theorem of multivariate function, Taylor's expansion of $F(e) + G(e(t - \tau))$ at the zero point is shown as follows,

$$F(e) + G(e(t - \tau)) = Ce + [\text{H.O.T.}]_1 + De(t - \tau) + [\text{H.O.T.}]_2 \quad (12)$$

where, $C = F'(e) \in \mathbb{R}^{n \times n}$, $D = G'(e(t - \tau)) \in \mathbb{R}^{n \times n}$, $[\text{H.O.T.}]_1$ and $[\text{H.O.T.}]_2$ are the high-order expansion terms of $F(e)$ and $G(e(t - \tau))$, respectively, so we can obtain the linearization equation of the controlled error system (11) at the zero point as follows,

$$\dot{e} = (A + R + CI)e + (B + DI)e(t - \tau) \quad (13)$$

In order to make $\|e\| \rightarrow 0$ as $t \rightarrow +\infty$, what conditions need to be satisfied? The answer will be given in Theorem 2.

Next, we introduce a lemma, which is necessary to prove theorem 2 firstly, and then present the proof process of Theorem 2.

Lemma 1 Let $x, y \in \mathbb{R}^n$ are any vectors, the following inequality

$$2x^T y \leq x^T x + y^T y \quad (14)$$

holds.

Theorem 2 If there is a real matrix $R \in \mathbb{R}^{n \times n}$ in (9) which makes error system (13) satisfy the following equation,

$$P(A + R + CI) + (A + R + CI)^T P = -2I \quad (15)$$

where P is a positive matrix and satisfies the normal inequality as follows,

$$\|P\| \|B + DI\| < 1 \quad (16)$$

then $\|e\| \rightarrow 0$ as $t \rightarrow +\infty$.

Proof In order to prove that the zero solution of the error system (13) is asymptotically stable, a Lyapunov function is constructed as follows,

$$V(t) = e^T(t)Pe(t) + \int_{t-\tau}^t e^T(s)e(s)ds \quad (17)$$

It is obvious that $V(t)$ is positive. In the meantime, the derivative of the Lyapunov function (17) is described as follows,

$$\dot{V}(t) = e^T(t)[(A + R + CI)^T P + P(A + R + CI)]e(t) + 2e^T(t - \tau)(B + DI)^T Pe(t) + e^T(t)e(t) - e^T(t - \tau)e(t - \tau) \quad (18)$$

According to (14) and (16), the equation (18) changes into the following inequality,

$$\begin{aligned} \dot{V}(t) &\leq -2e^T(t)e(t) + e^T(t - \tau)e(t - \tau) + e^T(t)P^T(B + DI)(B + DI)^T Pe(t) + e^T(t)e(t) - e^T(t - \tau)e(t - \tau) \\ &= -e^T(t)e(t) + e^T(t)P^T(B + DI)(B + DI)^T Pe(t) \end{aligned} \quad (19)$$

According to (16), we can obtain a relational expression as follows,

$$[P^T(B + DI)(B + DI)^T P]^{1/2} = \|P^T(B + DI)\| \leq \|P\| \|B + DI\| < 1 \quad (20)$$

We square both sides of the inequality (20) and then substitute the square result into (19), which we easily find $\dot{V}(t) < 0$. According to the Lyapunov stability theorem, the linear error system (13) is asymptotically stable, that is $\|e\| \rightarrow 0$ as $t \rightarrow +\infty$. The end of the proof is here.

Thus, when trajectories of the chaotic system (8) are close to the equilibrium point $\bar{y}$, the chaotic system (8) can be stabilized to $\bar{y}$ under the linear controller (9).

3 Numeral Simulation

In this section, we simulate the stabilization of the chaotic system without and with time-delay, respectively; they are shown in Section 3.1 and 3.2 to illustrate the correctness of Theorem 1 and Theorem 2, respectively.

3.1 Simulation for the Chaotic System without Time-Delay

We take the three eight hyperchaotic system^[22] as example of a chaotic system without time-delay to carry out the simulation; its dynamical equations are shown as follows,

$$\begin{cases} \dot{x}_1 = a(x_2 - x_1) \\ \dot{x}_2 = bx_1 - x_1x_3 - x_2 + cx_4 \\ \dot{x}_3 = x_1x_2 - \theta x_3 \\ \dot{x}_4 = -kx_1 \end{cases} \quad (21)$$

where x_1, x_2, x_3 and x_4 are state variables, and a, b, c, θ , and k are system parameters. When $a=10, b=28, c=2, \theta=4$, and $k=8$, the system (21) is in a hyperchaotic state and has some attractors presented in Fig.1.

Without loss of generality, we stabilize the hyperchaotic system (21) to its only equilibrium point $(0, 0, 0, 0)$. According to the equations (1), (2), (3), (4) and (5), the error system of the controlled system (21) can be written as follows,

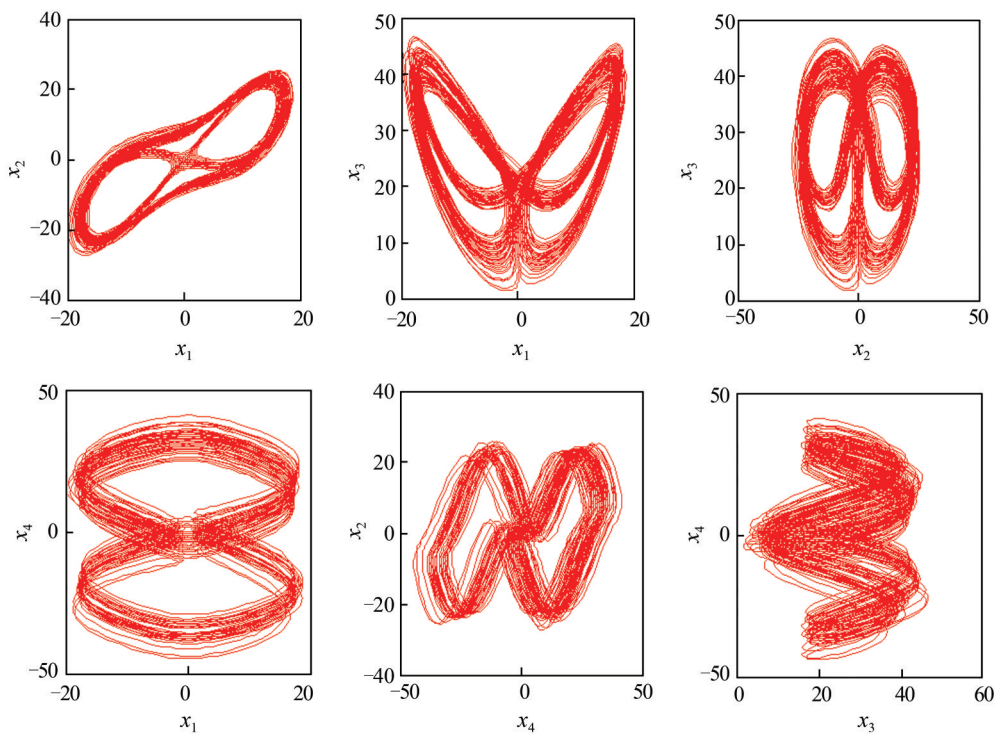


Fig.1 The attractors of the system (21)

$$\begin{pmatrix} \dot{e}_1 \\ \dot{e}_2 \\ \dot{e}_3 \\ \dot{e}_4 \end{pmatrix} = \begin{pmatrix} -a+r_1 & a & 0 & 0 \\ b & -1+r_2 & 0 & c \\ 0 & 0 & -\theta+r_3 & 0 \\ -k & 0 & 0 & r_4 \end{pmatrix} \begin{pmatrix} e_1 \\ e_2 \\ e_3 \\ e_4 \end{pmatrix} + \begin{pmatrix} 0 \\ -e_1 e_3 \\ e_1 e_2 \\ 0 \end{pmatrix} \quad (22)$$

According to Taylor's theorem of multivariate function, the Taylor expansion form of the nonlinear function $(0, -e_1 e_3, e_1 e_2, 0)^T$ at the zero point is described as follows,

$$(0, -e_1 e_3, e_1 e_2, 0)^T = (0, 0, 0, 0)^T + [\text{H.O.T.}] \quad (23)$$

Based on the formula (7), the Linearization system of the error system (22) at the zero point is shown as follows,

$$\begin{pmatrix} \dot{e}_1 \\ \dot{e}_2 \\ \dot{e}_3 \\ \dot{e}_4 \end{pmatrix} = \begin{pmatrix} -a+r_1 & a & 0 & 0 \\ b & -1+r_2 & 0 & c \\ 0 & 0 & -\theta+r_3 & 0 \\ -k & 0 & 0 & r_4 \end{pmatrix} \begin{pmatrix} e_1 \\ e_2 \\ e_3 \\ e_4 \end{pmatrix} \quad (24)$$

When $a=10$, $b=28$, $c=2$, $\theta=4$, and $k=8$, the characteristic equation of coefficient matrix in equation (24) is shown as follows,

$$\begin{cases} \lambda_1 = r_3 - 4 \\ \lambda^3 + p_2 \lambda^2 + p_1 \lambda + p_0 = 0 \end{cases} \quad (25)$$

where the coefficients of the characteristic equation (25) are shown as follows,

$$\begin{cases} p_0 = r_1 r_4 + 10 r_2 r_4 - r_1 r_2 r_4 + 270 r_4 - 160 \\ p_1 = r_1 r_2 + r_1 r_4 + r_2 r_4 - r_1 - 10 r_2 - 11 r_4 - 270 \\ p_2 = -(r_1 + r_2 + r_4 - 11) \end{cases} \quad (26)$$

According to the Routh-Hurwitz theorem, the sufficient condition of all the eigenvalues with negative real parts is described as follows,

$$\begin{cases} r_3 < 4 \\ p_0 > 0 \\ p_1 p_2 - p_0 > 0 \\ p_2 > 0 \end{cases} \quad (27)$$

By solving the inequality (27), we can obtain one of the solutions shown as follows,

$$\begin{cases} r_1 = -20 \\ r_2 = -10 \\ r_3 = 0 \\ r_4 = -10 \end{cases} \quad (28)$$

According to the controller (2) and equations (28), the controlled system (21) can be described as follows,

$$\begin{cases} \dot{x}_1 = 10(x_2 - x_1) - 20x_1, \\ \dot{x}_2 = 28x_1 - x_1x_3 - x_2 + 2x_4 - 10x_2, \\ \dot{x}_3 = x_1x_2 - 4x_3, \\ \dot{x}_4 = -8x_1 - 10x_4. \end{cases} \quad (29)$$

The time series curves of the system (21) at the controlled front and rear are shown in Fig. 2, where the time series curves of the system (21) are shown before the 20th second, and the ones of the system (29) are shown after the 20th second. The 20th second is a time cutoff for whether the system (21) is controlled.

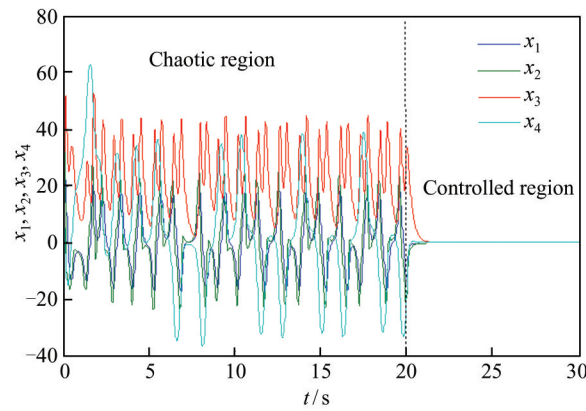


Fig.2 The time series curves of the hyperchaotic system (21) at controlled front and rear

From Fig.2, we can find that the system is in a hyperchaotic state in a chaotic region. Since the 20th second, the system has been controlled and comes into the controlled region; its time series curves quickly tend to be the only equilibrium point $(0, 0, 0, 0)$. All these illustrate that the correctness of stabilizing the chaotic system without time-delay in Theorem 1.

3.2 Simulation for the Chaotic System with Time-Delay

Rucklidge system with time-delay, whose bifurcation characteristics have been studied^[23] is introduced to illustrate the correctness of the Theorem 2. Its differential equations are written as follows,

$$\begin{cases} \dot{y}_1 = -ay_1 + by_2 - y_2y_3 \\ \dot{y}_2 = y_1(t - \tau) \\ \dot{y}_3 = y_2^2 - y_3 \end{cases} \quad (30)$$

where y_1, y_2 and y_3 are state variables and a, b are system parameters.

If $b \leq 0$, then the system (30) has one equilibrium point $(0, 0, 0)$, otherwise, it has three equilibrium points $(0, 0, 0)$, $(0, \sqrt{b}, b)$ and $(0, -\sqrt{b}, b)$. When $a=2$, $b=7.7$ and $\tau=0.12$, the subtraction value curves of the system (30) are shown in Fig.3 under two different initial values $(y_1(0)=2, y_2(0)=2.5, y_3(0)=1)$ and $(y'_1(0)=2.0001, y'_2(0)=2.5, y'_3(0)=1)$.

When the initial values of the system (30) take place subtle change; From Fig.3, we find the subtraction value curves cannot stabilize to zero. It indicates that the system (30) has a high sensitivity for the initial value. Hence, the Rucklidge system with time-delay is a chaotic system, and its chaotic attractors are presented in Fig. 4.

The system (30) can be written as follows,

$$\begin{pmatrix} \dot{y}_1 \\ \dot{y}_2 \\ \dot{y}_3 \end{pmatrix} = \begin{pmatrix} -2 & 7.7 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} y_1(t - \tau) \\ y_2(t - \tau) \\ y_3(t - \tau) \end{pmatrix} + \begin{pmatrix} -y_2y_3 \\ 0 \\ y_2^2 \end{pmatrix} \quad (31)$$

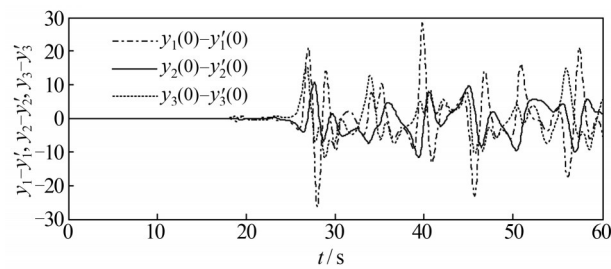


Fig.3 The subtraction value curves of the system (30)

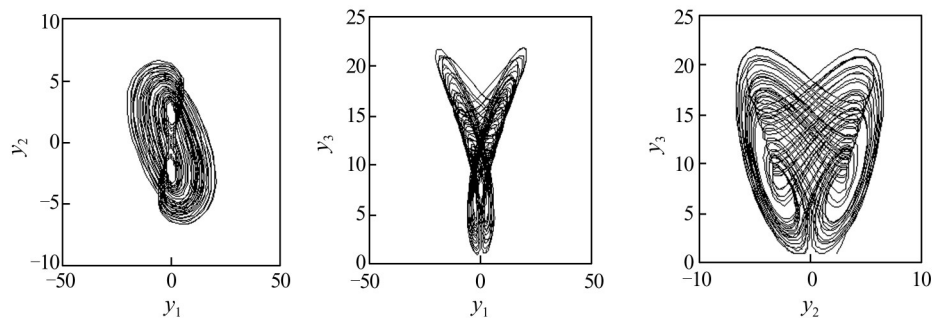


Fig.4 The chaotic attractors of the system (30)

When $a=2$, $b=7.7$ and $\tau=0.12$, the chaotic system (30) has three equilibrium points $(0,0,0)$, $(0, \sqrt{7.7}, 7.7)$ and $(0, -\sqrt{7.7}, 7.7)$. Without loss of generality, we arbitrarily select an equilibrium point $\bar{x}=(0, \sqrt{7.7}, 7.7)$. According to (11), when the system (31) is controlled, the error system at the point $\bar{x}=(0, \sqrt{7.7}, 7.7)$ is shown as follows,

$$\begin{pmatrix} \dot{e}_1 \\ \dot{e}_2 \\ \dot{e}_3 \end{pmatrix} = \begin{pmatrix} -2+r_{11} & 7.7+r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & -1+r_{33} \end{pmatrix} \begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} e_1(t-\tau) \\ e_2(t-\tau) \\ e_3(t-\tau) \end{pmatrix} + \begin{pmatrix} -e_2 e_3 - 7.7 e_2 - \sqrt{7.7} e_3 \\ 0 \\ e_2^2 + 2\sqrt{7.7} e_2 \end{pmatrix} \quad (32)$$

According to equations (12) and (32), we can calculate out

$$C = F'(e)|_{e=0} = \begin{pmatrix} 0 & -7.7 & -\sqrt{7.7} \\ 0 & 0 & 0 \\ 0 & 2\sqrt{7.7} & 0 \end{pmatrix}, D = G'(e(t-\tau))|_{e(t-\tau)=0} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad (33)$$

In order to calculate out R in equation (9), we give the norm of the matrix $V \in \mathbb{R}^{m \times n}$ as follows,

$$\|V\| = \max_{1 \leq j \leq n} \left\{ \sum_{i=1}^m |v_{ij}| \right\} \quad (34)$$

According to equation (34) and Theorem 2, we can calculate one positive solution of the matrix P in equation (15) as follows,

$$P = \begin{pmatrix} 0.5 & 0 & 0 \\ 0 & 0.5 & 0 \\ 0 & 0 & 0.5 \end{pmatrix} \quad (35)$$

Substituting equation (35) into equation (15), we can obtain the coefficient matrix of the linear feedback controller (9) as follows,

$$R = \begin{pmatrix} 0 & 0 & \frac{1}{2}\sqrt{7.7} \\ 0 & -2 & -\sqrt{7.7} \\ \frac{1}{2}\sqrt{7.7} & -\sqrt{7.7} & -1 \end{pmatrix} \quad (36)$$

According to equations (31), (9) and (36), the controlled Rucklidge system with time-delay is written as follows,

$$\begin{cases} \dot{y}_1 = -2y_1 + 7.7y_2 - y_2y_3 + \frac{1}{2}\sqrt{7.7}(y_3 - 7.7) \\ \dot{y}_2 = y_1(t - 0.12) - 2(y_2 - \sqrt{7.7}) - \sqrt{7.7}(y_3 - 7.7) \\ \dot{y}_3 = y_2^2 - y_3 + \frac{1}{2}\sqrt{7.7}y_1 - \sqrt{7.7}(y_2 - \sqrt{7.7}) - (y_3 - 7.7) \end{cases} \quad (37)$$

In the following simulation, we observe whether the controlled system (37) can be stabilized to the equilibrium point $\bar{x} = (0, \sqrt{7.7}, 7.7)$. The simulation result is shown in Fig.5, where the time series curves of the system (30) are shown before the 25th second, and the ones of the system (37) are shown after the 25th second. The 25th second is a time cutoff for whether the system (30) is controlled.

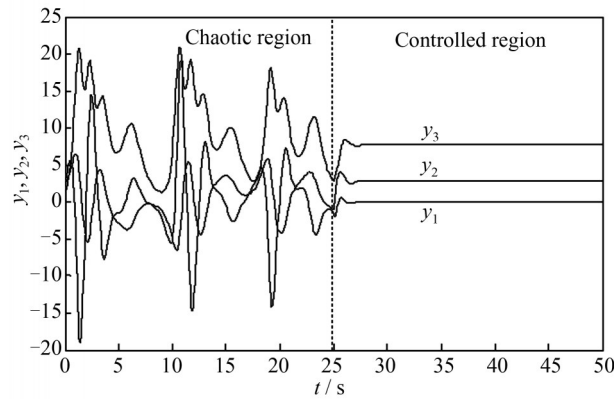


Fig.5 The time series curves of the chaotic system (30) at controlled front and rear

From Fig.5, we can find that the uncontrolled system (30) is in a chaotic state in the chaotic region between 0 to 25 s. Since the 25th second, the system has been controlled and comes into the controlled region; its time series curves quickly tend to the selected equilibrium point $(0, \sqrt{7.7}, 7.7)$. All these illustrate the correctness of stabilizing the chaotic system with time-delay in Theorem 2.

Remark 2 The linear feedback controller (9) constructed for the chaotic system with time-delay (8) is very correct and effective. The controller can quickly make the chaotic system with time-delay stabilize to its any equilibrium point.

4 Conclusion

By applying the nonlinear system's first-order approximation method and the linear system's stabilization condition, we structure a linear feedback controller to control the chaotic systems without time-delay. The chaotic behavior of the chaotic system is eliminated, and the chaotic system can be stabilized to the predetermined equilibriums under the constructed linear feedback controller.

In the case of chaotic system with time-delay, we also structure a linear feedback controller to control and stabilize it by the first-order approximation method of the nonlinear system, the Lyapunov stability theorem, and the matrix inequality theory.

The constructed linear feedback controllers have some advantages, such as the structure is simple, the circuit design is easy, the control process is convenient and the stabilized effect is good. At the same time, the designed linear feedback controller has universality for chaotic systems without and with time-delay.

References

- [1] Ott E, Grebogi C, Yorke J A. Controlling chaos [J]. *Physical Review Letters*, 1990, **64**(11): 1196-1199.

- [2] Shinbrot T, Ott E, Grebogi C, *et al.* Using chaos to direct trajectories to targets [J]. *Physical Review Letters*, 1990, **65**(26): 3215-3218.
- [3] Kostelich E J, Grebogi C, Ott E, *et al.* Higher dimensional targeting [J]. *Physical Review E*, 1993, **47**(1): 305-310.
- [4] Pyragas K. Continuous control of chaos by self-controlling feedback [J]. *Physics Letters A*, 1992, **170**: 421-428.
- [5] Ding Y T, Jiang W H, Wang H B. Delayed feedback control and bifurcation analysis of Rossler chaotic system[J]. *Nonlinear Dynamics*, 2010, **61**(4): 707-715.
- [6] Wang F, Zhang J G. Response and bifurcation of fractional duffing oscillator under combined recycling noise and time-delayed feedback control[J]. *Wuhan University Journal of Natural Sciences*, 2023, **28** (5): 421-432.
- [7] Sakamoto N, Van Der Schaft A J. Analytical approximation methods for the stabilizing solution of the Hamilton-Jacobi equation[J]. *IEEE Transactions on Automatic Control*, 2008, **53**(10): 2335-2350.
- [8] Yuasa Y, Sakamoto N, Umemura Y. Optimal control designs for systems with input saturations and rate limiters[C]//*SICE Annual Conference*, 2010: 2042-2045.
- [9] Sakamoto N. Case studies on the applications of the stable manifold approach for nonlinear optimal control design[J]. *Automatica*, 2013, **49**(2): 568-576.
- [10] Fujimoto R, Sakamoto N. The stable manifold approach for optimal swing up and stabilization of an inverted pendulum with input saturation[C]// *The 18th IFAC World Congress*, 2011, **18**: 8046-8051.
- [11] Habaguchi Y, Sakamoto N, Nagata K. Nonlinear optimal stabilization of unstable periodic orbits [J]. *International Federation of Automatic Control*, 2015, **48-18**: 215-220.
- [12] Vaidyanathan S, Volos C. Analysis and adaptive control of a novel 3-D conservative no-equilibrium chaotic system [J]. *Arch Control Sci*, 2015, **25**(3): 333-353.
- [13] Singh J P, Roy B K. Second order adaptive time varying sliding mode control for synchronization of hidden chaotic orbits in a new uncertain 4-D conservative chaotic system [J]. *Trans Inst Meas Control*, 2018, **40**(13): 3573-3586.
- [14] Ahn C K. Fuzzy delayed output feedback synchronization for time-delayed chaotic systems [J]. *Nonlinear Analysis: Hybrid Systems*, 2010, **4**(1):16-24.
- [15] Ahn C K. Neural network H_∞ chaos synchronization[J]. *Nonlinear Dynamics*, 2010, **60**(3): 295-302.
- [16] Song Q K, Huang T W. Stabilization and synchronization of chaotic systems with mixed time-varying delays via intermittent control with non-fixed both control period and control width[J]. *Neurocomputing*, 2015, **154**: 61-69.
- [17] Chen X Y, Liu Y, Ruan Q H, *et al.* Stabilization of nonlinear time-delay systems: Flexible delayed impulsive control[J]. *Applied Mathematical Modelling*, 2023, **114**: 488-501.
- [18] Li D K, Wei X M. Dislocated function projective partial synchronization between dynamical systems[J]. *Alexandria Engineering Journal*, 2023, **66**: 919-926.
- [19] Li R H, Chen W S, Li S. Finite-time stabilization for hyper-chaotic Lorenz system families via adaptive control[J]. *Applied Mathematical Modelling*, 2013, **37**(4): 1966-1972.
- [20] Yan H, Gao Y B. Synchronization for chaotic Lure systems via delayed feedback PD control [J]. *Journal of Nantong University*, 2017, **16**(4): 12-21(Ch).
- [21] Li D K. Generalized dislocated modified functional projective synchronization of chaotic systems [J]. *Journal of Southwest University*, 2019, **41**(11): 37-46(Ch).
- [22] Li D K. Hopf bifurcation and circuit implementation of a new hyperchaotic Lorenz system [J]. *Journal of Ningxia University*, 2016, **37**(3): 294-301(Ch).
- [23] He H J, Cui Y, Sun G. Time-delay Rucklidge system Hopf bifurcation analysis and circuit simulation[J]. *Chinese Journal of Computational Mechanics*, 2019, **36**(4): 542-547(Ch).

□